RIEMANN-HILBERT-BIRKHOFF INVERSE PROBLEM FOR SEMISIMPLE FLAT F-MANIFOLDS, AND CONVERGENCE OF ORIENTED ASSOCIATIVITY POTENTIALS

GIORDANO COTTI°

° *Faculdade de Ciências da Universidade de Lisboa*
Grupo de Física Matemática
Campo Grande Edifício C6, 1749-016 Lisboa, Portugal

Abstract. In this paper, we address the problem of classification of quasi-homogeneous formal power series providing solutions of the oriented associativity equations. Such a classification is performed by introducing a system of monodromy local moduli on the space of formal germs of homogeneous semisimple flat F-manifolds. This system of local moduli is well-defined on the complement of the *doubly resonant* locus, namely the locus of formal germs of flat F-manifolds manifesting both coalescences of canonical coordinates at the origin, and resonances of their *conformal dimensions*. It is shown how the solutions of the oriented associativity equations can be reconstructed from the knowledge of the monodromy local moduli via a Riemann-Hilbert-Birkhoff boundary value problem. Furthermore, standing on results of B. Malgrange and C. Sabbah, it is proved that any formal homogeneous semisimple flat F-manifold, which is not doubly resonant, is actually convergent. Our semisimplicity criterion for convergence is also reformulated in terms of solutions of Losev-Manin commutativity equations, growth estimates of correlators of F-cohomological field theories, and solutions of open Witten-Dijkgraaf-Verlinde-Verlinde equations.

Contents

° *E-mail*: gcotti@fc.ul.pt, gcotti@sissa.it

1. Introduction

Oriented associativity equations. In this paper, we address both the problem of classification and the convergence issues of formal solutions, in the ring of formal power series with complex coefficients, of the *oriented associativity equations* [LM00, LM04, Man05]. These consist of the overdetermined system of non-linear partial differential equations, in n functions $F^1(\boldsymbol{t}), \dots, F^n(\boldsymbol{t})$ depending on n variables $\boldsymbol{t} = (t^1, \dots, t^n)$, given by

$$\sum_\mu \frac{\partial F^\alpha}{\partial t^\beta \partial t^\mu}\frac{\partial F^\mu}{\partial t^\gamma \partial t^\varepsilon} = \sum_\mu \frac{\partial F^\alpha}{\partial t^\varepsilon \partial t^\mu}\frac{\partial F^\mu}{\partial t^\gamma \partial t^\beta}, \qquad \alpha, \beta, \gamma, \varepsilon = 1, \dots, n,$$

$$\sum_\mu A^\mu \frac{\partial F^\alpha}{\partial t^\mu \partial t^\beta} = \delta^\alpha_\beta, \quad A^\mu \in \mathbb{C}, \qquad \alpha, \beta = 1, \dots, n.$$

The oriented associativity equations are a natural generalization of Witten-Dijkgraaf-Verlinde -Verlinde (WDVV) associativity equations [Wit90, DVV91]. Their solutions $(F^1, \dots, F^n)$

reflect the rich geometry of *F-manifolds with a compatible flat structure*, for short *flat F-manifolds* [Man05].

Flat F-manifolds. In the early 1990s, B. Dubrovin introduced Frobenius manifolds as geometrical materialization of solutions of WDVV equations [Dub92, Dub96, Dub98, Dub99]. This notion turns up in many areas of Mathematics: for example Frobenius manifolds play a key role in mirror symmetry, singularity theory, quantum cohomology, integrable systems, and symplectic geometry.

It was soon realized, however, that weaker (i.e. with relaxed axioms) variants of the Frobenius structure are of interest *per se*. The core notion is that of F-manifolds, introduced by C. Hertling and Yu.I. Manin in [HM99]. Such a notion not only strictly includes the Frobenius structures, but it also greatly broadens the scope of the examples and applications. Examples of F-manifolds, indeed, arise not only in singularity theory [Her02], but also in quantum K-theory [Lee04], differential-graded deformation theory [Mer04, Mer06], and even information geometry [CM20].

Flat F-manifolds – introduced by Yu.I. Manin [Man05]– are an intermediate notion, weaker than Frobenius, but stronger than F-manifolds:

$$\text{Frobenius manifolds} \quad \subset \quad \text{flat } F\text{-manifolds} \quad \subset \quad F\text{-manifolds}.$$

Flat F-manifolds are equipped with the minimum amount of structures to share some of the deeper properties of Frobenius manifolds, including Dubrovin's deformed connection, Dubrovin's almost duality, and also operadic descriptions. See [Man04, Man05, LPR11, AL13, AL17]. These structures are also studied in [Get04], where they are called *Dubrovin manifolds*.

A flat F-manifold (in the analytic category) is a complex manifold M whose tangent spaces are equipped with an associative, commutative and unital algebra structure –analytically depending in the point– whose product $\circ$ is *compatible* with a given flat connection ∇. This means that each element of the pencil $(\nabla^z)_{z\in\mathbb{C}}$, defined by $\nabla^z_X Y = \nabla_X Y + zX\circ Y$, is required to be flat and torsionless.

The compatibility of $(\circ, \nabla)$ implies a *potentiality* condition: in ∇-flat local coordinates $\boldsymbol{t} = (t^1, \dots, t^n)$ on M, with $n = \dim_{\mathbb{C}} M$, the product $\circ$ descends from a vector potential: there exists $\boldsymbol{F} = (F^1, \dots, F^n)$ such that

$$\frac{\partial}{\partial t^\beta} \circ \frac{\partial}{\partial t^\gamma} = \sum_\alpha \frac{\partial^2 F^\alpha}{\partial t^\beta \partial t^\gamma} \frac{\partial}{\partial t^\alpha}, \quad \beta, \gamma = 1, \dots, n. \tag{1.1}$$

The associativity of $\circ$ is equivalent to the oriented associativity equations for $\boldsymbol{F}$. Vice-versa, starting from a solution $\boldsymbol{F}$ of the oriented associativity equations, we can define a flat F-structure via equation (1.1). If the starting solution $\boldsymbol{F}$ is a tuple of formal power series in $k[\![\boldsymbol{t}]\!]$ (with k a $\mathbb{Q}$-algebra), the resulting flat F-structure is said to be *formal* over k. It can be seen as a flat F-structure on the formal spectrum $\operatorname{Spf} k[\![\boldsymbol{t}]\!]$.

Homogeneity, semisimplicity, and double resonance. In this paper we consider only *quasi-homogenous* solutions $\boldsymbol{F}$ of the oriented associativity equations, i.e. satisfying a further condition of the form

$$\sum_{\alpha}[(1-q_\alpha)t^\alpha+r^\alpha]\frac{\partial F^\beta}{\partial t^\alpha}=(2-q_\beta)F^\beta(\boldsymbol{t})+\text{linear terms in }\boldsymbol{t},$$

for suitable complex numbers $q_\alpha, r^\alpha\in\mathbb{C}$. The resulting flat F-manifold is said to be *homogeneous*, or *conformal*. The vector field $E=\sum_\alpha[(1-q_\alpha)t^\alpha+r^\alpha]\frac{\partial}{\partial t^\alpha}$ is then an *Euler* vector field, i.e. it satisfies the conditions $\nabla\nabla E=0$ and $\mathfrak{L}_E(\circ)=\circ$. We say that $p\in M$ is *tame* if the spectrum of the operator $(E\circ)_p\in\mathrm{End}(T_pM)$ is simple, otherwise we say that p is *coalescing*.

An analytic flat F-manifold is said to be *semisimple* if there exists an open dense subset of points p whose corresponding algebra $(T_pM,\circ_p)$ is without nilpotent elements. This is equivalent to the existence of idempotents vectors $\pi_1,\dots,\pi_n\in T_pM$: $\pi_i\circ\pi_j=\pi_i\delta_{ij}$ for $i,j=1,\dots,n$. If a manifold is both homogenous and semisimple, the eigenvalues of the tensor $(E\circ)$ can be chosen as local holomorphic coordinates in a neighborhood of any semisimple point $p\in M$. Tame points are necessarily semisimple, whereas coalescing points may or may not be semisimple.

With each homogeneous semisimple (analytic/formal) flat F-manifolds we can associate a tuple $(\delta_1,\dots,\delta_n)$ of numerical invariants called *conformal dimensions*. Fix a semisimple point $p\in M$, and introduce the operator

$$\mu_p^0\in\mathrm{End}(T_pM),\quad \mu_p^0\left(\frac{\partial}{\partial t^\alpha}\right)=q_\alpha\frac{\partial}{\partial t^\alpha},\quad \alpha=1,\dots,n.$$

The conformal dimensions can be defined as the numbers $\delta_1,\dots,\delta_n\in\mathbb{C}$ satisfying

$$\mu_p^0(\pi_i)\circ\pi_i=\delta_i\pi_i,\quad i=1,\dots,n.$$

They are defined up to ordering, and they actually do not depend on the chosen semisimple point $p\in M$. The flat F-manifold is be said to be *conformally resonant* if $\delta_i-\delta_j\in\mathbb{Z}\setminus\{0\}$ for some i,j. The conformal dimensions of a Frobenius manifold are all equal, with common value $\frac{d}{2}$ (the number $d\in\mathbb{C}$ is the *charge* of the Frobenius structure). In particular, Frobenius manifolds are never conformally resonant.

In the formal case, all the conditions on points introduced above (tameness, coalescence, semisimplicity) are intended to be referred to the origin $\boldsymbol{t}=0$, the only geometric point of the formal spectrum $\mathrm{Spf}\, k[\![\boldsymbol{t}]\!]$.

A (germ of) pointed flat F-manifold (M,p) is be said to be *doubly resonant* if M is conformally resonant, and p is coalescing.

Results. One of the main aspects of Dubrovin's analytic theory of Frobenius manifolds is their isomonodromic approach. Under the quasi-homogeneity assumption of the WDVV potential, the semisimple part of a Frobenius manifold can be locally identified with the space of isomonodromic deformation parameters of ordinary differential equations on $\mathbb{P}^1$ with rational coefficients, see [Dub98, Dub99].

In this paper, we extend to the case of homogenous semisimple flat F-manifolds both Dubrovin's analytical theory as well as its refinement developed in [CG17, CDG20]. The key ingredient is a family $(\widehat{\nabla}^\lambda)_{\lambda\in\mathbb{C}}$ of flat extended deformed connections on π^*TM, with $\pi\colon M\times\mathbb{C}^*\to\mathbb{C}^*$, whose restrictions to $M\times\{z\}$ equal $\nabla^z=\nabla+z(-\circ-)$. These families of flat connections $\widehat{\nabla}^\lambda$ on homogeneous flat F-manifolds were first introduced by Yu.I. Manin [Man05].

Remark 1.1. In [Sab07, Chapter VII, §1] the notion of *Saito structure without metric* is introduced. These structures are defined in terms of several data on a complex manifold M. Among them there is a flat meromorphic connection $\widehat{\nabla}$ on the bundle π^*TM on $M\times\mathbb{P}^1$. It turns out that the notion of Saito structure without metric is equivalent to the notion of homogeneous flat F-manifold, and that $\widehat{\nabla}$ is one of the connections $\widehat{\nabla}^\lambda$ above. See also [KMS20], in which it is shown that the space of isomonodromic deformation parameters of extended Okubo systems can be equipped with Saito structures without metrics.

For any germ (M,p), semisimple and not doubly resonant, we introduce a tuple of numerical data, the *monodromy data* of the flat F-manifold. These data split into two pieces: a pair (μ^λ, R) of matrices called "monodromy data at $z=0$", and a 4-tuple (S_1,S_2,Λ,C) of matrices called "monodromy data at $z=\infty$". Precise definitions are given in Section 4. In the case of Frobenius manifolds, all these data are subjected to several constraints: the final amount of data coincides with the 4-tuple (μ,R,S,C) of monodromy data introduced by Dubrovin in [Dub98, Dub99]. If M is analytic, the monodromy data define *local invariants* of M: if $p_1,p_2\in M$ are sufficiently close, the data of the germs $(M,p_1),(M,p_2)$ are equal.

Theorem 1.2 (Cf. Theorems 6.10, 6.15)**.** *Any homogeneous semisimple (analytic/formal) pointed germ of flat F-manifold, which is not doubly resonant, is uniquely determined by its monodromy data. In particular, the vector potential $\boldsymbol{F}$ can be explicitly reconstructed from the monodromy data via a Riemann-Hilbert-Birkhoff boundary value problem.*

We show that the totality of local isomorphism classes of germs of n-dimensional flat F-manifolds can be parametrized by points of a "stratified" space, whose generic stratum has dimension n^2. The monodromy data provide a system of local coordinates. The Frobenius structures correspond a locus of generic dimension $\frac{1}{2}(n^2-n)$. See Theorem 6.19.

The re-construction procedure of the flat F-structure is based on a crucial property of a joint system of "generalized" Darboux-Egoroff equations [Lor14]: solutions $\Gamma(\boldsymbol{u})$ of this system of non-linear partial differential equations are uniquely determined by its initial value Γ_o at one point $\boldsymbol{u}_o\in\mathbb{C}^n$ (with possibly $u_o^i=u_o^j$ for $i\neq j$), provided there is no conformal resonance. See Lemma 6.16.

We underline that both the initial values $(\boldsymbol{u}_o,\Gamma_o)$ and the monodromy data provide a system of local coordinates on the space of germs of flat F-manifolds. The reconstruction procedure of $\boldsymbol{F}$ in terms of the initial values, however, is generally impossible, the dependance being typically transcendental (e.g. for $n=3$, the general Darboux-Egoroff system reduces to the full-parameters family of Painlevé equations PVI, see [Lor14]). This makes the monodromy data "preferable" as a system of coordinates for the classification of flat F-structures.

There is a further advantage in choosing the monodromy data as a system of local moduli. Indeed, they make possible the study of convergence issues.

Theorem 1.3 (Cf. Theorem 6.17)**.** *Let $\boldsymbol{F}\in\mathbb{C}[\![\boldsymbol{t}]\!]^{\times n}$ be a quasi-homogeneous solution of the oriented associativity equations. If $\boldsymbol{F}$ defines a semisimple formal flat F-manifold, which is not doubly resonant, then $\boldsymbol{F}$ is a tuple of convergent functions.*

For the proof, we invoke results of B. Malgrange [Mal83a, Mal83b, Mal86] and C. Sabbah [Sab18] on the solvability of families of Riemann-Hilbert-Birkhoff problems. More precisely, we use their equivalent formulations given in [Cot20c]. Notice that Theorem 1.3 generalizes [Cot20c, Theorem 5.1].

Cohomological field theories. Frobenius structures are intrinsically correlated to the notion of cohomological field theories. The geometry of Frobenius structures reflects properties of the cohomology rings $H^\bullet(\overline{\mathcal{M}}_{0,n},\mathbb{C})$, with $n \geqslant 3$, and they can indeed be defined in terms of $H^\bullet(\overline{\mathcal{M}}_{0,n},\mathbb{C})$-valued poly-linear maps. See [KM94, Man99, Pan18].

At the level of flat F-manifolds, such a construction has been generalized in two different (a posteriori equivalent) ways. The first one is due to A. Losev and Yu.I. Manin [LM00, LM04], the second one to A. Buryak and P. Rossi [BR18]. See also [ABLR20a, ABLR20b, ABLR21].

In [LM00], a new compactification $\overline{L}_n$ of $\mathcal{M}_{0,n}$ is introduced. The boundary strata represent isomorphisms classes of stable n-pointed *chains* of projective lines, in which the marked points do not play a symmetric role. In [LM04], a notion of *genus 0 extended modular operad*, and $\mathcal{L}$-*algebras* over it are studied. Moreover, it is shown that the differential equations satisfied by generating functions of correlators of $\mathcal{L}$-algebras lead to two differential geometric pictures: the first one is the study of pencils of flat connections, based on the *commutativity equations*, the second one is the study of flat F-manifolds, based on the oriented associativity equations. These two pictures are actually locally equivalent, if a further amount of data – a *primitive element* – is given. See also the constructions in [Los97, Los98].

Notice that the compactifications $\overline{L}_n$ and $\overline{\mathcal{M}}_{0,n}$, and their higher genus analogs, both arise in the more general construction of [Has03] as compactified moduli spaces of *weighted* pointed stable curves, for two different choices of the weights. See also [Man04].

In [BR18] A. Buryak and P. Rossi introduced the notion of F-*cohomological field theories* (F-CohFT) as a generalization of both cohomological field theories [KM94, Man99], and partial cohomological field theories [LRZ15]. An F-CohFT is defined by the datum of a family of $H^\bullet(\overline{\mathcal{M}}_{g,n},\mathbb{C})$-valued poly-linear maps on a tensor product $V^*\otimes V^{\otimes n}$ (V is an arbitrary $\mathbb{C}$-vector space), which satisfy some natural $\mathfrak{S}_n$-equivariance and gluing properties. Given an F-CohFT, its *genus zero sector* (or *tree level*) defines a formal flat F-structure on V.

In Section 7, we review all these cohomological field theoretical approaches to flat F-manifolds, their equivalences, and we rephrase our semisimplicity criterion of convergence in terms of solutions of Losev-Manin commutativity equations, and growth estimates of correlators of F-CohFT's. Furthermore, in Appendix B we prove[1] that any formal flat F-manifold over $\mathbb{C}$ descend from a unique F-CohFT in the sense of P. Rossi and A. Buryak. This is in complete analogy with the Frobenius manifolds case, see [Man99].

Structure of the paper. In Section 2 we present some preliminary material and basic properties of flat F-manifolds, in both formal and analytic categories. We recall definitions of homogeneity, semisimplicity of flat F-manifolds. We also introduce the notion of local isomorphisms, pointed germs, and irreducibility of flat F-manifolds.

[1]The author is not aware of a proof of this fact in literature.

In Section 3, we firstly describe how to reconstruct oriented associativity potentials from deformed coordinates on a flat F-manifolds. We then introduce a family of flat extended deformed connections $\widehat{\nabla}^\lambda$, and we develop the analytical theory of its flat (co)sections. We study the differential system of $\widehat{\nabla}^\lambda$-flatness in three different frames: the flat frame, the idempotent frame, and a normalized idempotent frame (the normalizing factors are the so-called *Lamé coefficients*). We also introduce Darboux-Tsarev and Darboux-Egoroff systems of equations.

In Section 4, after introducing the notion of spectrum of a flat F-manifolds, we define the monodromy data of a homogeneous semisimple flat F-manifold. We also describe their mutual constraints. In Section 5, we then clarify the dependence of the monodromy data on all the choices normalizations involved in their definition. Different choices of normalizations affect the numerical values of the data via the action of suitable groups. We show that the analytic continuation of the flat F-structure is described by a braid group action on the tuple of monodromy data.

Section 6 contains the main results of the paper. After introducing the notion of admissible data and the related Riemann-Hilbert-Birkhoff (RHB) boundary value problem, we recall some results of B. Malgrange and C. Sabbah as formulated in [Cot20c]. We show that germs of flat F-structures can be constructed starting from solutions of RHB problems. Moreover, we show that any germ is of such a form: it can be reconstructed from its monodromy data. It is also proved that any formal germ of homogenous semisimple flat F-manifold (over $\mathbb{C}$), which is not doubly resonant, is actually convergent.

In Section 7, we review equivalent approaches for defining flat F-manifolds. We recall the notions of Losev-Manin commutativity equations, Losev-Manin cohomological field theories (LM-CohFT), and F-cohomological field theories (F-CohFT) in the sense of A. Buryak and P. Rossi. We discuss the equivalence of these notions. Furthermore, we also discuss relations with the *open* WDVV (OWDVV) equations. Our semisimplicity criterion is reformulated in terms of growth estimates of correlators of LM-CohFT and F-CohFT, and convergence of solutions of OWDVV equations.

In Appendix A, we provide a proof for the following characterization of irreducibility of flat F-structures in terms of Euler vector fields: a flat F-manifold is irreducible if and only if any two arbitrary Euler vector fields differ by a scalar multiple of the unit vector field.

In Appendix B, we prove that any formal flat F-manifold descends from a unique tree-level F-CohFT.

Acknowledgements. The author is thankful to D. Guzzetti, C. Hertling, A.R. Its, P. Lorenzoni, Yu.I. Manin, D. Masoero, A.T. Ricolfi, P. Rossi, V. Roubtsov, A. Varchenko, C. Sabbah, M. Smirnov, D. Yang for several valuable discussions. The author is thankful to the Hausdorff Research Institute for Mathematics (HIM) in Bonn, Germany, where this project was started, for providing excellent working conditions during the JTP "New Trends in Representation Theory". This research was supported by HIM (Bonn, Germany), and by the FCT Project PTDC/MAT-PUR/ 30234/2017 "Irregular connections on algebraic curves and Quantum Field Theory".

2. Flat F-manifolds

2.1. **Analytic flat F-manifolds.** Let M be a complex analytic manifold with dimension $\dim_{\mathbb{C}} M = n$. Denote by TM, T^*M the holomorphic tangent and cotangent bundles, and by $\mathscr{T}_M$, Ω^1_M their sheaves of sections. If $E \to M$ is a holomorphic bundle with sheaf of sections $\mathscr{E}$, we denote by $\Gamma(E) = \Gamma(M, \mathscr{E})$ the space of global sections. By usual abuse of notations, we will write $X \in \mathscr{E}$ for $X \in \Gamma(U, \mathscr{E})$ for some (or arbitrary) open set $U \subseteq M$.

Let (M, ∇, c, e) be the datum of

(1) a connection $\nabla\colon \mathscr{T}_M \to \Omega^1_M \otimes \mathscr{T}_M$ on TM;
(2) a section $c \in \Gamma\left(TM \otimes \bigodot^2 T^*M\right)$;
(3) a vector field $e \in \Gamma(TM)$ such that
 (a) $c(-, e, -) = c(-, -, e) \in \Gamma(\operatorname{End} TM)$ is the identity morphism,
 (b) $\nabla e = 0$.

Denote by $X \circ Y := c(-, X, Y)$ the commutative product defined by the tensor c, and introduce the one-parameter family of connections $(\nabla^z)_{z \in \mathbb{C}}$ defined by $\nabla^z_X Y := \nabla_X Y + zX \circ Y$ for $X, Y \in \mathscr{T}_M$.

Definition 2.1. We say that (M, ∇, c, e) is a *flat F-manifold* if the connection ∇^z is flat and torsionless for any $z \in \mathbb{C}$.

Let $\boldsymbol{t} = (t^1, \dots, t^n)$ be a system of ∇-flat local coordinates on M. Set $\partial_\alpha := \frac{\partial}{\partial t^\alpha}$, for $\alpha = 1, \dots, n$ and define $c^\alpha_{\beta\gamma} := c(dt^\alpha, \partial_\beta, \partial_\gamma)$. The flatness and torsionless of ∇^z is equivalent to the associativity of $\circ$, and the symmetry of $\partial_\alpha c^\delta_{\beta\gamma}$ in (α, β, γ). Hence, there locally exist analytic functions $\boldsymbol{F} = (F^1, \dots, F^n) \in \mathcal{O}^n_M$ such that

$$c^\alpha_{\beta\gamma} = \frac{\partial^2 F^\alpha}{\partial t^\beta \partial t^\gamma}, \quad \alpha, \beta, \gamma = 1, \dots, n.$$

In what follows the Einstein summation rule is used over repeated Greek indices. Let $A^\alpha \in \mathbb{C}$ be constants such that $e = A^\alpha \partial_\alpha$. From the associativity of $\circ$ and the properties of e, we have

$$A^\mu \frac{\partial^2 F^\alpha}{\partial t^\mu \partial t^\beta} = \delta^\alpha_\beta, \qquad \alpha, \beta = 1, \dots, n, \tag{2.1}$$

$$\frac{\partial^2 F^\alpha}{\partial t^\mu \partial t^\beta} \frac{\partial^2 F^\mu}{\partial t^\gamma \partial t^\delta} = \frac{\partial^2 F^\alpha}{\partial t^\mu \partial t^\gamma} \frac{\partial^2 F^\mu}{\partial t^\beta \partial t^\delta}, \qquad \alpha, \beta, \gamma, \delta = 1, \dots, n. \tag{2.2}$$

Equations (2.1), (2.2) are called *oriented associativity equations*, and $\boldsymbol{F}$ is the *oriented associativity potential* of the flat F-structure.

A flat F-manifold is said to be *homogeneous* if there it is equipped with an *Euler vector field*, i.e. a vector field $E \in \Gamma(TM)$ such that

$$\nabla\nabla E = 0, \quad \mathfrak{L}_E c = c.$$

Lemma 2.2. *We have $[e, E] = e$.*

Proof. The condition $\mathfrak{L}_E c = c$ is equivalent to $[E, Y \circ Z] - [E, Y] \circ Z - [E, Z] \circ Y = Y \circ Z$, for $Y, Z \in \mathscr{T}_M$. If $Y = Z = e$, the identity follows. $\square$

Homogeneous flat F-manifold are also called *Saito structures without metric*, [Sab07, Ch. VII]. We assume that the $(1,1)$-tensor ∇E is diagonalizable, and in diagonal form in the $\boldsymbol{t}$-coordinates:

$$E=\sum_{\alpha=1}^{n}((1-q_\alpha)t^\alpha+r^\alpha)\frac{\partial}{\partial t^\alpha},\quad q_\alpha,r^\alpha\in\mathbb{C}. \tag{2.3}$$

The condition $\mathfrak{L}_E c=c$ is thus equivalent to

$$E^\mu\frac{\partial F^\alpha}{\partial t^\mu}=(2-q_\alpha)F^\alpha+A^\alpha_\beta t^\beta+B^\alpha,\quad A^\alpha_\beta,B^\alpha\in\mathbb{C}. \tag{2.4}$$

Definition 2.3. A flat F-manifold (M,∇,c,e) is a *Frobenius manifold* if there exist a symmetric non-degenerate $\mathscr{O}_M$-bilinear form $\eta\in\Gamma(\bigodot^2 T^*M)$, called *metric*, such that

$$\nabla\eta=0,\quad\text{and}\quad\eta(X\circ Y,Z)=\eta(X,Y\circ Z),\quad X,Y,Z\in\mathscr{T}_M. \tag{2.5}$$

In such a case, ∇ is the Levi-Civita connection of η. A vector field $E\in\Gamma(TM)$ is *Euler* if it satisfies the conditions

$$\mathfrak{L}_E c=c,\quad\mathfrak{L}_E\eta=(2-d)\eta, \tag{2.6}$$

where the number $d\in\mathbb{C}$ is the *conformal dimension* (or *charge*) of the Frobenius manifold.

Remark 2.4. An Euler vector field E for a Frobenius manifold is automatically an Euler vector field for the underlying flat F-manifold. The condition $\nabla\nabla E=0$ is indeed implied by the conformal Killing condition (2.6), and the flatness of ∇.

Remark 2.5. In the case a flat F-manifold is actually Frobenius, the oriented associativity potentials $\boldsymbol{F}=(F^1,\dots,F^n)$, solutions of (2.1) and (2.2), can be shown to locally descend from a single WDVV potential $F(\boldsymbol{t})$, i.e. a solution of the system of equations

$$\begin{aligned}
&A^\mu\frac{\partial^3 F}{\partial t^\mu\partial t^\alpha\partial t^\beta}=\eta_{\alpha\beta}=\text{const.},\quad\eta=(\eta_{\alpha\beta})_{\alpha,\beta},\quad\eta^{-1}=(\eta^{\alpha\beta})_{\alpha,\beta} && \alpha,\beta=1,\dots,n,\\
&\frac{\partial^3 F}{\partial t^\alpha\partial t^\beta\partial t^\mu}\eta^{\mu\nu}\frac{\partial^3 F}{\partial t^\nu\partial t^\gamma\partial t^\delta}=\frac{\partial^3 F}{\partial t^\delta\partial t^\beta\partial t^\mu}\eta^{\mu\nu}\frac{\partial^3 F}{\partial t^\nu\partial t^\gamma\partial t^\alpha}, && \alpha,\beta,\gamma,\delta=1,\dots,n.
\end{aligned}$$

The potentials F^α's are the components of the η-gradient of F, that is $F^\alpha(\boldsymbol{t})=\eta^{\alpha\beta}\partial_\beta F(\boldsymbol{t})$.

2.2. **Formal flat F-manifolds.** Let

- k be a commutative $\mathbb{Q}$-algebra,
- H be a free k-module of finite rank,
- $K:=k[\![H^*]\!]$ be the completed symmetric algebra of $H^*:=\mathrm{Hom}_k(H,k)$.

Fix a basis $(\Delta_1,\dots,\Delta_n)$ of H, and denote by $\boldsymbol{t}=(t^1,\dots,t^n)$ the dual coordinates. The algebra K is then identified with the algebra of formal power series $k[\![\boldsymbol{t}]\!]$. Denote by $\mathrm{Der}_k(K)$ the K-module of k-linear derivations of K. Put $\partial_\alpha=\frac{\partial}{\partial t^\alpha}\colon K\to K$. The module $\mathrm{Der}_k(K)$ is a free K-module with basis $(\partial_1,\dots,\partial_n)$. We will write Φ_α for $\partial_\alpha\Phi$ for $\Phi\in K$.

Elements of $H_K:=K\otimes_k H$ will be identified with derivations on K, by $\Delta_\alpha\mapsto\partial_\alpha$.

Definition 2.6. A *formal flat F-manifold structure* on H is given by an n-tuple $\boldsymbol{\Phi}=(\Phi_1,\dots,\Phi_n)\in K^n$, satisfying the oriented associativity equations (2.1), (2.2), where $A^\mu\in k$.

Define the K-linear multiplication $\circ$ on H_K by

$$\Delta_\alpha \circ \Delta_\beta := c^\gamma_{\alpha\beta}\Delta_\gamma, \quad c^\gamma_{\alpha\beta} := \frac{\partial^2 \Phi^\gamma}{\partial t^\alpha \partial t^\beta} \quad \alpha, \beta = 1, \dots, n.$$

The oriented associativity equations imply that $\circ$ is associative, and that $e := A^\mu \Delta_\mu$ is the unit of the algebra $(H_K, \circ)$. A vector $E \in H_K$ is an *Euler* vector if it is if the form (2.3), and the pair $(E, \mathbf{\Phi})$ satisfies equations (2.4).

Let $\mathrm{Diff}_1(H_K, H_K)$ denote the set of $\mathscr{D} \in \mathrm{Hom}_k(H_K, H_K)$ such that

$$ab\mathscr{D}(p) - b\mathscr{D}(ap) - a\mathscr{D}(bp) + \mathscr{D}(abp) = 0, \quad a, b \in K, \quad p \in H_K.$$

Both $\mathrm{Der}_k(K)$ and $\mathrm{Diff}_1(H_K, H_K)$ are naturally equipped with a K-module structure. A (formal) connection on H_K is defined by a K-linear morphism $\nabla\colon \mathrm{Der}_k(K) \to \mathrm{Diff}_1(H_K, H_K)$, $u \mapsto \nabla_u$ satisfying the Leibniz rule

$$\nabla_u(ap) = u(a)p + a\nabla_u p, \quad a \in K, \quad p \in H_K.$$

The torsion and curvature of ∇ are the K-bilinear morphisms $T, R\colon \mathrm{Der}_k(K) \times \mathrm{Der}_k(K) \to \mathrm{Hom}_K(H_K, H_K)$ defined by

$$\begin{aligned} T(u, v) &:= \nabla_u v - \nabla_v u - [u, v], && u, v \in \mathrm{Der}_k(K) \cong H_K, \\ R(u, v) &:= [\nabla_u, \nabla_v] - \nabla_{[u,v]}, && u, v \in \mathrm{Der}_k(K). \end{aligned}$$

We can thus introduce the one-parameter family $(\nabla^z)_{z\in k}$ of (formal) connection given by $\nabla^z_{\partial_\alpha}\partial_\beta := z\partial_\alpha \circ \partial_\beta$.

Remark 2.7. If $\boldsymbol{F} := (F^1, \dots, F^n)$ is a (formal/analytic) solution of the oriented associativity equations (2.2), then also $\widetilde{\boldsymbol{F}} := (\lambda_1 F^1, \dots, \lambda_n F^n)$ is a solution for any $(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$. If the original flat F-manifold has a unit $e = A^\mu\partial_\mu$, then the rescaled flat F-manifold structure has unit $e' = \frac{1}{\lambda_\mu}A^\mu\partial_\mu$.

Remark 2.8. If E is an Euler vector field for a given flat F-manifold structure, then also $E - \lambda e$ is an Euler vector field, for $\lambda \in \mathbb{C}$. Under two further assumptions of *semisimplicity* and *irreducibility*, one can prove that any Euler vector field is of this form. See Theorem 2.17.

2.3. **Local isomorphisms and pointed germs.** Let $(M_i, \nabla_i, c_i, e_i)$, with $i = 1, 2$, be two analytic flat F-manifolds. A biholomorphism $\varphi\colon M_1 \to M_2$ is an *isomorphism* of flat F-manifolds if

1. $d\varphi(\ker \nabla_1) \subseteq \ker(\varphi^*\nabla_2)$, where $d\varphi\colon \mathscr{T}_{M_1} \to \varphi^*\mathscr{T}_{M_2}$ is the differential of φ, and $\varphi^*\nabla_2$ is the pulled-back connection on $\varphi^* TM_2$,
2. for each $p \in M_1$, the map $d\varphi_p\colon T_pM_1 \to T_{\varphi(p)}M_2$ is an isomorphism of unital algebras.

Lemma 2.9. *Let M_1 and M_2 be two isomorphic flat F-manifolds. Given two systems of local flat coordinates, $\boldsymbol{t}$ on M_1 and $\tilde{\boldsymbol{t}}$ on M_2, the corresponding local potentials $\boldsymbol{F}_1(\boldsymbol{t})$ and $\boldsymbol{F}_2(\tilde{\boldsymbol{t}})$ are related by*

$$\begin{gathered} F_2^\alpha(\tilde{\boldsymbol{t}}) = G^\alpha_\lambda F_1^\lambda(\boldsymbol{t}) + \textit{linear terms in } \boldsymbol{t}, \qquad \alpha = 1, \dots, n, \\ \tilde{\boldsymbol{t}} = \varphi(\boldsymbol{t}) = \boldsymbol{G}\boldsymbol{t} + \boldsymbol{c}, \end{gathered}$$

where $\boldsymbol{G} \in GL(n,\mathbb{C})$ and $\boldsymbol{c} \in \mathbb{C}^n$ is a constant vector.

Vice-versa, $\boldsymbol{F}_1(\boldsymbol{t}), \boldsymbol{F}_2(\boldsymbol{t})$ be two solutions of (2.1) and (2.2). If they define isomorphic flat F-structures, then there exist $\lambda_\alpha \in \mathbb{C}^*$ such that

$$F_2^\alpha(\boldsymbol{t}) = \lambda_\alpha F_1^\alpha(\boldsymbol{t}) + \textit{linear terms in } \boldsymbol{t}, \quad \alpha = 1,\dots,n. \qquad \square$$

A *pointed flat F-manifold* is a pair (M,p), where M is a flat F-manifold, and $p \in M$ is a fixed *base point*. Isomorphisms between pointed flat F-manifold will always be assumed to be base point preserving. Given (M,p) we will always consider flat coordinates $\boldsymbol{t}$ vanishing at p.

Two pointed flat F-manifolds (M_1,p_1), (M_2,p_2) are *locally isomorphic* if there exist open neighborhoods $\Omega_1 \subseteq M_1$ of p_1, and $\Omega_2 \subseteq M_2$ of p_2 respectively, with isomorphic induced flat F-structures, i.e. $(\Omega_1,p_1) \cong (\Omega_2,p_2)$.

A *pointed germ* is a (local isomorphism) equivalence class of pointed flat F-manifolds.

Any analytic pointed flat F-manifold (M,p) induces a formal flat F-manifold $(H,\boldsymbol{\Phi})$ over $k=\mathbb{C}$. Choose flat coordinates $\boldsymbol{t}$ vanishing at p, and set $H := T_pM$. Let $\mathcal{O}_{M,p}$ be the local ring of germs at p, and $\mathfrak{m}$ be its maximal ideal. The formal potential Φ^α is given by the image of F^α in the completion $\widehat{\mathcal{O}_{M,p}} := \varprojlim \left(\mathcal{O}_{M,p}/\mathfrak{m}^\ell\right)$ of the local ring $\mathcal{O}_{M,p}$: this means that Φ^α is defined by the Taylor series expansion of F^α at p in coordinates $\boldsymbol{t}$. Moreover, the formal flat F-structure $(H,\boldsymbol{\Phi})$ is also equipped with a flat unit $e|_p$. If M has Euler vector field E, then $(H,\boldsymbol{\Phi})$ has Euler vector field $E|_p$. We will say that the formal flat F-structure constructed in this way, starting from an analytic one, is *convergent*.

Vice-versa, let us assume that $(H,\boldsymbol{\Phi})$ is a formal flat F-structure over $k=\mathbb{C}$ (with Euler vector field). If the common domain of convergence $\Omega \subseteq H$ of the power series $\Phi^\alpha \in k[\![\boldsymbol{t}]\!]$ is non-empty, it is easily seen that Ω is equipped with an analytic flat F-structure (with Euler vector field).

2.4. **Semisimple flat F-manifolds.** Let (M,∇,c,e) be an analytic flat F-manifold. A point $p \in M$ is called *semisimple* if the algebra $(T_pM,\circ_p)$ is without nilpotent elements. This is equivalent to

- the existence of idempotent vectors $\pi_1,\dots,\pi_n \in T_pM$, i.e. such that $\pi_i \circ_p \pi_j = \pi_i\delta_{ij}$,
- the existence of $v \in T_pM$ such that $v\circ_p\colon T_pM \to T_pM$ has simple spectrum.

Semisimplicity is an open property: if $p \in M$ is semisimple, then there exists an open neighborhood $\mathcal{V}$ of p, such that any point of $\mathcal{V}$ is semisimple. Moreover, if $\mathcal{V}$ is small enough, we have well defined local idempotent holomorphic vector fields $\pi_1,\dots,\pi_n \in \Gamma(\mathcal{V},\mathscr{T}_M)$. See e.g. [Her02, Ch. II] for a detailed discussion.

Let $(H,\boldsymbol{\Phi})$ be a formal flat F-manifold. Denote by $\circ_0$ the product on H defined by structure constants $c^\alpha_{\beta\gamma}(0) := \partial^2_{\alpha\beta}\Phi|_{\boldsymbol{t}=0}$. We will say that $(H,\boldsymbol{\Phi})$ is

- *semisimple at the origin* if we have an isomorphisms of k-algebras $(H,\circ_0) \cong k^n$,
- *formally semisimple* if we have an isomorphism of K-algebras $(H,\circ) \cong K^n$.

Formal semisimplicity is thus equivalent to the existence of vectors $\pi_i \in H_K$ such that $\pi_i \circ \pi_j = \pi_i\delta_{ij}$.

Lemma 2.10. *A formal flat F-manifold is formally semisimple iff it is semisimple at the origin.*

Proof. The proof of [Cot20c, Lemma 4.2] works verbatim. $\square$

Remark 2.11. In both formal and analytic case, we have $e = \sum_{i=1}^{n} \pi_i$.

Proposition 2.12. *For both formal and analytic semisimple flat F-manifolds, the idempotents vectors $\pi_1, \dots, \pi_n$ are pairwise commuting, i.e. $[\pi_i, \pi_j] = 0$. Hence, there exist local coordinates $\boldsymbol{u} = (u^1, \dots, u^n)$ such that $\pi_i = \frac{\partial}{\partial u^i}$ for $i = 1, \dots, n$. The local coordinates $\boldsymbol{u}$ will be called* canonical. $\square$

In the formal case, the functions u^i are formal functions, i.e. elements of K. Canonical coordinates $\boldsymbol{u}$ are defined up to permutations and shifts by constants. We set $\partial_i := \frac{\partial}{\partial u^i}$ for $i = 1, \dots, n$. If an Euler vector field is given, then the shift freedom can be actually frozen.

Proposition 2.13. *A vector field $E \in \Gamma(TM)$ satisfies $\mathfrak{L}_E c = c$ iff in canonical coordinates it has the form $E = \sum_j (u^j + c^j)\partial_j$. Up to shifts of canonical coordinates $\boldsymbol{u}$, we have $E = \sum_j u^j \partial_j$.* $\square$

Hence the eigenvalues of the tensor $(E\circ) \in \Gamma(\operatorname{End} TM)$ may and will be chosen as local canonical coordinates.

Definition 2.14. A point $p \in M$ will be called

- *tame* if the operator $E\circ_p \colon T_pM \to T_pM$ has simple spectrum
- *coalescing*, otherwise.

If a point is tame, then it is necessarily semisimple, and with pairwise distinct canonical coordinates.

The same definition can adapted to the formal case, relatively at the origin $\boldsymbol{t} = 0$, by looking at the spectrum of $E\circ_0 \colon H \to H$.

At coalescing points p, we have $\boldsymbol{u}(p) \in \Delta$ where $\Delta \subseteq \mathbb{C}^n$ denotes the *big diagonal*

$$\Delta := \bigcup_{i \neq j} \{u_i = u_j\}.$$

For a given flat F-manifold M, denote by $\operatorname{Aut}(M)$ the group of isomorphisms $\varphi \colon M \to M$. For semisimple flat F-manifolds, Proposition 2.12 allows to compute the connected component $\operatorname{Aut}(M)_0$ of the identity.

Proposition 2.15. *If M is a semisimple flat F-manifold, then the connected component $\operatorname{Aut}(M)_0$ of the identity is a commutative n-dimensional Lie group. Moreover, it acts locally transitively on M.*

Proof. The Lie algebra of $\operatorname{Aut}(M)$ can be identified with the Lie algebra of vector fields $X \in \Gamma(TM)$ on M such that $\mathfrak{L}_X c = 0$. This is equivalent to the condition

$$[X, Y \circ Z] - [X, Y] \circ Z - [X, Z] \circ Y = 0, \quad Y, Z \in \mathscr{T}_M.$$

In local canonical coordinates $\boldsymbol{u}$, set $X = \sum_i X^i \partial_i$ and take $Y = Z = \partial_j$. We have $\partial_j X^k = 0$ for all j, k. Hence, locally X is a constant linear combination of the idempotent vector fields ∂_j. In local canonical coordinates, the flow of X reads as shifts $u^i \mapsto u^i + c^i$. $\square$

Remark 2.16. In [AL13] a notion of *bi-flat* F-manifold is introduced. This consists of the datum of two flat F-manifolds structures $(\nabla, \circ, e)$ and $(\nabla^*, *, E)$, on the same manifold M, satisfying the following compatibility conditions:

(1) E is $\circ$-invertible on M,
(2) $\mathfrak{L}_E(\circ) = \circ$,
(3) $X * Y = E^{-1} \circ X \circ Y$, for all $X, Y \in \mathscr{T}_M$,
(4) $(d_\nabla - d_{\nabla^*})(X\circ) = 0$, for all $X \in \mathscr{T}_M$ (here d_∇ is the exterior covariant derivative).

In [AL17], in the tame semisimple case (pairwise distinct canonical coordinates), it is proved that a bi-flat F structure is actually *equivalent* to the datum of a homogeneous flat F-manifold with invertible Euler vector field E.

2.5. **Irreducible flat F-manifolds.** If M_1, M_2 are two flat F-manifolds, their product $M_1 \times M_2$ is naturally equipped with a flat F-structure, called the *sum* $M_1 \oplus M_2$. If M_1, M_2 are homogenous, then also $M_1 \oplus M_2$ is homogeneous.

We say that a flat F-manifold M is *irreducible* if no pointed germ (M, p) is locally isomorphic to a pointed sum $(M_1 \oplus M_2, p')$.

In the semisimple homogeneous case, we have the following characterization of irreducibility.

Theorem 2.17. *Let M be a formal/analytic semisimple and homogeneous flat F-manifold. The following conditions are equivalent:*

(1) *M is irreducible;*
(2) *if $E_1, E_2 \in \Gamma(TM)$ are two Euler vector fields, then $E_2 = E_1 - \lambda e$ for some $\lambda \in \mathbb{C}$.* □

The proof of this result can be found in Appendix A.

Remark 2.18. Theorem 2.17 underlines how much selective is the condition $\nabla\nabla E = 0$. In the category of F-manifolds (not necessarily flat), Euler vector fields are defined[2] by the condition $\mathfrak{L}_E c = c$ only. For semisimple F-manifolds, given an Euler vector field E, all other Euler fields are of the form $E + \sum_{i=1}^n \mathbb{C}\pi_i$. See [Her02, Ex. 2.12(ii)].

3. Extended deformed connections

3.1. **∇^z-flat coordinates and oriented associativity potentials.** In both the analytic and the formal context (over k), we can look for ∇^z-flat coordinates of the flat F-structure, i.e. functions $\tilde{t}^\alpha(\boldsymbol{t}, z)$ such that $\nabla^z d\tilde{t}^\alpha = 0$. Assume they are of the form

$$\tilde{t}^\alpha(\boldsymbol{t}, z) := \sum_{p=0}^{\infty} h_p^\alpha(\boldsymbol{t}) z^p \in k[\![\boldsymbol{t}, z]\!], \quad h_0^\alpha(\boldsymbol{t}) = t^\alpha, \quad \alpha = 1, \dots, n.$$

Theorem 3.1. *The functions h_p^α satisfy the recursive equations*

$$h_0^\alpha(\boldsymbol{t}) = t^\alpha, \quad \partial_\gamma \partial_\beta h_{p+1}^\alpha = c_{\gamma\beta}^\lambda \partial_\lambda h_p^\alpha, \quad p \in \mathbb{N}.$$

Proof. The ∇^z-flatness equations for a one-form $\xi = \xi_\alpha dt^\alpha$ are $\partial_\gamma \xi_\beta = z c_{\gamma\beta}^\lambda \xi_\lambda$. □

[2] A more general notion of Euler vector field of *weight* $d \in \mathbb{C}$ is discussed in [Her02, Man99]: these are vector fields E such that $\mathfrak{L}_E c = d \cdot c$. If $d \neq 0$, one can always rescale E in order to be of weight 1.

Corollary 3.2. *The functions $h_1^\alpha(\boldsymbol{t})$ equal the oriented associativity potentials $F^\alpha(\boldsymbol{t})$ up to linear terms.*

Proof. We have $\partial_\gamma\partial_\beta h_1^\alpha = c_{\beta\gamma}^\alpha$. □

3.2. **Family of extended deformed connections.** Following [Man05, Section 3], we introduce a one-parameter family $(\widehat{\nabla}^\lambda)_\lambda$ of flat connections, which "rigidify" the family $(\nabla^z)_z$. See also [BB19, Section 4.3], [ABLR20a, Section 1.4].

Analytic case. Let (M, ∇, c, e, E) be a homogenous flat F-manifold. Introduce the $(1,1)$-tensors $\mathcal{U}, \mu^{(\lambda)} \in \Gamma(\mathrm{End}(TM))$, with $\lambda \in \mathbb{C}$, by the formulae

$$\mathcal{U}(X) = E \circ X, \quad \mu^\lambda(X) := (1-\lambda)X - \nabla_X E, \quad X \in \mathscr{T}_M.$$

By equation (2.3), in $\boldsymbol{t}$-coordinates we have $\mu^\lambda = \mathrm{diag}(q_1 - \lambda, \dots, q_n - \lambda)$.

Denote by $\pi\colon M \times \mathbb{C}^* \to M$ the canonical projection on the first factor. If $\mathscr{T}_M$ denotes the tangent sheaf of M, then $\pi^*\mathscr{T}_M$ is the sheaf of sections of π^*TM, and $\pi^{-1}\mathscr{T}_M$ is the sheaf of sections of π^*TM constant along the fibers of π. All the tensors $c, e, E, \mathcal{U}, \mu$ can be lifted to the pullback bundle π^*TM, and we denote these lifts with the same symbols. Consequently, also the connection ∇ can be uniquely lifted on π^*TM in such a way that $\nabla_{\frac{\partial}{\partial z}} Y = 0$ for $Y \in \pi^{-1}\mathscr{T}_M$.

The extended deformed connection $\widehat{\nabla}^\lambda$, with $\lambda \in \mathbb{C}$, is the connection on π^*TM defined by the formulae

$$\widehat{\nabla}^\lambda_{\frac{\partial}{\partial t^\alpha}} Y = \nabla_{\frac{\partial}{\partial t^\alpha}} Y + z\frac{\partial}{\partial t^\alpha} \circ Y, \qquad \widehat{\nabla}^\lambda_{\frac{\partial}{\partial z}} Y = \nabla_{\frac{\partial}{\partial z}} Y + \mathcal{U}(Y) - \frac{1}{z}\mu^\lambda(Y), \tag{3.1}$$

where $Y \in \pi^*\mathscr{T}_M$.

Formal case. Let k be a commutative $\mathbb{Q}$-algebra and $(H, \boldsymbol{\Phi}, E)$ a formal homogeneous flat F-manifold manifold over k. Denote by $k(\!(z)\!)$ the k-algebra of formal Laurent series in an auxiliary indeterminate z. Set $K(\!(z)\!) := k[\![\boldsymbol{t}]\!](\!(z)\!)$ and $H_{K(\!(z)\!)} := H \otimes_k K(\!(z)\!)$.

In what follows we assume that the K-linear operator $\nabla^0 E\colon \mathrm{Der}_k(K) \cong H_K \to H_K$ is (diagonalizable and) in diagonal form in the basis $(\Delta_1, \dots, \Delta_n)$. Define the K-linear operators $\mathcal{U}, \mu^\lambda$, with $\lambda \in k$, by the formulae

$$\begin{aligned} \mathcal{U}\colon H_K \to H_K, &\qquad X \mapsto E \circ X,\\ \mu^\lambda\colon \mathrm{Der}_k(K) \cong H_K \to H_K, &\qquad X \mapsto (1-\lambda) - \nabla_X E. \end{aligned}$$

All the tensors $\circ, \mathcal{U}, \mu^\lambda$ can be $K(\!(z)\!)$-linearly extended to $H_{K(\!(z)\!)}$. We will denote such an extension by the same symbols.

Denote by $\mathrm{Diff}_1(H_{K(\!(z)\!)}, H_{K(\!(z)\!)})$ the set of morphisms $\mathscr{D} \in \mathrm{Hom}_k(H_{K(\!(z)\!)}, H_{K(\!(z)\!)})$ such that

$$ab\mathscr{D}(p) - b\mathscr{D}(ap) - a\mathscr{D}(bp) + \mathscr{D}(abp) = 0, \quad a, b \in K(\!(z)\!), \quad p \in H_{K(\!(z)\!)}.$$

Both $\mathrm{Der}_k(K(\!(z)\!))$ and $\mathrm{Diff}_1(H_{K(\!(z)\!)}, H_{K(\!(z)\!)})$ are naturally equipped with an $K(\!(z)\!)$-module structure.

The extended deformed connection $\widehat{\nabla}^\lambda\colon \mathrm{Der}_k(K(\!(z)\!)) \to \mathrm{Diff}_1(H_{K(\!(z)\!)}, H_{K(\!(z)\!)})$ is the $K(\!(z)\!)$-linear operator defined by the formulae

$$\widehat{\nabla}^\lambda_{\frac{\partial}{\partial t^\alpha}} X = \nabla^z_{\frac{\partial}{\partial t^\alpha}} X, \qquad \widehat{\nabla}^\lambda_{\frac{\partial}{\partial z}} X = \frac{\partial}{\partial z} X + \mathcal{U}(Y) - \frac{1}{z}\mu^\lambda(X),$$

where $Y \in H_{K(\!(z)\!)}$.

In both the analytic and formal pictures, the following result holds.

Theorem 3.3. *The connection $\widehat{\nabla}^\lambda$ is flat for any $\lambda \in \mathbb{C}$ (resp $\lambda \in k$).*

Proof. The flatness of $\widehat{\nabla}^\lambda$ is equivalent to the following conditions: $\partial_\alpha c^\delta_{\beta\gamma}$ is completely symmetric in (α, β, γ), the product $\circ$ is associative, $\nabla\nabla E = 0$, and $\mathfrak{L}_E c = c$. This can be easily checked by a straightforward computation. $\square$

Remark 3.4. For $\lambda = \frac{1}{2}d$, the connection $\widehat{\nabla}^\lambda$ equals the extended deformed connection $\widehat{\nabla}$ as defined by Dubrovin [Dub96, Dub98, Dub99]. In that case, the tensor $\mathcal{U}$ (resp. $\mu = \mu^{(\frac{d}{2})}$) is η-self-adjoint (resp. η-anti-self-adjoint). It follows that if $\zeta_1, \zeta_2 \in \pi^*\mathscr{T}_M$ are two $\widehat{\nabla}$-flat vector fields, then the pairings $\langle \zeta_1, \zeta_2 \rangle_\pm := \eta\left(\zeta_1(\boldsymbol{t}, e^{\pm\pi\sqrt{-1}}z), \zeta_2(\boldsymbol{t}, z)\right)$ do not depend on $(\boldsymbol{t}, z)$. See also [CDG20, Section 2].

3.3. **$\widehat{\nabla}^\lambda$-flat covectors.** In both the analytic and formal pictures, the extended connections $\widehat{\nabla}^\lambda$ induce connections on the whole tensor algebra of π^*TM (resp. $H_{K(\!(z)\!)}$). So, for example, let ξ denote a $\widehat{\nabla}^\lambda$-flat section of the bundle $\pi^*(T^*M)$. In the co-frame $(dt^\alpha)_{\alpha=1}^n$, the equation $\widehat{\nabla}^\lambda \xi = 0$ can be written, in more convenient matrix notations, as the joint system of differential equations

$$\frac{\partial \xi}{\partial t^\alpha} = z\mathcal{C}^T_\alpha \xi, \quad \frac{\partial \xi}{\partial z} = \left(\mathcal{U} - \frac{1}{z}\mu^\lambda\right)^T \xi, \tag{3.2}$$

where $\xi = (\xi_1, \dots, \xi_n)^T$ is a column vector of components w.r.t. $(dt^\alpha)_\alpha$, and

$$(\mathcal{C}_\alpha)^\gamma_\beta := c^\gamma_{\alpha\beta}, \quad (\mathcal{U})^\beta_\alpha = E^\varepsilon c^\beta_{\alpha\varepsilon}, \quad (\mu^\lambda)^\beta_\alpha = (q_\alpha - \lambda)\delta_{\alpha\beta}.$$

We will refer to the second of equations (3.2) as the *∂_z-equation* of the flat F-manifold.

3.4. **Matrices $\widetilde{\Psi}, \widetilde{V}_i, \widetilde{V}^\lambda, \widetilde{\Gamma}$.** Assume that (M, ∇, c, e, E) is a semisimple homogeneous flat F-manifold, and introduce the Jacobian matrix $\widetilde{\Psi}$ by

$$\widetilde{\Psi}^i_\alpha := \frac{\partial u^i}{\partial t^\alpha}, \quad i, \alpha = 1, \dots, n.$$

In canonical coordinates $\boldsymbol{u}$, under the gauge transformation $\tilde{x} = (\widetilde{\Psi}^{-1})^T \xi$, the system (3.2) becomes

$$\frac{\partial \tilde{x}}{\partial u^i} = \left(zE_i - \widetilde{V}_i\right)^T \tilde{x}, \quad \frac{\partial \tilde{x}}{\partial z} = \left(U - \frac{1}{z}\widetilde{V}^\lambda\right)^T \tilde{x}, \tag{3.3}$$

where

$$(E_i)_{jk} = \delta_{ij}\delta_{ik}, \quad U := \mathrm{diag}(u^1, \dots, u^n), \quad \widetilde{V}_i := \partial_i \widetilde{\Psi} \cdot \widetilde{\Psi}^{-1}, \quad \widetilde{V}^\lambda := \widetilde{\Psi} \cdot \mu^\lambda \cdot \widetilde{\Psi}^{-1}.$$

Proposition 3.5. *The following facts hold true for both formal and analytic semisimple homogeneous flat F-manifolds.*

(1) *There exist an off-diagonal matrix $\widetilde{\Gamma}$ such that*

$$\widetilde{V}_i = \widetilde{V}_i' + [\widetilde{\Gamma}, E_i], \qquad i = 1, \dots, n, \tag{3.4}$$

$$\widetilde{V}^\lambda = (\widetilde{V}^\lambda)' + [\widetilde{\Gamma}, U]. \tag{3.5}$$

In particular, $\widetilde{\Gamma}^i_j = -(\widetilde{V}_i)^i_j$ for $i \neq j$.

(2) *We have*

$$[E_i, \widetilde{V}^\lambda] = [U, \widetilde{V}_i], \quad \partial_i \widetilde{V}^\lambda = [\widetilde{V}_i, \widetilde{V}^\lambda]. \tag{3.6}$$

(3) *The diagonal entries of the matrix $\widetilde{V}^\lambda$ are constant w.r.t. $\boldsymbol{u}$.*

(4) *We have $\partial_i \widetilde{V}_j' = \partial_j \widetilde{V}_i'$.*

Proof. The compatibility condition $\partial_i \partial_j = \partial_j \partial_i$ implies the constraints

$$\partial_i \widetilde{V}_j - \partial_j \widetilde{V}_i = [\widetilde{V}_i, \widetilde{V}_j], \quad [E_i, E_j] = 0, \tag{3.7}$$

$$[E_i, \widetilde{V}_j] = [E_j, \widetilde{V}_i], \tag{3.8}$$

Identities (3.7) are trivially satisfied, by definition of the matrices E_i and $\widetilde{V}_i$. From (3.8), we deduce

$$(\delta_{jh} - \delta_{jk})(\widetilde{V}_i)^h_k = (\delta_{ih} - \delta_{ik})(\widetilde{V}_j)^h_k \quad \underset{h=j,\, j\neq k}{\Longrightarrow} \quad (\widetilde{V}_i)^j_k = (\delta_{ij} - \delta_{ik})(\widetilde{V}_j)^j_k = [\widetilde{\Gamma}, E_i]^j_k,$$

where $\widetilde{\Gamma} = (\widetilde{\Gamma}^j_k)_{j,k}$ is defined by $\widetilde{\Gamma}^j_k := -(\widetilde{V}_j)^j_k$. This proves (3.4). The compatibility condition $\partial_i \partial_z = \partial_z \partial_i$ implies the constraints

$$[E_i, U] = 0, \tag{3.9}$$

$$[E_i, \widetilde{V}^\lambda] = [U, \widetilde{V}_i], \quad \partial_i \widetilde{V}^\lambda = [\widetilde{V}_i, \widetilde{V}^\lambda]. \tag{3.10}$$

Identity (3.9) is trivially satisfied. Identity (3.5) follows from (3.4) and the first of (3.10). The constancy of $(\widetilde{V}^\lambda)'$ follows from the second identity of (3.10), and equations (3.4), (3.5). Finally, from the first identity of (3.7) we have $\partial_i \widetilde{V}_j' - \partial_j \widetilde{V}_i' = [\widetilde{V}_i, \widetilde{V}_j]' = 0$, by (3.4), (3.5). $\square$

3.5. **Darboux-Tsarev equations, and conformal dimensions.** Let us introduce the Christoffel symbols K^h_{ij} by $\nabla_{\partial_i} \partial_j = \sum_h K^h_{ij} \partial_h$.

Lemma 3.6. *We have $K^h_{ij} = -(\widetilde{V}_i)^h_j$.*

Proof. The claim follows from the following computation:

$$\nabla_{\partial_i} \partial_j = \nabla_{\partial_i} [(\widetilde{\Psi}^{-1})^\alpha_j \partial_\alpha] = [\partial_i (\widetilde{\Psi}^{-1})^\alpha_j] \partial_\alpha = -\sum_\ell (\widetilde{\Psi}^{-1})^\alpha_\ell \; \partial_i \widetilde{\Psi}^\ell_\beta \; (\Psi^{-1})^\beta_j \partial_\alpha = -\sum_\ell (\widetilde{V}_i)^\ell_j \partial_\ell. \quad \square$$

Proposition 3.7. *The following identities hold true:*

$$K^h_{ij} = 0, \qquad i, j, h \text{ distinct}, \tag{3.11}$$

$$K^i_{ij} = K^i_{ji} = -K^i_{jj} = \widetilde{\Gamma}^i_j, \qquad i \neq j, \tag{3.12}$$

$$K^i_{ii} = -\textstyle\sum_{h \neq i} \widetilde{\Gamma}^i_h. \tag{3.13}$$

Moreover, the functions $\widetilde{\Gamma}^i_j$ *satisfy the Darboux-Tsarev equations*

$$\partial_k\widetilde{\Gamma}^i_j = -\widetilde{\Gamma}^i_j\widetilde{\Gamma}^i_k + \widetilde{\Gamma}^i_j\widetilde{\Gamma}^j_k + \widetilde{\Gamma}^i_k\widetilde{\Gamma}^k_j \qquad i,j,k \text{ distinct}, \tag{3.14}$$

$$\textstyle\sum_k \partial_k\widetilde{\Gamma}^i_j = 0, \qquad i \neq j. \tag{3.15}$$

Proof. In canonical coordinates we have $c^i_{jk} = \delta^i_j\delta^i_k$. Consequently, $(\nabla_\ell c)^i_{jk} = \sum_p \delta^p_j\delta^p_k K^i_{p\ell} - \delta^i_k K^i_{\ell j} - \delta^i_j K^i_{\ell k}$. We have $K^h_{ij} = K^h_{ji}$ because ∇ is torsion free. Hence, from the symmetry $(\nabla_\ell c)^i_{jk} = (\nabla_j c)^i_{\ell k}$, one obtains (3.11), and the first two equalities of (3.12). The equality $K^i_{ij} = \widetilde{\Gamma}^i_j$ follows from Lemma 3.6. Equation (3.13) follows from the condition $\nabla e = 0$. By flatness of ∇, the components $R^i_{jk\ell}$ of the Riemann tensor equal zero. By definition we have $R^i_{jk\ell} = \partial_k K^i_{j\ell} - \partial_\ell K^i_{jk} + \sum_p K^i_{kp}K^p_{j\ell} - \sum_p K^i_{\ell p}K^p_{kj}$. Darboux-Tsarev equation (3.14) is equivalent to $R^i_{ji\ell} = 0$. From the identity $\nabla_{\partial_i}\partial_\alpha = \sum_j(\partial_i\widetilde{\Psi}^j_\alpha)\partial_j + \sum_j \widetilde{\Psi}^j_\alpha\nabla_{\partial_i}\partial_j$, summing over i, we obtain

$$0 = \nabla_e\partial_\alpha = \textstyle\sum_i \nabla_{\partial_i}\partial_\alpha = \sum_j \left(\sum_i \partial_i\widetilde{\Psi}^j_\alpha\right)\partial_j + \sum_j \widetilde{\Psi}^j_\alpha \underbrace{\nabla_e\partial_j}_{\nabla_{\partial_j}e+[e,\partial_j]=0}, \tag{3.16}$$

$$\Longrightarrow \textstyle\sum_i \partial_i\widetilde{\Psi} = 0. \tag{3.17}$$

We have

$$\partial_k\widetilde{\Gamma}^i_j = -\partial^2_{ki}\widetilde{\Psi}^i_\alpha(\widetilde{\Psi}^{-1})^\alpha_j + \partial_i\widetilde{\Psi}^i_\alpha \sum_h (\widetilde{\Psi}^{-1})^\alpha_h\ \partial_k\widetilde{\Psi}^h_\gamma\ (\widetilde{\Psi}^{-1})^\gamma_j.$$

Summing over k, and using (3.17), we obtain (3.15). □

Corollary 3.8. *For $i = 1,\dots,n$, the matrix $\widetilde{V}_i$ has the following structure*

$$\widetilde{V}_i = \begin{pmatrix} -\widetilde{\Gamma}^1_i & & & & \widetilde{\Gamma}^1_i & & & \\ & -\widetilde{\Gamma}^2_i & & & \widetilde{\Gamma}^2_i & & & \\ & & \ddots & & \vdots & & & \\ & & & -\widetilde{\Gamma}^{i-1}_i & \widetilde{\Gamma}^{i-1}_i & & & \\ -\widetilde{\Gamma}^i_1 & -\widetilde{\Gamma}^i_2 & \dots & -\widetilde{\Gamma}^i_{i-1} & \sum_{h\neq i}\widetilde{\Gamma}^i_h & -\widetilde{\Gamma}^i_{i+1} & \dots & -\widetilde{\Gamma}^i_n \\ & & & & \widetilde{\Gamma}^{i+1}_i & -\widetilde{\Gamma}^{i+1}_i & & \\ & & & & \vdots & & \ddots & \\ & & & & \widetilde{\Gamma}^n_i & & & -\widetilde{\Gamma}^n_i \end{pmatrix}.$$

□

Proposition 3.9. *We have $\widetilde{V}^\lambda = -\lambda\cdot\mathbf{1} + \sum_i u^i\widetilde{V}_i$.*

Proof. For any i, we have

$$\nabla_{\partial_i}E = \nabla_{\partial_i}\sum_j u^j\partial_j = \partial_i + \sum_j u^j\nabla_{\partial_i}\partial_j = \partial_i + \sum_{j,h} u^j K^h_{ij}\partial_h = \partial_i + \sum_{j,h} u^j K^h_{ji}\partial_h.$$

By Lemma 3.6 one concludes. □

Corollary 3.10. *The following identities hold true:*

$$\sum_j u^j \partial_j \widetilde{V}_i = -\widetilde{V}_i, \tag{3.18}$$

$$\sum_j u^j \partial_j \widetilde{\Gamma} = -\widetilde{\Gamma}, \tag{3.19}$$

$$(u^i - u^j)\partial_i \widetilde{\Gamma}^i_j = \sum_{\ell \neq i,j} (u^j - u^\ell)\{-\widetilde{\Gamma}^i_j \widetilde{\Gamma}^i_\ell + \widetilde{\Gamma}^i_j \widetilde{\Gamma}^j_\ell + \widetilde{\Gamma}^i_\ell \widetilde{\Gamma}^\ell_j\} - \widetilde{\Gamma}^i_j, \tag{3.20}$$

$$(u^j - u^i)\partial_j \widetilde{\Gamma}^i_j = \sum_{\ell \neq i,j} (u^i - u^\ell)\{-\widetilde{\Gamma}^i_j \widetilde{\Gamma}^i_\ell + \widetilde{\Gamma}^i_j \widetilde{\Gamma}^j_\ell + \widetilde{\Gamma}^i_\ell \widetilde{\Gamma}^\ell_j\} - \widetilde{\Gamma}^i_j. \tag{3.21}$$

Proof. Equation (3.18) follows from the second equation of (3.6) and the first equation of (3.7). Equation (3.19) is easily deduced. Equations (3.20) and (3.21) follow from (3.14), (3.15), and (3.19). ☐

Remark 3.11. In this section we started from a given semisimple flat F-manifold and we obtained a solution $\widetilde{\Gamma}^i_j$ of the Darboux-Tsarev equations. The opposite construction works as well: in [AL15] it is shown that the datum of

- a solution $\widetilde{\Gamma}^i_j$ of (3.14),(3.15),
- the connection ∇ with Christoffel symbols K^i_{jk} given by (3.11),(3.12),(3.13),
- the structure constants $c^i_{jk} = \delta^i_j \delta^i_k$,
- the vector field $e := \sum_i \partial_i$,

locally defines a (tame) semisimple flat F-manifold structure on $\mathbb{C}^n \setminus \Delta$.

Conformal dimensions. By Propositions 3.5-(3), and 3.9, there exist complex numbers $\delta_1, \dots, \delta_n \in \mathbb{C}$ such that

$$(\widetilde{V}^\lambda)' = -\lambda \cdot \mathbf{1} + \mathrm{diag}(\delta_1, \dots, \delta_n).$$

Definition 3.12. The numbers $\delta_1, \dots, \delta_n$ are called *conformal dimensions* of the (formal or analytic) semisimple flat F-manifold. We will say that a (formal/analytic) semisimple flat F-manifold is *conformally resonant* if $\delta_i - \delta_j \in \mathbb{Z} \setminus 0$ for some i, j.

Remark 3.13. We have $\delta_i = \sum_k u^k (\widetilde{V}_k)^i_i = \sum_{k \neq i} (u^i - u^k)\widetilde{\Gamma}^i_k$.

Remark 3.14. In the case of Frobenius manifolds, all conformal dimensions equal $\frac{1}{2}d$, where d is the conformal dimension of equation (2.6). In particular, a Frobenius manifold is never conformally resonant.

3.6. **Lamé coefficients, matrices $\Psi, V_i, V^\lambda, \Gamma$, and Darboux-Egoroff equations.** For any $j = 1, \dots, n$ define the one form

$$\omega_j(\boldsymbol{u}) := -\sum_{i=1}^n \widetilde{V}_i(\boldsymbol{u})^j_j du^i.$$

Proposition 3.15. *The one-forms ω_j are closed. There locally exist functions $H_j(\boldsymbol{u})$ such that*

$$d \log H_j = \omega_j, \quad j = 1, \dots, n.$$

Proof. It follows from point (4) of Proposition 3.5. □

The functions H_j are called *Lamé coefficients*, and they are defined up to scalar rescaling $H_j \mapsto \lambda_j H_j$, $\lambda_j \in \mathbb{C}^*$.

Arrange the Lamé coefficients in the diagonal matrix $H := \mathrm{diag}(H_1, \dots, H_n)$, and define the matrices

$$\Psi := H\widetilde{\Psi}, \quad V_i := H\widetilde{V}_i H^{-1} + \partial_i H \cdot H^{-1}, \quad V^\lambda := H\widetilde{V}^\lambda H^{-1}, \quad \Gamma := H\widetilde{\Gamma}H^{-1}.$$

Proposition 3.16. *The functions $H_1, \dots, H_n$ satisfy the following system*

$$\partial_j H_i = \Gamma^i_j H_j, \quad i \neq j, \quad \partial_i H_i = -\sum_{k\neq i} \Gamma^i_k H_k, \quad \sum_j u^j \partial_j H_i = -\delta_i H_i.$$

Proof. It easily follows from the definitions and identities of the previous section. □

Under the gauge transformation $x = (H^{-1})^T \tilde{x}$, the system (3.3) becomes

$$\frac{\partial x}{\partial u^i} = (zE_i - V_i)^T x, \quad \frac{\partial x}{\partial z} = \left(U - \frac{1}{z}V^\lambda\right)^T x. \tag{3.22}$$

Proposition 3.17. *The following identities hold true:*

$$V_i = [\Gamma, E_i], \tag{3.23}$$

$$V^\lambda = (V^\lambda)' + [\Gamma, U], \qquad (V^\lambda)' = (\widetilde{V}^\lambda)' = \mathrm{diag}(\delta_1 - \lambda, \dots, \delta_n - \lambda), \tag{3.24}$$

$$\partial_i \Psi \cdot \Psi^{-1} = V_i, \qquad [E_i, V^\lambda] = [U, V_i], \quad \partial_i V^\lambda = [V_i, V^\lambda]. \tag{3.25}$$

Proof. It easily follows from the definitions and identities of the previous section. □

Proposition 3.18. *The matrix Γ satisfy the Darboux-Egoroff equations*

$$\partial_k \Gamma^i_j = \Gamma^i_k \Gamma^k_j, \qquad i, j, k \text{ distinct}, \tag{3.26}$$

$$\textstyle\sum_k \partial_k \Gamma^i_j = 0, \qquad i \neq j, \tag{3.27}$$

$$\textstyle\sum_k u^k \partial_k \Gamma^i_j = (\delta_j - \delta_i - 1)\Gamma^i_j, \qquad i \neq j, \tag{3.28}$$

$$(u^j - u^i)\partial_i \Gamma^i_j = \textstyle\sum_{k\neq i,j} (u^k - u^j)\Gamma^i_k \Gamma^k_j - (\delta_j - \delta_i - 1)\Gamma^i_j, \tag{3.29}$$

$$(u^i - u^j)\partial_j \Gamma^i_j = \textstyle\sum_{k\neq i,j} (u^k - u^i)\Gamma^i_k \Gamma^k_j - (\delta_j - \delta_i - 1)\Gamma^i_j. \tag{3.30}$$

Proof. It easily follows from the definitions, the Darboux-Tsarev system for $\widetilde{\Gamma}$, and the homogeneity identities (3.19) of the previous section. □

Remark 3.19. For $n = 3$, the Darboux-Egoroff joint system of equations (3.26), (3.27), (3.28) is equivalent to the full family of Painlevé equations PVI. See remarkable formulas of [Lor14, Theorem 4.1].

Remark 3.20. In the case of Frobenius manifolds, there is a canonical choice for the Lamé coefficients: $H_i = \eta(\partial_i, \partial_i)^{\frac{1}{2}}$ for $i = 1, \dots, n$. The resulting coefficients Γ^i_j are the rotation coefficients of the metric η. They satisfy the further symmetry condition $\Gamma^i_j = \Gamma^j_i$.

Remark 3.21. In the light of Remark 3.20, given a semisimple flat F-manifold with a fixed choice of the Lamé coefficients H_i, we can define a metric by $\eta := \sum_{i=1}^n H_i^2 du^i$. Such a metric clearly is compatible with the product, in the sense that the second of equations (2.5) is satisfied. The flatness of η is the obstruction for the flat F-manifold to be actually Frobenius. For a more invariant description of the metric η, see [ABLR20a, Prop. 1.8].

In [ABLR20a] a further notion of *semisimple Riemannian F-manifold* is introduced. In *loc. cit.* it is also proved the local equivalence of semisimple flat F-manifolds and semisimple Riemannian F-manifolds. Notice that the notion of semisimple Riemannian F-manifolds given in [ABLR20a] relaxes the axioms of analog structures introduced in [DS11, LPR11]. See also the recent preprint [ABLR21].

From a given homogeneous semisimple flat F-manifold, we obtained a joint system of equations (3.22). In the analytic case, such a joint system defines[3] an *isomonodromic system*, because of integrability equations (3.25).

Vice-versa, one can start from such an isomonodromic system to construct the homogeneous semisimple flat F-manifold structure. This is exactly the point of view of the definition of *Saito structures without metric* given in [Sab07, Ch. VII, §1.a].

Notice that one can actually work with a companion Fuchsian system, obtained via a Fourier-Laplace transform. This is the point of view of [KMS20], in which Saito structures without metric are constructed on the space of isomonodromic deformation parameters for extended Okubo systems.

In both [Sab07, KMS20], coalescences of the parameters of deformations (the entries of $U = \operatorname{diag}(u^1, \dots, u^n)$ of equation (3.22)) are not taken into account. In our equation (3.22), on the contrary, we allow coalescences, provided that the matrix Ψ is not singular, i.e. provided that the geometric point of the flat F-structure is semisimple. In the recent paper [Guz20], D. Guzzetti extended the results of [BJL81, Guz16] to the case of Fuchsian systems with *confluent singularities*. Furthermore, in [Guz20] a notion of *isomonodromic Laplace transform* is introduced: with such an analytic tool, the study of the correspondence

$$\text{Monodromy data of an irregular system} \longleftrightarrow \text{Monodromy data of a Fuchsian system},$$

originally developed in [BJL81], has been extended to the isomonodromic case (possibly with coalescences/confluences). This also gives a new proof of the results of [CG18, CDG19].

The point of view of the current paper differs from the perspective of [Sab07, KMS20], via a Riemann-Hilbert correspondence. In Section 6, we will show a one-to-one correspondence between (local isomorphism classes of) homogeneous semisimple flat F-structures and solvable Riemann-Hilbert-Birkhoff problems.

4. Monodromy moduli of admissible germs of semisimple flat F-manifolds

4.1. **μ-nilpotent operators and μ-parabolic group.** Let (V, μ) be the datum of a n-dimensional complex vector space, and a diagonalizable operator $\mu\colon V \to V$. Denote by $\operatorname{spec}(\mu) = (\mu_1, \dots, \mu_n)$ the spectrum of μ, and by V_{μ_α} the eigenspace corresponding to the eigenvalue μ_α.

[3] This will be explained in details the next section.

We say that $A \in \operatorname{End}(V)$ is *μ-nilpotent* if

$$AV_{\mu_\alpha} \subseteq \bigoplus_{m \geqslant 1} V_{\mu_\alpha + m} \quad \text{for all } \mu_\alpha \in \operatorname{spec}(\mu).$$

In particular such an operator is nilpotent in the usual sense. Denote by $\mathfrak{c}(\mu)$ the set of all μ-nilpotent operators. It is easy to see that the set $\mathfrak{c}(\mu)$ is a Lie algebra w.r.t. the commutator $[-,-]$ in $\operatorname{End}(V)$. We can decompose a μ-nilpotent operator A in components A_k, $k \geqslant 1$, such that

$$A_k V_{\mu_\alpha} \subseteq V_{\mu_\alpha + k} \quad \text{for any } \mu_\alpha \in \operatorname{spec}(\mu),$$

so that the following identities hold:

$$z^\mu A z^{-\mu} = A_1 z + A_2 z^2 + A_3 z^3 + \dots, \quad [\mu, A_k] = kA_k \quad \text{for } k = 1, 2, 3, \dots.$$

Lemma 4.1. *Let (V, μ) as above, and let us fix a basis $(v_i)_{i=1}^n$ of eigenvectors of μ.*

1. *The operator $A \in \operatorname{End}(V)$ is μ-nilpotent if and only if its associate matrix w.r.t. the basis $(v_i)_{i=1}^n$ satisfies the condition $(A)^\alpha_\beta = 0$ unless $\mu_\alpha - \mu_\beta \in \mathbb{N}^*$.*
2. *If $A \in \operatorname{End}(V)$ is a μ-nilpotent operator, then the matrices associated with its components $(A_k)_{k \geqslant 1}$ w.r.t. the basis $(v_i)_{i=1}^n$ satisfy the condition $(A_k)^\alpha_\beta = 0$ unless $\mu_\alpha - \mu_\beta = k$, with $k \in \mathbb{N}^*$.* □

Define the *μ-parabolic* group to be the Lie group $\mathcal{C}(\mu)$ of operator $G \in GL(V)$ such that $G = \mathbf{1} + A$, with $A \in \mathfrak{c}(\mu)$. We have the canonical identification of Lie algebras $T_{\mathbf{1}}\mathcal{C}(\mu) = \mathfrak{c}(\mu)$, and the canonical adjoint action $\operatorname{Ad}\colon \mathcal{C}(\mu) \to \operatorname{Aut} \mathfrak{c}(\mu)$ defined by

$$\operatorname{Ad}_G(A) := GAG^{-1}, \quad G \in \mathcal{C}(\mu), \quad A \in \mathfrak{c}(\mu).$$

Remark 4.2. Consider the space $V^* := \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Each $f \in \operatorname{End}_{\mathbb{C}}(V)$ induces a dual map $f^* \in \operatorname{End}_{\mathbb{C}}(V^*)$, defined by $f^*(w) := w \circ f$, where $w \in V^*$. This defines an *anti-isomorphism* of Lie algebras

$$(-)^*\colon \operatorname{End}_{\mathbb{C}}(V) \to \operatorname{End}_{\mathbb{C}}(V^*), \quad [f_1, f_2]^* = -[f_1^*, f_2^*].$$

The image of $\mathfrak{c}(\mu)$ coincides with $\mathfrak{c}(-\mu^*)$.

4.2. **Spectrum of a flat F-manifold.** Consider an analytic pointed flat F-manifold (M, p). For any $\in \mathbb{C}$, we have a pair (T_pM, μ_p^λ) satisfying all the assumption of Section 4.1. We can consequently introduce the Lie algebra $\mathfrak{c}(\mu_p^\lambda)$, and the Lie group $\mathcal{C}(\mu_p^\lambda)$.

If (V_1, μ_1), (V_2, μ_2) are two pairs as in Section 4.1, a *morphism of pairs* $f\colon (V_1, \mu_1) \to (V_2, \mu_2)$ is the datum of a linear morphism $f\colon V_1 \to V_2$, compatible with the operators μ_1, μ_2, i.e., $\mu_2 \circ f = f \circ \mu_1$.

Given $\lambda_1, \lambda_2 \in \mathbb{C}$, it is easy to see that the pairs $(T_pM, \mu_p^{\lambda_1})$ and $(T_pM, \mu_p^{\lambda_2})$ are isomorphic. Moreover, given $p_1, p_2 \in M$, the two pairs attached to the germs (M, p_1) and (M, p_2) are (non-canonically) isomorphic: using the connection ∇, for any path $\gamma : [0, 1] \to M$ with $\gamma(0) = p_1$ and $\gamma(1) = p_2$, the parallel transport along γ provides an isomorphism of the pairs at p_1 and p_2.

As a result, with any homogeneous flat F-manifold manifold (M, ∇, c, e, E) (not necessarily semisimple), we can canonically associate an isomorphism class $[(V, \mu)]$ of pairs as above, which will be called the *spectrum* of M.

Fix a system of flat coordinates $\boldsymbol{t}=(t^1,\dots,t^n)$ diagonalizing $\mu^\lambda=\operatorname{diag}(q_1-\lambda,\dots,q_n-\lambda)$. We can thus introduce the λ-independent matrix Lie algebras

$$\mathfrak{c}(\mu):=\left\{R\in\mathfrak{gl}(n,\mathbb{C})\colon R^\alpha_\beta=0\text{ unless }q_\alpha-q_\beta\in\mathbb{Z}_{>0}\right\},$$
$$\mathfrak{c}(-\mu^*):=\left\{R\in\mathfrak{gl}(n,\mathbb{C})\colon R^\alpha_\beta=0\text{ unless }q_\alpha-q_\beta\in\mathbb{Z}_{<0}\right\},$$

which are canonically anti-isomorphic via transposition, see Remark 4.2. We also denote by $\mathcal{C}(\mu)$ and $\mathcal{C}(-\mu^*)$ the corresponding parabolic Lie groups.

4.3. **Solutions in Levelt normal forms and monodromy data at $z=0$.** We now introduce some formal invariant of the given analytic flat F-manifold, by studying *Levelt normal forms* of solutions of the joint system of differential equations (3.2).

Theorem 4.3.
(1) *There exist $n\times n$-matrix valued functions $(G_p(\boldsymbol{t}))_{p\geqslant1}$, analytic in $\boldsymbol{t}$, and a $\boldsymbol{t}$-independent matrix $R\in\mathfrak{c}(-\mu^*)$, such that the matrix*

$$\Xi(\boldsymbol{t},z)=G(\boldsymbol{t},z)z^{-\mu^\lambda}z^R,\quad G(\boldsymbol{t},z)=\mathbf{1}+\sum_{p=1}^\infty G_p(\boldsymbol{t})z^p,$$

is a (formal) solution of the joint system (3.2).
(2) *The series $G(\boldsymbol{t},z)$ converges to an analytic function in $(\boldsymbol{t},z)$. The matrix $\Xi(\boldsymbol{t}_o,z)$ is a fundamental system of solutions of the ∂_z-equation of (3.2) for any fixed $\boldsymbol{t}_o$.*

Proof. Consider n functions $\tilde t^\alpha(\boldsymbol{t},z)=\sum_{p=0}^\infty h^\alpha_p(\boldsymbol{t})z^p$ such that $\nabla^z d\tilde t^\alpha=0$, and $\tilde t^\alpha(\boldsymbol{t},0)=t^\alpha$. This translates in the following recursive equations for the coefficients h^α_p:

$$h^\alpha_0(\boldsymbol{t})=t^\alpha,\quad \partial_\gamma\partial_\beta h^\alpha_{p+1}=c^\varepsilon_{\gamma\beta}\partial_\varepsilon h^\alpha_p,\quad p\geqslant0.$$

Introduce the Jacobian matrix $J(\boldsymbol{t},z):=(J^\alpha_\beta)_{\alpha,\beta}$, with $J^\alpha_\beta(\boldsymbol{t},z):=\frac{\partial\tilde t^\alpha}{\partial t^\beta}$. Under the gauge transformation $\xi=J^T\tilde\xi$, the joint system (3.2) becomes

$$\begin{aligned}\frac{\partial\tilde\xi}{\partial t^\alpha}&=\left(zJ\mathcal{C}_\alpha J^{-1}-\frac{\partial J}{\partial t^\alpha}J^{-1}\right)^T\tilde\xi=0,\\ \frac{\partial\tilde\xi}{\partial z}&=\left[J\left(\mathcal{U}-\frac{1}{z}\mu^\lambda\right)J^{-1}-\frac{\partial J}{\partial z}J^{-1}\right]^T\tilde\xi\\ &=\left(-\frac{1}{z}(\mu^\lambda)^T+U_1^T+zU_2^T+z^2U_3^T+\dots\right)\tilde\xi,\end{aligned}\tag{4.1}$$

for suitable matrices U_k. From the compatibility $\partial_\alpha\partial_z=\partial_z\partial_\alpha$, it follows that the matrices U_k are $\boldsymbol{t}$-independent. Up to a further gauge transformation $\tilde\xi\mapsto G(z)\tilde\xi$, of the form $G(z)=1+\sum_{k=1}^\infty G_kz^k$, the differential equation (4.1) can be put in a *normal form*

$$\begin{gathered}\frac{\partial\tilde\xi}{\partial z}=\left(-\frac{1}{z}\mu^\lambda+R_1+zR_2+z^2R_3+\dots\right),\\ (R_k)^\alpha_\beta\neq0\text{ only if }\mu^\lambda_\alpha-\mu^\lambda_\beta=-k,\quad k\geqslant1.\end{gathered}\tag{4.2}$$

Indeed, from the recursion relations

$$R_n = U_n^T + nG_n - [G_n, \mu^\lambda] + \sum_{k=1}^{n-1}(G_{n-k}U_k^T - R_k G_{n-k})$$

we determine the entries $(R_n)^\alpha_\beta$ for $(\mu^\lambda)_\alpha - (\mu^\lambda)_\beta = -n$, and $(G_n)^\alpha_\beta$ for $(\mu^\lambda)_\alpha - (\mu^\lambda)_\beta \neq -n$. We set $(G_n)^\alpha_\beta = 0$ for $(\mu^\lambda)_\alpha - (\mu^\lambda)_\beta = -n$. See also [Gan59]. A fundamental system of solutions of (4.2) is given by $\tilde{\xi}(z) = z^{-\mu^\lambda} z^R$, where $R := \sum_k R_k$. The proof of the convergence of the series $G(\boldsymbol{t}, z)$ is standard, the reader can consult e.g. [Was95, Sib90]. $\square$

Corollary 4.4. *The monodromy matrix M_0, defined by $\Xi(\boldsymbol{t}, e^{2\pi\sqrt{-1}}z) = \Xi(\boldsymbol{t}, z)M_0$, is independent of $\boldsymbol{t}$. We have $M_0 := \exp(-2\pi\sqrt{-1}\mu^\lambda)\exp(2\pi\sqrt{-1}R)$.* $\square$

Solutions $\Xi(\boldsymbol{t}, z)$ of the form above will be said to be in *Levelt normal form.*

Theorem 4.5. *Assume that*

$$\Xi(\boldsymbol{t}, z) = G(\boldsymbol{t}, z)z^{-\mu^\lambda} z^R, \quad G(\boldsymbol{t}, z) = \mathbf{1} + \sum_{p=1}^\infty G_p(\boldsymbol{t})z^p, \quad R \in \mathfrak{c}(-\mu^*)$$

$$\widetilde{\Xi}(\boldsymbol{t}, z) = \widetilde{G}(\boldsymbol{t}, z)z^{-\mu^\lambda} z^{\widetilde{R}}, \quad \widetilde{G}(\boldsymbol{t}, z) = \mathbf{1} + \sum_{p=1}^\infty \widetilde{G}_p(\boldsymbol{t})z^p, \quad \widetilde{R} \in \mathfrak{c}(-\mu^*),$$

are two solutions of the joint system (3.2), in Levelt normal form. Then there exists a unique $C \in \mathcal{C}(-\mu^*)$ such that $\widetilde{R} = C^{-1}RC$, the function $p_C(z) := z^{-\mu^\lambda} z^R C z^{-\widetilde{R}} z^{\mu^\lambda}$ is polynomial in z, and $\widetilde{G}(\boldsymbol{t}, z) = p_C(z)G(\boldsymbol{t}, z)$.

Proof. By assumption there exist a unique invertible matrix $C \in M_n(\mathbb{C})$ such that $\widetilde{\Xi} = \Xi C$. This implies that

$$G^{-1}\widetilde{G} = z^{-\mu^\lambda} z^R C z^{-\widetilde{R}} z^{\mu^\lambda}.$$

We deduce that the r.h.s. is a series in z of the form $z^{-\mu^\lambda} z^R C z^{\widetilde{R}} z^{\mu^\lambda} = \mathbf{1} + H_1 z + H_2 z^2 + \dots$. Actually, this sum is finite (i.e. a polynomial in z) because both R and $\widetilde{R}$ are nilpotent. We can also re-write this identity as follows

$$z^R C z^{-\widetilde{R}} = z^{\mu^\lambda}(\mathbf{1} + H_1 z + H_2 z^2 + \dots)z^{-\mu^\lambda}. \tag{4.3}$$

The l.h.s. is a polynomial in $\log z$, the r.h.s. contains only powers of z. Hence, both sides should actually be independent of z. The (α, β)-entry of the r.h.s. equals

$$\delta^\alpha_\beta + (H_1)^\alpha_\beta\, z^{(\mu^\lambda)_\alpha - (\mu^\lambda)_\beta + 1} + (H_2)^\alpha_\beta\, z^{(\mu^\lambda)_\alpha - (\mu^\lambda)_\beta + 2} + \dots,$$

which is z-independent iff $(H_k)^\alpha_\beta = 0$ for $(\mu^\lambda)_\alpha - (\mu^\lambda)_\beta \neq -k$. Set $z = 1$ in (4.3): we have $C = \mathbf{1} + \sum_k H_k$. This shows that $C \in \mathcal{C}(-\mu^*)$.

We have just shown that both sides of (4.3) are z-independent and they equal C. The l.h.s. of equation (4.3) can also be written as $Cz^{C^{-1}RC}z^{-\widetilde{R}}$. Thus, $Cz^{C^{-1}RC}z^{-\widetilde{R}} = C$. It follows that $\widetilde{R} = C^{-1}RC$. $\square$

Definition 4.6. We call *monodromy data at $z = 0$* of the flat F-manifold the datum $(\lambda, \mu^\lambda, [R])$, where $[R]$ is the adjoint orbit, in the Lie algebra $\mathfrak{c}(-\mu^*)$, of the exponents of solutions of (3.2) in Levelt normal form.

Remark 4.7. The notion of spectrum of a flat F-manifold generalizes the corresponding notion for Frobenius manifolds given in [Dub99, CDG20]. In the Frobenius manifolds case, the group $\mathcal{C}(\mu)$ is replaced by its subgroup $\mathcal{G}(\eta,\mu)$ called *(η,μ)-parabolic orthogonal group*: this is due to the fact that solutions in Levelt normal form satisfy a further η-orthogonality requirement described in Remark 3.4. Notice that in the Frobenius case we have $-\mu^*=\eta\mu\eta^{-1}$. See [CDG20, Section 2.1].

4.4. **Admissible germs and monodromy data at $z=\infty$.** Let (M,∇,c,e,E) be an analytic *semisimple* homogenous flat F-manifold. Under semisimplicity assumption, the joint system of differential equations (3.2) is gauge equivalent to the joint system (3.22). By studying this system, we are going to introduce another set of invariants of pointed germs of the flat F-manifold.

Semisimple, doubly resonant, and admissible germs. An analytic pointed germ (M,p) will be called

- *(tame/coalescing) semisimple* if the base point p is (tame/coalescing) semisimple,
- *doubly resonant* if p is coalescing and M is conformally resonant,
- *admissible* if it is semisimple but not doubly resonant.

In this section, we will consider admissible pointed germs (M,p).

Remark 4.8. According to Definition 2.14, the specification "tame/coalescing" depends on the choice of the Euler vector field. In case M is irreducible, it does not depend on such a choice. This follows from Theorem 2.17.

Formal solutions. Let (M,p) be an admissible germ. Fix an ordering $\boldsymbol{u}_o=(u_o^1,\dots,u_o^n)\in\mathbb{C}^n$ of the operator $\mathcal{U}(p)\colon T_pM\to T_pM$. Consider the ∂_z-equation of the joint system (3.22) specialized at $\boldsymbol{u}=\boldsymbol{u}_o$.

Theorem 4.9. *There exist unique $n\times n$-matrices $(\mathring{A}_k)_{k\geqslant1}$ such that the matrix*

$$\mathring{X}_{\rm for}(z)=\left(\mathbf{1}+\sum_{k=1}^{\infty}\frac{\mathring{A}_k}{z^k}\right)z^{\Lambda}e^{zU_o},\quad \Lambda:=\lambda\cdot\mathbf{1}-\operatorname{diag}(\delta_1,\dots,\delta_n),\quad U_o=\operatorname{diag}(u_o^1,\dots,u_o^n),$$

is a formal solution of the ∂_z-equation of (3.22), specialized at $\boldsymbol{u}=\boldsymbol{u}_o$. Moreover, we have $\mathring{A}_1''=\Gamma(\boldsymbol{u}_o)^T$.

Proof. The matrix $\mathring{X}_{\rm for}(z)$ is a solution of the ∂_z-equation of (3.22) if and only if we have

$$(1-k)\mathring{A}_{k-1}+A_{k-1}\Lambda=[U_o,\mathring{A}_k]-V(\boldsymbol{u}_o)^T\mathring{A}_{k-1},\quad k\geqslant1,\quad \mathring{A}_0:=\mathbf{1}. \tag{4.4}$$

We can compute recursively the matrices $\mathring{A}_k$. Let us start with $\mathring{A}_1$.

- For (i,j), with $i\neq j$, and so that $u_o^i\neq u_o^j$: from (4.4) specialized at $k=1$, we deduce

$$(u_o^i-u_o^j)(\mathring{A}_1)^i_j=V^j_i=(u_o^i-u_o^j)\Gamma(\boldsymbol{u}_o)^j_i\quad\Longrightarrow\quad(\mathring{A}_1)^i_j=\Gamma(\boldsymbol{u}_o)^j_i.$$

- For (i,j), with $i\neq j$, and so that $u_o^i=u_o^j$: from (4.4) specialized at $k=2$, we deduce

$$(\mathring{A}_1)^i_j=\frac{1}{1-\delta_i+\delta_j}\sum_{\ell\neq i}V(\boldsymbol{u}_o)^\ell_i(\mathring{A}_1)^\ell_j=\frac{1}{1-\delta_i+\delta_j}\sum_{\ell\neq i,j}(u_o^i-u_o^\ell)\Gamma(\boldsymbol{u}_o)^j_\ell\Gamma(\boldsymbol{u}_o)^\ell_i=\Gamma(\boldsymbol{u}_o)^j_i.$$

In the last equality, we used identity (3.29) specialized at $\boldsymbol{u} = \boldsymbol{u}_o$.
• For the diagonal entries: from (4.4) specialized at $k = 2$, we deduce

$$(\mathring{A}_1)^i_i = \sum_{\ell\neq i} V(\boldsymbol{u}_o)^\ell_i(\mathring{A}_1)^\ell_i = \sum_{\ell\neq i}(u^i_o - u^\ell_o)\Gamma(\boldsymbol{u}_o)^i_\ell\Gamma(\boldsymbol{u}_o)^\ell_i.$$

This completes the computation of $\mathring{A}_1$, and also proves that $\mathring{A}_1'' = \Gamma(\boldsymbol{u}_o)^T$.

Assume now to have computed $\mathring{A}_1, \mathring{A}_2, \dots, \mathring{A}_{h-1}$. The matrix $\mathring{A}_h$ can be computed by repeating the same procedure. Namely, from (4.4), with $k = h$, one can compute the entries $(\mathring{A}_h)^i_j$ for $i \neq j$ such that $u^i_o \neq u^j_o$. From (4.4), with $k = h+1$, one can compute the remaining entries of $\mathring{A}_h$. ☐

Theorem 4.10. *Let $\Omega \subseteq \mathbb{C}^n$ be a simply connected open neighborhood of $\boldsymbol{u}_o$. If Ω is sufficiently small, then:*
(1) *For any $\boldsymbol{u} \in \Omega$ there exist unique $n \times n$-matrices $(A_k(\boldsymbol{u}))_{k\geqslant 1}$ such that the matrix*

$$X_{\text{for}}(\boldsymbol{u}, z) = \left(\mathbf{1} + \sum_{k=1}^\infty \frac{A_k(\boldsymbol{u})}{z^k}\right) z^\Lambda e^{zU} \tag{4.5}$$

is a formal solution of the ∂_z-equation of (3.22). Moreover, we have $A_1(\boldsymbol{u})'' = \widetilde{\Gamma}(\boldsymbol{u})^T$.
(2) *We have $A_k(\boldsymbol{u}_o) = \mathring{A}_k$ for $k \geqslant 1$, and $X_{\text{for}}(\boldsymbol{u}_o, z) = \mathring{X}(z)$.*

Proof. Point (1) can be proved following the same computations as for Theorem 4.9. Point (2) follows by uniqueness. ☐

Remark 4.11. From the computations above, it is clear that the coefficients A_k are holomorphic at point $\boldsymbol{u} \in \Omega$ such that $u^i \neq u^j$ for $i \neq j$. Below we will prove that the coefficients A_k are actually holomorphic on the whole Ω.

The identity $X_{\text{for}}(\boldsymbol{u}, ze^{2\pi\sqrt{-1}}) = X_{\text{for}}(\boldsymbol{u}, z)e^{2\pi\sqrt{-1}\Lambda}$ justifies the following terminology.

Definition 4.12. The matrix $\Lambda = \operatorname{diag}(\lambda - \delta_1, \dots, \lambda - \delta_n)$ is called *formal monodromy* matrix.

Admissible directions τ. Let $q \in M$ be an arbitrary point, and fix an ordering $\boldsymbol{u}(q) := (u^1(q), \dots, u^n(q)) \in \mathbb{C}^n$ of the eigenvalues of the operator $\mathcal{U}(q)\colon T_qM \to T_qM$. Denote by $\operatorname{Arg}(z) \in]-\pi, \pi]$ the principal branch of the argument of the complex number z. Set

$$\mathscr{S}(q) := \left\{\operatorname{Arg}\left(-\sqrt{-1}(\overline{u^i(q)} - \overline{u^j(q)}\right) + 2\pi k\colon k \in \mathbb{Z},\ i, j \text{ are s.t. } u^i(q) \neq u^j(q)\right\}.$$

Any element $\tau \in \mathbb{R} \setminus \mathscr{S}(q)$ will be called an *admissible direction at q*.

Remark 4.13. The notion of admissibility only depends on the *set* $\{u^1(q), \dots, u^n(q)\}$.

Asymptotic solutions. Though the formal series defining X_{for} are typically divergent, X_{for} contains asymptotical information about genuine analytic solutions of the ∂_z-equation of (3.22).

Let (M, p) be an admissible germ, Ω as in Theorem 4.10, and τ an admissible direction at p. Consider a sufficiently small simply connected open neighborhood $\widetilde{\Omega} \subseteq M$ of p such that:

(1) $\widetilde{\Omega} \subseteq M_{ss}$, i.e. any point $q \in \widetilde{\Omega}$ is semisimple,

(2) a coherent choice of ordering $\boldsymbol{u}$ of eigenvalues of $\mathcal{U}$ is fixed on $\widetilde{\Omega}$, so that $\boldsymbol{u}\colon \widetilde{\Omega} \to \mathbb{C}^n$ defines a local system of canonical coordinates, with $\boldsymbol{u}(p) = \boldsymbol{u}_o$,
(3) $\boldsymbol{u}(\widetilde{\Omega}) \subseteq \Omega$,
(4) τ is admissible at any $q \in \widetilde{\Omega}$.

Theorem 4.14. *If $\widetilde{\Omega}$ is as above, then the following facts hold.*
(1) *For any $q \in \widetilde{\Omega}$ there exist three fundamental systems of solutions X_1, X_2, X_3, of the ∂_z-equation of (3.22) specialized at $\boldsymbol{u} = \boldsymbol{u}(q)$, uniquely determined by the asymptotics*

$$X_i(\boldsymbol{u}, z) \sim X_{\rm for}(\boldsymbol{u}, z), \quad |z| \to +\infty, \quad \tau-(3-h)\pi < \arg z < \tau+(h-2)\pi, \quad h = 1, 2, 3. \tag{4.6}$$

(2) *The functions X_i are holomorphic w.r.t. $\boldsymbol{u} \in \boldsymbol{u}(\widetilde{\Omega})$, and the asymptotics (4.6) holds true uniformly in $\boldsymbol{u}$.*
(3) *The functions $X_i(\boldsymbol{u}, z)$ are solutions of the joint system of differential equations (3.22).*
(4) *The solutions X_1 and X_3 satisfy the identity $X_3(\boldsymbol{u}, ze^{2\pi\sqrt{-1}}) = X_1(\boldsymbol{u}, z)e^{2\pi\sqrt{-1}\Lambda}$, for $z \in \widehat{\mathbb{C}^*}$.*

Proof. Let us temporarily assume that q is tame, i.e. $u^i(q) \neq u^j(q)$ for $i \neq j$. For the proof of points (1) and (2), see e.g. [BJL79, Was95]. Fix $h \in \{1, 2, 3\}$, and set $W_i(\boldsymbol{u}, z) := \partial_i X_h(\boldsymbol{u}, z) - (zE_i - V_i)^T X_h(\boldsymbol{u}, z)$ for $i = 1, \dots, n$ and $h = 1, 2, 3$. A simple computation, invoking the identities (3.25) shows that $W_i(\boldsymbol{u}, z)$ is a solution of the ∂_z-equation of (3.22). Hence, there exist a matrix $C(\boldsymbol{u})$ such that $W_i(\boldsymbol{u}, z) = X_h(\boldsymbol{u}, z)C(\boldsymbol{u})$. Denote by $F(\boldsymbol{u}, z) = \mathbf{1} + z^{-1}A_1(\boldsymbol{u}) + O(z^{-2})$ the formal power series in (4.5). For $|z| \to +\infty$ in the sector

$$\mathcal{V}_{\tau,h} := \left\{ z \in \widehat{\mathbb{C}^*} \colon \tau - (3-h)\pi < \arg z < \tau + (h-2)\pi \right\},$$

the function $W_i(\boldsymbol{u}, z)$ has asymptotics

$$\begin{aligned} W_i(\boldsymbol{u}, z) &\sim \partial_i F(\boldsymbol{u}, z) z^\Lambda e^{zU} + F(\boldsymbol{u}, z) z z^\Lambda \overbrace{e^{zE_i u_i}}^{=E_i e^{zU}} - zE_i F(\boldsymbol{u}, z) z^\Lambda e^{zU} + V_i^T F(\boldsymbol{u}, z) z^\Lambda e^{zU} \\ &= \left(\partial_i F(\boldsymbol{u}, z) + zF(\boldsymbol{u}, z)E_i - zE_i F(\boldsymbol{u}, z) + V_i^T F(\boldsymbol{u}, z)\right) z^\Lambda e^{zU}. \end{aligned}$$

But we also have $W_i(\boldsymbol{u}, z) \sim F(\boldsymbol{u}, z) z^\Lambda e^{zU} C(\boldsymbol{u})$. As a consequence, we deduce

$$z^\Lambda e^{zU} C(\boldsymbol{u}) e^{-zU} z^{-\Lambda} = \text{formal power series in } \frac{1}{z}. \tag{4.7}$$

For $j \neq k$, the sector $\mathcal{V}_{\tau,h}$ contains rays of points z along which $\mathrm{Re}(z(u^j - u^k)) > 0$. Hence, necessarily, we deduce that the (j,k)-entry of $C(\boldsymbol{u})$ vanishes, otherwise we would have a divergence on the l.h.s. of (4.7). So the matrix $C(\boldsymbol{u})$ is diagonal, and

$$\begin{aligned} C(\boldsymbol{u}) &= z^\Lambda e^{zU} C(\boldsymbol{u}) e^{-zU} z^{-\Lambda} \\ &= F(\boldsymbol{u}, z)^{-1} \left(\partial_i F(\boldsymbol{u}, z) + zF(\boldsymbol{u}, z)E_i - zE_i F(\boldsymbol{u}, z) + V_i^T F(\boldsymbol{u}, z)\right) \\ &= z(E_i - E_i) + (A_1 E_i - E_i A_1 + V_i^T) + O\left(\frac{1}{z}\right) = O\left(\frac{1}{z}\right), \end{aligned}$$

where we used the identity $V_i^T = [E_i, \Gamma^T] = [E_i, A_1]$. Hence $C(\boldsymbol{u}) = 0$. This proves point (3) in the case q is tame.

The coefficients A_k of Theorem 4.10 are holomorphic at $\boldsymbol{u}$ such that $u^i \neq u^j$. Moreover, from the computations above, we deduce that

$$[A_{k+1}, E_i] = [A_1, E_i]A_k - \partial_i A_k, \quad k \geqslant 1. \tag{4.8}$$

This formula recursively determines the off-diagonal matrix A''_{k+1} in terms of $A_1, \dots, A_k$. On the other hand, the diagonal entries of A_{k+1} can be computed as in Theorem 4.10, so that

$$(k+1)(A_{k+1})^i_i = \sum_{\ell \neq i} V^\ell_i (A_{k+1})^\ell_i = \sum_{\ell \neq i} (u^i - u^\ell)\Gamma^i_\ell (A_{k+1})^\ell_i. \tag{4.9}$$

Since $A''_1 = \Gamma^T$ is holomorphic also at coalescing points $\boldsymbol{u}$, an inductive argument shows that all the matrices A_k are holomorphic at coalescing point, by using formulae (4.8) and (4.9). See also [CDG19, Prop. 19.3]. The system (3.22) is a completely integrable Pfaffian system with holomorphic coefficients on $\boldsymbol{u}(\widetilde{\Omega})$: the solutions $X_i(z, \boldsymbol{u})$ can be $\boldsymbol{u}$-analytically continued as single-valued holomorphic functions on $\boldsymbol{u}(\widetilde{\Omega})$, see [CDG19, Cor. 19.1]. The assumptions of [CDG19, Th. 14.1] are thus satisfied, and (1),(2),(3) hold true also at coalescing points. Finally, notice that the two functions $X_1(\boldsymbol{u}, z)e^{2\pi\sqrt{-1}\Lambda}$ and $X_3(\boldsymbol{u}, ze^{2\pi\sqrt{-1}})$ have the same asymptotics on the sector $\tau - 2\pi < \arg z < \tau - \pi$. By uniqueness, it follows point (4). $\square$

Remark 4.15. For any $h = 1, 2, 3$, the precise meaning of the uniform asymptotic relation (4.6) is the following: for any compact $K \subseteq \boldsymbol{u}(\widetilde{\Omega})$, for any $\ell \in \mathbb{N}$, and for any unbounded closed subsector $\overline{\mathcal{V}}$ of $\mathcal{V}_{\tau,h} := \left\{ z \in \widehat{\mathbb{C}^*} : \tau - (3-h)\pi < \arg z < \tau + (h-2)\pi \right\}$, there exists a constant $C_{h,K,\ell,\overline{\mathcal{V}}} \in \mathbb{R}_{>0}$ such that

$$z \in \overline{\mathcal{V}} \setminus \{0\} \quad \Longrightarrow \quad \sup_{u \in K} \left\| X_h(\boldsymbol{u}, z)e^{-zU}z^{-\Lambda} - \left(\mathbf{1} + \sum_{m=1}^{\ell-1} \frac{A_m(\boldsymbol{u})}{z^m} \right) \right\| < \frac{C_{h,K,\ell,\overline{\mathcal{V}}}}{|z|^\ell}.$$

Stokes and central connection matrices. Let (M, p) be an admissible germ, and τ an admissible direction at p. Let $\Xi(\boldsymbol{t}, z)$ be a solution in Levelt form of the joint system (3.2), and $X_h(\boldsymbol{u}, z)$, with $h = 1, 2, 3$, be the solutions of the joint system (3.22) as in Theorem 4.14. Let $\boldsymbol{t}_o = \boldsymbol{t}(p)$ and $\boldsymbol{u}_o = \boldsymbol{u}(p)$ the values of the flat and canonical coordinates at p, respectively.

We define the *Stokes matrices* $\mathring{S}_1, \mathring{S}_2$ at p to be the matrices defined by

$$X_2(\boldsymbol{u}_o, z) = X_1(\boldsymbol{u}_o, z)\mathring{S}_1, \quad X_3(\boldsymbol{u}_o, z) = X_2(\boldsymbol{u}_o, z)\mathring{S}_2. \tag{4.10}$$

We define the *central connection matrix* $\mathring{C}$ at p to be the matrix defined by

$$X_2(\boldsymbol{u}_o, z) = (\Psi(\boldsymbol{u}_o)^{-1})^T \cdot \Xi(\boldsymbol{t}_o, z) \cdot \mathring{C}. \tag{4.11}$$

Proposition 4.16. *We have*

(1) *the matrices* $\mathring{S}_1, \mathring{S}_2, \mathring{C}$ *are invertible, with* $\det \mathring{S}_1 = \det \mathring{S}_2 = 1$,
(2) $(\mathring{S}_1)_{ii} = (\mathring{S}_2)_{ii} = 1$,
(3) *if* $i \neq j$, *then* $(\mathring{S}_1^{-1})_{ij} = 0$ *if* $\operatorname{Re}\left(e^{\sqrt{-1}(\tau-\pi)}(u^i_o - u^j_o)\right) > 0$,
(4) *if* $i \neq j$, *then* $(\mathring{S}_2)_{ij} = 0$ *if* $\operatorname{Re}\left(e^{\sqrt{-1}\tau}(u^i_o - u^j_o)\right) > 0$,

(5) *we have*

$$\mathring{S}_1^{-1}e^{2\pi\sqrt{-1}\Lambda}\mathring{S}_2^{-1}=\mathring{C}^{-1}e^{-2\pi\sqrt{-1}\mu^\lambda}e^{2\pi\sqrt{-1}R}\mathring{C}. \tag{4.12}$$

Proof. The proof of points (1)-(4) is standard, see [Was95]. For point (5) follows from point (4) of Theorem 4.14. ☐

Proposition 4.17 ([CDG19, CG18])**.** *If p is a coalescing point, define the partition $\{1,\dots,n\} = \coprod_{r\in J} I_r$ such that for any $r\in J$ we have $\{i,j\}\subseteq I_r$ if and only if $u_o^i=u_o^j$. We then have the further vanishing condition*

$$(S_1)_{ij}=(S_1)_{ji}=(S_2)_{ij}=(S_2)_{ji}=0 \quad \text{if } i,j\in I_r \text{ for some } r\in J. \qquad ☐$$

For a $\mathcal{D}$-modules theoretical proof of Proposition 4.17 see the recent preprint [Sab21].

Definition 4.18. We call *monodromy data at $z=\infty$* of the admissible germ (M,p) the 4-tuple of matrices $(\mathring{S}_1,\mathring{S}_2,\Lambda,\mathring{C})$.

Remark 4.19. In the case of Frobenius manifolds, with the standard choice $\lambda=\frac{d}{2}$, we have $\Lambda=0$ and $\mathring{S}_1^{-1}=\mathring{S}_2^T$. This follow from the (anti-)self-adjointness properties of $\mathcal{U}$ and μ. For detailed proofs see [CDG20, Th. 2.42]. In the notations of *loc. cit.* we have $\mathring{S}_1=S$ and $\mathring{S}_2=S_-^{-1}$. Moreover, the Stokes matrices are uniquely determined by the metric, the central connection matrix, and the monodromy data at $z=0$:

$$S=C^{-1}e^{-\pi\sqrt{-1}R}e^{-\pi\sqrt{-1}\mu}\eta^{-1}(C^{-1})^T,\qquad S_-=S^T=C^{-1}e^{\pi\sqrt{-1}R}e^{\pi\sqrt{-1}\mu}\eta^{-1}(C^{-1})^T. \tag{4.13}$$

This is a direct consequence of the symmetries of the joint system (3.2), see Remark 3.4.

In the next paragraphs we show that the monodromy data at $z=\infty$ define local invariants of the germ, i.e. that they are locally constant w.r.t. small perturbations of both the point p and the admissible direction τ.

Isomonodromicity Property. Let $\widetilde{\Omega}$ be an open neighborhood of p as above. By Theorems 4.3 and 4.14, if we let vary the point q in $\widetilde{\Omega}$, we have well defined solutions $\Xi(\boldsymbol{t}(q),z), X_i(\boldsymbol{u}(q),z)$, $i=1,2,3$, of the joint systems (3.2) and (3.22) respectively. We can thus introduce the Stokes and central connections matrices (S_1,S_2,C) as functions of $q\in\widetilde{\Omega}$ by the formulae

$$X_2(\boldsymbol{u}(q),z) = X_1(\boldsymbol{u}(q),z)\,S_1(\boldsymbol{u}(q)), \tag{4.14}$$
$$X_3(\boldsymbol{u}(q),z) = X_2(\boldsymbol{u}(q),z)\,S_2(\boldsymbol{u}(q)), \tag{4.15}$$
$$X_2(\boldsymbol{u}(q),z) = (\Psi(\boldsymbol{u}(q))^{-1})^T\cdot\Xi(\boldsymbol{t}(q),z)\cdot C(\boldsymbol{u}(q)). \tag{4.16}$$

Theorem 4.20. *The functions S_1,S_2,C are constant on $\widetilde{\Omega}$. In particular, we have $S_1(q)=\mathring{S}_1, S_2(q)=\mathring{S}_2, C(q)=\mathring{C}$, for all $q\in\widetilde{\Omega}$.*

Proof. Let us prove the statement for S_1. We have

$$\begin{aligned}\partial_i S_1(\boldsymbol{u}) &= \partial_i\left[X_1(\boldsymbol{u}(q),z)^{-1}X_2(\boldsymbol{u}(q),z)\right]\\ &= -X_1(\boldsymbol{u}(q),z)^{-1}\cdot\partial_i X_1(\boldsymbol{u}(q),z)\cdot X_1(\boldsymbol{u}(q),z)^{-1}\cdot X_2(\boldsymbol{u}(q),z)\\ &\quad +X_1(\boldsymbol{u}(q),z)^{-1}\partial_i X_2(\boldsymbol{u}(q),z)\\ &= -X_1(\boldsymbol{u}(q),z)^{-1}\cdot(zE_i-V_i)^T\cdot X_2(\boldsymbol{u}(q),z)\\ &\quad +X_1(\boldsymbol{u}(q),z)^{-1}\cdot(zE_i-V_i)^T\cdot X_2(\boldsymbol{u}(q),z)=0.\end{aligned}$$

The proof for S_2, C is similar. $\square$

Small perturbations of the admissible direction. Let (M,p) be an admissible germ, and τ be an admissible direction at p.

Theorem 4.21. *If $\tau'\in\mathbb{R}$ is such that $|\tau-\tau'|<\inf_{\varphi\in\mathscr{S}(p)}|\tau-\varphi|$, then τ' is admissible at p. Moreover, the monodromy data at p computed w.r.t. τ and τ' are equal.*

Proof. The first claim is straightforward. Let us prove the second claim. There exist fundamental systems of solutions $X_i(\boldsymbol{u}(p),z)$ and $X_i'(\boldsymbol{u}(p),z)$, for $i=1,2,3$, such that

$$
\begin{aligned}
X_h(\boldsymbol{u}(p),z)&\sim X_{\rm for}(\boldsymbol{u}(p),z),\quad |z|\to+\infty,\quad z\in\mathcal{V}_{\tau,h},\quad h=1,2,3,\\
X_h'(\boldsymbol{u}(p),z)&\sim X_{\rm for}(\boldsymbol{u}(p),z),\quad |z|\to+\infty,\quad z\in\mathcal{V}_{\tau',h},\quad h=1,2,3,
\end{aligned}
$$

by Theorem 4.14, see also Remark 4.15. We prove that $X_h=X_h'$ for all $h=1,2,3$.

Let K_h be the connection matrix s.t. $X_h'=X_h\,K_h$. We have

$$
z^{\Lambda}e^{zU}K_he^{-zU}z^{-\Lambda}\sim\mathbf{1},\quad |z|\to+\infty,\quad z\in\mathcal{V}_{\tau,h}\cap\mathcal{V}_{\tau',h}.
$$

By taking the (j,k)-entry, for any $\ell\in\mathbb{N}$ we have

$$
(K_h)_{jk}e^{z(u^j(p)-u^k(p))}z^{\delta_k-\delta_j}=\delta_{jk}+O\left(|z|^{-\ell}\right),\quad |z|\to+\infty,\quad z\in\mathcal{V}_{\tau,h}\cap\mathcal{V}_{\tau',h}.
$$

Assume $j\neq k$. If $u^j(p)=u^k(p)$, then necessarily $(K_h)_{jk}=0$. If $u^j(p)\neq u^k(p)$, notice that in $\mathcal{V}_{\tau,h}\cap\mathcal{V}_{\tau',h}$ there are rays along which $\mathrm{Re}(z(u^j(p)-u^k(p)))$ is negative, and also rays along which it is positive. So, we necessarily have $(K_h)_{jk}=0$. This proves that K_h is diagonal. It follows that $K_h=\mathbf{1}$ for $h=1,2,3$. $\square$

4.5. **Monodromy data for a formal admissible germ.** In Sections 4.2, 4.3, 4.4, the flat F-manifold structure on M is assumed to be analytic. The notion of admissible germs and of their monodromy data can however extended to the formal case.

Let $(H,\boldsymbol{\Phi})$ be a semisimple formal F-manifold over $\mathbb{C}$, with Euler field E. Associated with it we have two joint systems of differential equations (3.2) and (3.22) whose coefficients are matrix-valued formal power series in the coordinates $\boldsymbol{t}$ and $\boldsymbol{u}$, respectively.

We will say that $(H,\boldsymbol{\Phi})$ is

- *doubly resonant*, if the origin is coalescing and the formal flat F-manifold is conformally resonant,
- *admissible* if it is semisimple but not doubly resonant.

The ∂_z-equations of the joint systems (3.2) and (3.22) can be specialized at $\boldsymbol{t}=0$ and $\boldsymbol{u}=\boldsymbol{u}_o$, respectively. For these specialized systems of equations we can introduce a triple $(\lambda,\mu^\lambda,[R])$ of monodromy data at $z=0$, and a 4-tuple $(\mathring{S}_1,\mathring{S}_2,\Lambda,\mathring{C})$ of monodromy data at $z=\infty$, exactly as in the case of an analytic germ (M,p).

The system $(\lambda,\mu^\lambda,[R],\mathring{S}_1,\mathring{S}_2,\Lambda,\mathring{C})$ will be referred to as the *monodromy data* of the formal structure $(H,\boldsymbol{\Phi})$. *A priori*, Theorem 4.20 cannot be adapted to this formal picture, but Theorem 4.21 still holds true, and its proof works verbatim.

In Section 6.4, we will prove that an admissible formal germ is actually convergent: it defines an analytic flat F-manifold, so that all the results of Sections 4.2, 4.3, 4.4 apply.

5. Normalizations, and analytic continuation

5.1. **Choices of normalizations.** The monodromy data of an admissible germ (M,p) are defined up to several non-canonical choices:

1. the choice of $\lambda \in \mathbb{C}$,
2. the choice of a base point in the universal cover $\widehat{\mathbb{C}^*}$,
3. the choice of the solution Ξ in Levelt normal form,
4. the choice of Lamé coefficients $(H_1,\dots,H_n)$,
5. the choice of ordering of canonical coordinates $(u^1(p),\dots,u^n(p))$,
6. the choice of an admissible direction $\tau \in \mathbb{R}\setminus \mathscr{S}(p)$.

Different choices of normalizations affect the numerical values of the monodromy data. These transformations of the data can be described by actions of corresponding suitable groups:

1. the group $\mathbb{C}$,
2. the deck transformation group $\mathrm{Deck}(\widehat{\mathbb{C}^*}) \cong \mathbb{Z}$,
3. the group $\mathcal{C}(-\mu^*)$,
4. the torus $(\mathbb{C}^*)^n$,
5. the symmetric group $\mathfrak{S}_n$,
6. the braid group $\mathcal{B}_n$.

We first describe actions (1)-(5), and postpone the description of action (6) in the next sections.

• *Action of* $\mathbb{C}$: the transformation $\lambda \mapsto \lambda'$ implies the following transformations of the monodromy data by translations

$$\mu^\lambda \mapsto \mu^{\lambda'} = \mu^\lambda + (\lambda-\lambda')\mathbf{1}, \qquad \Lambda \mapsto \Lambda - (\lambda-\lambda')\mathbf{1}.$$

For irreducible flat F-manifolds, the choice of λ is equivalent to the choice of an Euler vector field, see Theorem 2.17.

• *Action of* $\mathrm{Deck}(\widehat{\mathbb{C}^*}) \cong \mathbb{Z}$: a different choice of the base point in $\widehat{\mathbb{C}^*}$ is equivalent to the choice of a different determination of the logarithm (i.e. of the argument $\arg z$). In particular, by changing $\log z \mapsto \log z + 2\pi k\sqrt{-1}$ with $k \in \mathbb{Z}$, we have the transformations

$$S_1 \mapsto e^{-2\pi k\sqrt{-1}\Lambda}\, S_1\, e^{2\pi k\sqrt{-1}\Lambda}, \quad S_2 \mapsto e^{-2\pi k\sqrt{-1}\Lambda}\, S_2\, e^{2\pi k\sqrt{-1}\Lambda},$$

$$C \mapsto M_0^{-k}\, C\, e^{2\pi k\sqrt{-1}\Lambda}, \quad M_0 := e^{-2\pi\sqrt{-1}\mu^\lambda} e^{2\pi\sqrt{-1}R}, \qquad k \in \mathbb{Z}.$$

• *Action of* $\mathcal{C}(-\mu^*)$: for $A \in \mathcal{C}(-\mu^*)$, the change of solutions $\Xi \mapsto \Xi A$ implies the transformation of the central connection matrix

$$C \mapsto A^{-1}C.$$

• *Action of* $(\mathbb{C}^*)^n$: for $(h_1,\dots,h_n) \in (\mathbb{C}^*)^n$, consider the transformation $(H_1,\dots,H_n) \mapsto (H_1h_1,\dots,H_nh_n)$. The monodromy data transform as follows

$$S_1 \mapsto hS_1h^{-1}, \quad S_2 \mapsto hS_2h^{-1}, \quad C \mapsto Ch^{-1},$$

$$\text{where } h := \mathrm{diag}(h_1,\dots,h_n).$$

• *Action of* $\mathfrak{S}_n$: for $\sigma \in \mathfrak{S}_n$, consider the permutation of canonical coordinates $(u^1, \dots, u^n) \mapsto (u^{\sigma(1)}, \dots, u^{\sigma(n)})$. The monodromy data transform as follows

$$S_1 \mapsto PS_1P^{-1}, \quad S_2 \mapsto PS_2P^{-1}, \quad C \mapsto CP^{-1}, \quad \Lambda \mapsto P\Lambda P^{-1},$$

$$\text{where} \quad P = (P_{ij})_{i,j}, \quad P_{ij} := \delta_{\sigma(i)j}.$$

5.2. **Triangular and lexicographical orders.** If an admissible direction τ at p is fixed, we will say that the canonical coordinates $(u^i(p))_{i=1}^n$ at p are in *triangular order w.r.t. the admissible direction* τ if the Stokes matrix S_1 is upper triangular, and S_2 is lower triangular.

On the one hand, in general, triangular orders at p are not unique. This happens for example if p is a semisimple coalescing point. In such a case, we have $(S_1)_{ij} = (S_1)_{ji} = 0$ if $u_i = u_j$ with $i \neq j$, by Proposition 4.17. If S_1 is upper triangular, then so is PS_1P^{-1} for P corresponding to the transposition $i \leftrightarrow j$. Similarly, the lower triangular structure of S_2 is preserved.

On the other hand, we always have a distinguished triangular order, called *lexicographical w.r.t.* τ. Introduce the following rays in the complex plane

$$L_j := \{u^j(p) + \rho e^{\sqrt{-1}(\frac{\pi}{2} - \tau)} \colon \rho \in \mathbb{R}_+\}, \quad j = 1, \dots, n.$$

The ray L_j originates from the point $u^j(p)$, and it is oriented from $u^j(p)$ to ∞.

The canonical coordinates $(u^1(p), \dots, u^n(p))$ are in lexicographical order if L_j is to the left of L_k (w.r.t. the orientation above), for any $1 \leqslant j < k \leqslant n$.

The lexicographical order is the unique triangular order at p if the number of nonzero entries of S_1 or S_2 is maximal, i.e. $\frac{n(n-1)}{2}$.

5.3. **Braid group action on matrices.** Denote by U_n and L_n the groups of unipotent upper and lower triangular $n \times n$-matrices, and by $\mathfrak{t}$ the Lie algebra of diagonal $n \times n$-matrices.

The (abstract) *Artin braid group* $\mathcal{B}_n$ *with* n*-strings* is the group with $n - 1$ generators $\beta_1, \dots, \beta_{n-1}$ satisfying the relations

$$\beta_i\beta_j = \beta_j\beta_i, \quad \text{if } |i-j| > 1, \quad \beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}. \tag{5.1}$$

Given $\boldsymbol{g} = (g_1, g_2, g_3) \in U_n \times L_n \times \mathfrak{t}$, define $3(n-1)$ block-diagonal matrices $B_1^{(i)}(\boldsymbol{g}), B_2^{(i)}(\boldsymbol{g})$, $B_3^{(i)}(\boldsymbol{g})$, with $i = 1, \dots, n-1$, as follows:

$$\begin{aligned} B_1^{(i)}(\boldsymbol{g}) &:= \mathbf{1}_{i-1} \oplus \begin{bmatrix} (g_1)_{i,i+1} & 1 \\ 1 & 0 \end{bmatrix} \oplus \mathbf{1}_{n-i-1}, \\ B_2^{(i)}(\boldsymbol{g}) &:= \mathbf{1}_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & (g_2)_{i+1,i} \end{bmatrix} \oplus \mathbf{1}_{n-i-1}, \\ B_3^{(i)}(\boldsymbol{g}) &:= \mathbf{1}_{i-1} \oplus \begin{bmatrix} \hbar \cdot (g_1)_{i,i+1} & 1 \\ 1 & 0 \end{bmatrix} \oplus \mathbf{1}_{n-i-1}, \end{aligned} \tag{5.2}$$

where

$$\hbar := e^{2\pi\sqrt{-1}[(g_3)_{i+1} - (g_3)_i]}.$$

For any $\beta_i \in \mathcal{B}_n$, define the triple $\boldsymbol{g}^{\beta_i} \in U_n \times L_n \times \mathfrak{t}$ by

$$\boldsymbol{g}^{\beta_i} := \left(B_1^{(i)}(\boldsymbol{g})^{-1} g_1 B_2^{(i)}(\boldsymbol{g}), \quad B_2^{(i)}(\boldsymbol{g})^{-1} g_2 B_3^{(i)}(\boldsymbol{g}), \quad P_i\, g_3\, P_i\right), \tag{5.3}$$

where P_i is the permutation matrix $i \leftrightarrow i+1$.

Lemma 5.1. *The braid group $\mathcal{B}_n$ acts on $U_n \times L_n \times \mathfrak{t}$ by mapping $(\beta_i, \boldsymbol{g}) \mapsto \boldsymbol{g}^{\beta_i}$ for $i = 1, \dots, n-1$.*

Proof. By a direct computation, one checks that $g^\beta = \mathrm{id}$ for any relator β in (5.1). □

Example. Let $n = 3$, and

$$\boldsymbol{g} = \left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix}, \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_3 & 0 \\ 0 & 0 & d_3 \end{pmatrix}\right).$$

We have

$$\boldsymbol{g}^{\beta_1} = \left(\begin{pmatrix} 1 & \alpha & c \\ 0 & 1 & b-ac \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ ae^{2\sqrt{-1}(d_2-d_1)\pi} & 1 & 0 \\ a\beta e^{2\sqrt{-1}(d_2-d_1)\pi} + \gamma & \beta & 1 \end{pmatrix}, \begin{pmatrix} d_2 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_3 \end{pmatrix}\right),$$

$$\boldsymbol{g}^{\beta_2} = \left(\begin{pmatrix} 1 & b & a+b\gamma \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \beta-\alpha\gamma & 1 & 0 \\ \alpha & ce^{2\sqrt{-1}(d_3-d_2)\pi} & 1 \end{pmatrix}, \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_3 & 0 \\ 0 & 0 & d_2 \end{pmatrix}\right).$$

If $\beta = (\beta_1\beta_2)^3$, the triple $\boldsymbol{g}^\beta = (g_1', g_2', g_3')$ equals

$$g_1' = \begin{pmatrix} 1 & ae^{2\sqrt{-1}(d_2-d_1)\pi} & be^{2\sqrt{-1}(d_3-d_1)\pi} \\ 0 & 1 & ce^{2\sqrt{-1}(d_3-d_2)\pi} \\ 0 & 0 & 1 \end{pmatrix} = e^{-2\pi\sqrt{-1}g_3}\, g_1\, e^{2\pi\sqrt{-1}g_3},$$

$$g_2' = \begin{pmatrix} 1 & 0 & 0 \\ e^{2\sqrt{-1}(d_1-d_2)\pi}\alpha & 1 & 0 \\ e^{2\sqrt{-1}(d_1-d_3)\pi}\beta & e^{2\sqrt{-1}(d_2-d_3)\pi}\gamma & 1 \end{pmatrix} = e^{-2\pi\sqrt{-1}g_3}\, g_2\, e^{2\pi\sqrt{-1}g_3},$$

$$g_3' = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} = g_3.$$

5.4. **Braid mutations of monodromy data.** Let M an analytic homogeneous semisimple flat F-manifold, and denote by M' the open set of tame semisimple points $p \in M$, i.e. at which the spectrum of the operator $\mathcal{U}(p)\colon T_pM \to T_pM$ is simple.

Consider the following two different settings:

(I) Assume $g\colon [0,1] \to M'$ to be a continuous path such that $g([0,1])$ is contained in a simply connected open set, on which a coherent choice of normalizations (1)-(5) can be done. Assume also that
  - τ is admissible at both $g(0)$ and $g(1)$,
  - there exists $\bar{t} \in [0,1]$ such that τ is not admissible at $g(\bar{t})$.

(II) Assume $p \in M'$ is a semisimple point, and fix some choice of normalizations (1)-(5). Let $\tau_0, \tau_1 \in \mathbb{R} \setminus \mathscr{S}(p)$, and $\tau\colon [0,1] \to M$ to be a continuous map such that

- $\tau(0) = \tau_0$ and $\tau(1) = \tau_1$,
- there exists $\bar{t} \in [0,1]$ such that $\tau(\bar{t})$ is not admissible at p.

In both cases (I) and (II), for each $t \in \{0,1\}$, we can introduce a set $\mathcal{M}_t$ of monodromy data.

Problem: In both settings (I) and (II), how to describe the transformation $\mathcal{M}_0 \mapsto \mathcal{M}_1$?

The matrices μ^λ, R will not depend on t, due to the results of Section 4.3. Hence, we need to describe how the matrices (S_1, S_2, C, Λ) will transform. In this section, we prove that this is described by an action of the braid group, which on the triple (S_1, S_2, Λ) reduces to (5.3).

Remark 5.2. Pictures (I) and (II) are "dual" to each other. In (I), we have a fixed $\tau \in \mathbb{R}$ and a variable set $\mathscr{S}(g(t))$ of non-admissible directions such that $\tau \in \mathscr{S}(g(0)) \cap \mathscr{S}(g(1))$.

In (II), we have a fixed set $\mathscr{S}(p) \subseteq \mathbb{R}$ of non-admissible directions and a continuous map $\tau\colon [0,1] \to \mathbb{R}$ with $\tau(0), \tau(1) \in \mathbb{R} \setminus \mathscr{S}(p)$. In both cases, we have to face a *wall-crossing phenomenon*: the fixed (resp. variable) point τ is not admissible for some values of the time parameter.

Given $\boldsymbol{u} \in \mathbb{C}^n$ introduce a family of *Stokes rays* in the universal cover $\widehat{\mathbb{C}^*}$: for any pair (i,j) such that $u^i \neq u^j$ set

$$\tau_{ij}(\boldsymbol{u}) := \frac{3\pi}{2} - \operatorname{Arg}(u^i - u^j), \qquad R_{ij}^{(k)}(\boldsymbol{u}) := \{z \in \widehat{\mathbb{C}^*}\colon \arg z = \tau_{ij}(\boldsymbol{u}) + 2\pi k\}, \quad k \in \mathbb{Z}.$$

Also, for any $\tau \in \mathbb{R}$ introduce the *admissible ray*

$$\ell_\tau := \{z \in \widehat{\mathbb{C}^*}\colon \arg z = \tau\}.$$

Both Stokes and admissible rays are equipped with the natural orientation, from 0 to ∞. Any continuous transformations of $\boldsymbol{u}$ and τ induce continuous rotations of the Stokes and admissible rays. In the case of settings (I) and (II), the oriented ray crosses some of the Stokes rays during the transformation. We will call *elementary* any such transformation of rays, along which ℓ_τ crosses one Stokes ray $R_{ij}^{(k)}$ only.

Let us focus on the picture (I). Fix $\boldsymbol{u}_o \in \mathbb{C}^n \setminus \Delta$ with components in τ-lexicographical order. Consider a continuous map $b_i\colon [0,1] \to (\mathbb{C}^n \setminus \Delta)$, with $i = 1, \dots, n-1$, such that:

(1) $b_i(0) = \boldsymbol{u}_o$,
(2) $b_i(t)^h = b_i(0)^h$ for all $h \neq i, i+1$,
(3) $b_i(t)^i$ counter-clockwise rotates w.r.t. $b_i(t)^{i+1}$ in the plane $\mathbb{C}$,
(4) $b_i(0)^i = b_i(1)^{i+1}$ and $b_i(1)^i = b_i(0)^{i+1}$.

The map b_i can be seen as a loop on $\operatorname{Conf}_n(\mathbb{C}) := (\mathbb{C}^n \setminus \Delta)/\mathfrak{S}_n$, the configuration space of n pairwise distinct points in $\mathbb{C}$. The space $\operatorname{Conf}_n(\mathbb{C})$ is aspherical (i.e. $\pi_k(\operatorname{Conf}_n(\mathbb{C})) = 0$ for $k \geqslant 2$), and its fundamental group is isomorphic to the braid group $\mathcal{B}_n$, see [KT08]. Consider the homotopy classes $[b_i]$ in $\pi_1(\operatorname{Conf}(\mathbb{C}^n), \{u^i(0)\}) \cong \mathcal{B}_n$. It is easily seen that

$$[b_i] * [b_j] = [b_j] * [b_i], \quad |i-j| > 1, \qquad [b_i] * [b_{i+1}] * [b_i] = [b_{i+1}] * [b_i] * [b_{i+1}],$$

where $*$ denotes the concatenation of loops. We identify $[b_i]$ with the elementary braid β_i.

In the case of picture (II), any of the maps b_i's can be seen as a map with target M, by working in a local chart with canonical coordinates in τ-lexicographical order. It is an elementary transformation: one of the Stokes rays $R_{i,i+1}^{(k)}$ clockwise crosses the ray ℓ_τ.

In summary, elementary transformations of type (I) can be identified with elements of $\mathcal{B}_n$. Dually, by exchanging orientations (counter-clockwise $\leftrightarrow$ clockwise), we can identify β_i with the type (II) transformation defined by a counter-clockwise rotation of ℓ_τ across one of the Stokes rays $R^{(k)}_{i,i+1}$.

Let (S_1, S_2, Λ, C) be the 4-tuple of Stokes, formal monodromy, and central connection matrices computed

- w.r.t. the point $g(0)$, in case (I);
- w.r.t. the line $\tau(0)$, in case (II);

In both cases (I) and (II), the monodromy data are always computed w.r.t. the lexicographical order of canonical coordinates, so that $(S_1, S_2, \Lambda) \in U_n \times L_n \times \mathfrak{t}$.

Theorem 5.3. *Along the elementary transformation β_i, with $i = 1, \dots, n-1$, the monodromy data transform as follows:*

$$(S_1, S_2, \Lambda) \mapsto (S_1, S_2, \Lambda)^{\beta_i}, \quad C \mapsto CB^{-1},$$

where

$$B = B_2^{(i)}(S_1, S_2, \Lambda) = \mathbf{1}_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & (S_2)_{i+1,i} \end{bmatrix} \oplus \mathbf{1}_{n-i-1}.$$

Cf. equations (5.2) and (5.3).

Proof. Whatever is the case under consideration, (I) or (II), let us consider the initial "frozen" configuration of Stokes and admissible rays, $R^{(h)}_{ij}$ and ℓ_τ.

Label the Stokes rays as follows: let $R^{(1)}$ be the first Stokes ray on the left of ℓ_τ, $R^{(0)}$ the first Stokes ray on the right of ℓ_τ, and extend the numeration $R^{(k)}$, with $k \in \mathbb{Z}$, so that the label k increases in counter-clockwise order.

Let m be the number of Stokes rays in any sector of $\widehat{\mathbb{C}^*}$ defined by

$$(2j-1)\pi < |\arg z - \tau| < 2\pi j, \qquad j \in \mathbb{Z}.$$

The number m also equals the number of Stokes rays in the sectors

$$2\pi j < |\arg z - \tau| < (2j+1)\pi, \qquad j \in \mathbb{Z}.$$

For generic initial points $\boldsymbol{u}_o \in \mathbb{C}^n \setminus \Delta$, we have $m = \frac{n(n-1)}{2}$, but some Stokes rays may coincide[4].

Define $\Pi^{(k)}$, with $k \in \mathbb{Z}$, to be the sector in $\widehat{\mathbb{C}^*}$ from the ray $R^{(k-1)}$ to the ray $R^{(m+k)}$. For each $k \in \mathbb{Z}$, there exists a unique solution $X^{(k)}(\boldsymbol{u}_o, z)$ of the ∂_z-equation of (3.22), specialized at $\boldsymbol{u} = \boldsymbol{u}_o$, such that

$$X^{(k)}(\boldsymbol{u}_o, z) \sim X_{\rm for}(\boldsymbol{u}_o, z), \qquad |z| \to +\infty, \quad z \in \Pi^{(k)}.$$

Introduce invertible matrices K_k, called *Stokes factors*, such that

$$X^{(k+1)}(\boldsymbol{u}_o, z) = X^{(k)}(\boldsymbol{u}_o, z)\, K_k, \quad k \in \mathbb{Z}.$$

[4]This happens for example if there are three indices (i, j, k) such that $u_o^i, u_o^j, u_o^k \in \mathbb{C}$ are collinear, or there are four indices (i, j, k, l) such that u_o^i, u_o^j and u_o^k, u_o^l define two parallel lines.

The matrices K_k have the following structure: all diagonal entries are 1, and the entry $(K_k)_{ab}$ is non-zero only if $R^{(m+k)}$ is one of the rays $R_{ab}^{(h)}$ with $h \in \mathbb{Z}$, see [BJL79].

Recall that the Stokes matrices S_1, S_2 are defined in terms of solutions X_1, X_2, X_3, see equation (4.10). We have

$$X_1(\boldsymbol{u}_o, z) \equiv X^{(1-m)}(\boldsymbol{u}_o, z), \quad X_2(\boldsymbol{u}_o, z) \equiv X^{(1)}(\boldsymbol{u}_o, z), \quad X_3(\boldsymbol{u}_o, z) \equiv X^{(1+m)}(\boldsymbol{u}_o, z).$$

Hence, we deduce

$$X_2(\boldsymbol{u}_o, z) = X^{(0)}(\boldsymbol{u}_o, z)\, K_0 = X^{(-1)}(\boldsymbol{u}_o, z)\, K_{-1}K_0 = \dots = \underbrace{X^{(1-m)}(\boldsymbol{u}_o, z)}_{X_1(\boldsymbol{u}_o, z)}\, K_{1-m} \dots K_{-1}K_0.$$

From equation (4.10), we deduce

$$S_1 = K_{1-m} \dots K_{-1}K_0.$$

Analogously, we have

$$S_2 = K_1 \dots K_m.$$

Up to now we have considered a "static" picture, at the initial time $t = 0$ of the transformation β_i. By letting the time parameter t vary, the Stokes rays and/or the ray ℓ_τ rotate. In particular, immediately before the collision of Stokes and oriented rays, we have $R_{i,i+1}^{(h)} = R^{(1)}$ for a suitable $h \in \mathbb{Z}$. After the collision we have $R_{i,i+1}^{(h)} \equiv R^{(0)}$. Hence, after the transformation β_i, we have the following transformation of Stokes matrices

$$\begin{aligned} S_1 &\mapsto S_1' = K_{-m} \dots K_0 K_1 = K_{1-m}^{-1}\, S_1\, K_1, \\ S_2 &\mapsto S_2' = K_2 \dots K_m K_{m+1} = K_1^{-1}\, S_2\, K_{1+m}. \end{aligned}$$

Similarly, the central connection matrix transforms as follows

$$C \mapsto C' = C\, K_1^{-1}$$

The only non-zero off-diagonal entries of K_1, K_{1-m}, K_{1+m} are

$$(K_{1-m})_{i,i+1} = (S_1)_{i,i+1}, \qquad (K_1)_{i+1,i} = (S_2)_{i+1,i},$$
$$(K_{1+m})_{i,i+1} = [e^{-2\pi\sqrt{-1}\Lambda} S_1 e^{2\pi\sqrt{-1}\Lambda}]_{i,i+1}.$$

The last identity follows from point (4) of Theorem 4.14. Finally, we also need to recover the lexicographical order, which is lost after the transformation β_i. By applying the permutation $i \leftrightarrow i+1$, we complete the proof. □

Remark 5.4. Theorem 5.3 generalizes the braid group action in Dubrovin's analytic theory of Frobenius manifolds, see [Dub99, Th. 4.8]. The action of the braid group on $U_n \times L_n \times \mathfrak{t}$ is just the simplest case of a more general picture described in [Boa01, Boa02]. The starting point is the observation that the "monodromy manifold"[5] $U_n \times L_n \times \mathfrak{t}$ is isomorphic to the dual Poisson-Lie group G^* of $G = GL(n, \mathbb{C})$. In [Boa02, Section 2] P. Boalch generalized the notion of Stokes multipliers and isomonodromic deformations for general connected complex reductive groups G. It was also proved that G^* can be identified with the space of meromorphic connections on principal G-bundles over the disc. In such a case one has an action

[5] In [Boa01, Boa02], meromorphic connections on a closed disk $D \subseteq \mathbb{C}$, with an irregular singularity only, are studied. We have a connections on $\mathbb{C}^*$ with two singularities, and we need a further piece of information for a global description of the monodromy: the central connection matrix.

of $\pi_1(\mathfrak{t}_{\rm reg})$ on G^*, where $\mathfrak{t}_{\rm reg}$ is the regular subset of a Cartan subalgebra $\mathfrak{t}\subseteq\mathfrak{g}$. This induces an action on G^* of the full braid group $\pi_1(\mathfrak{t}_{\rm reg}/W)$, with W the Weyl group. Such an action coincides with the De Concini-Kac-Procesi action of $\pi_1(\mathfrak{t}_{\rm reg}/W)$ on G^*, obtained in [DKP92] as classical limit of the *quantum Weyl group action* on the corresponding quantum group, due to Lusztig [Lus90], Kirillov and Reshetikhin [KR90], Soibelman [Soi90].

Action of the center $Z(\mathcal{B}_n)$. Consider the shift of the admissible direction $\tau\mapsto\tau+2\pi$. We have the following facts:

(1) In the generic case (i.e. for canonical coordinates in general position), the number of Stokes rays in any sector of $\mathbb{C}^*$ of width 2π equals $n(n-1)$. An elementary braid acts whenever the line ℓ_τ crosses a Stokes ray. So, in total, we expect that a complete rotation of ℓ_τ correspond to the product of $n(n-1)$ elementary braids.

(2) The effect of the shift $\tau\mapsto\tau+2\pi$ on the monodromy data can be identified with a transformation of different nature, namely a different choice of normalization (2). This consists in a different choice of the branch of the logarithm. From this it follows that the braid corresponding to $\tau\mapsto\tau+2\pi$ must commute with any other braids.

From point (2), we deduce that the braid corresponding to $\tau\mapsto\tau+2\pi$ is an element of the center

$$Z(\mathcal{B}_n)\cong\mathbb{Z}\cong{\rm Deck}(\widehat{\mathbb{C}^*}).$$

The center $Z(\mathcal{B}_n)$ is the cyclic group generated by the braid $\beta=(\beta_1\dots\beta_{n-1})^n$. From point (1), and the action of ${\rm Deck}(\widehat{\mathbb{C}^*})$, we deduce the following result.

Proposition 5.5. *The braid corresponding to the shift $\tau\mapsto\tau+2\pi$ of the admissible direction is the generator β of the center $Z(\mathcal{B}_n)$. It acts as follows:*

$$(S_1,S_2,\Lambda)^\beta=\left(e^{-2\pi\sqrt{-1}\Lambda}S_1e^{2\pi\sqrt{-1}\Lambda},\quad e^{-2\pi\sqrt{-1}\Lambda}S_2e^{2\pi\sqrt{-1}\Lambda},\quad\Lambda\right)$$
$$C\mapsto M_0^{-1}Ce^{2\pi\sqrt{-1}\Lambda}.$$
□

This proposition extends a computation (for $n=3$) of the Example of Section 5.3.

Corollary 5.6. *The generator $\beta=(\beta_1\dots\beta_{n-1})^n$ of the center $Z(\mathcal{B}_n)$ acts on $U_n\times L_n\times\mathfrak{t}$ as follows:*

$$\boldsymbol{g}^\beta=\left(e^{-2\pi\sqrt{-1}g_3}\,g_1\,e^{2\pi\sqrt{-1}g_3},\quad e^{-2\pi\sqrt{-1}g_3}\,g_2\,e^{2\pi\sqrt{-1}g_3},\quad g_3\right).\tag{5.4}$$

Proof. Let $\boldsymbol{g}'$ be the r.h.s. of (5.4). A simple computation shows that $B_j^{(i)}(\boldsymbol{g})e^{2\pi\sqrt{-1}g_3}=B_j^{(i)}(\boldsymbol{g}')$ for $i=1,\dots,n-1$ and $j=1,2,3$. Thus, if there exists $\tilde\beta\in\mathcal{B}_n$ such that $\boldsymbol{g}^{\tilde\beta}=\boldsymbol{g}'$ then $\tilde\beta\in Z(\mathcal{B}_n)$, i.e. $\tilde\beta=\beta^k$ for some $k\in\mathbb{Z}$. We have $k=1$, by Proposition 5.5. □

5.5. **Analytic continuation of the flat F-structure.** There is a more global point of view from which one can reinterpret the results of the previous sections. It both makes transparent the appearance of a braid group action on the monodromy data, and clarifies the "duality" of settings (I) and (II) of the previous section. Moreover, it also describes the analytic continuation of the flat F-manifold structure.

Admissibility sets $\mathcal{A}_n,\mathcal{A}_n^O,\mathcal{A}_n^Z,\widehat{\mathcal{A}_n}$. Consider the configuration space ${\rm Conf}_n(\mathbb{C}):=(\mathbb{C}^n\setminus\Delta)/\mathfrak{S}_n$ of n points in the plane, together with the following covering space:

- the ordered $\mathfrak{S}_n$-covering $\mathrm{Conf}_n^O(\mathbb{C}) := \mathbb{C}^n \setminus \Delta$,
- the covering $\mathrm{Conf}_n^Z(\mathbb{C})$ associated to the center $Z(\pi_1(\mathrm{Conf}_n(\mathbb{C}))) \cong \mathbb{Z}$,
- and the universal cover $\widehat{\mathrm{Conf}_n(\mathbb{C})}$.

The fundamental group $\pi_1(\mathrm{Conf}_n^O(\mathbb{C}))$ is isomorphic to the group $\mathcal{P}_n$ of pure braids. Notice that $Z(\mathcal{B}_n) = Z(\mathcal{P}_n) \cong \mathbb{Z}$. The coverings spaces above fit into the chain

$$\widehat{\mathrm{Conf}_n(\mathbb{C})} \to \mathrm{Conf}_n^Z(\mathbb{C}) \to \mathrm{Conf}_n^O(\mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C}).$$

The fundamental group of the space $\mathrm{Conf}_n^Z(\mathbb{C})$ is isomorphic to $\mathcal{B}_n/Z(\mathcal{B}_n) \cong M_n(\mathbb{R}^2)$, the mapping class group of the n-punctured plane, see [Bir74].

Remark 5.7. The space $\widehat{\mathrm{Conf}_n(\mathbb{C})}$ has been described for the first time by S. Kaliman [Kal75, Kal77, Kal93]: in *loc. cit.* it is proved that it is isomorphic to $\mathbb{C}^2 \times \mathcal{T}(0, n+1)$, where $\mathcal{T}(0, n+1)$ denotes the Teichmüller space of the Riemann sphere with $n+1$ punctures. The space $\mathcal{T}(0, n+1)$ is homeomorphic to $\mathbb{R}^{2n-4}$, and it is biholomorphic to a holomorphically convex Bergmann domain in $\mathbb{C}^{n-2}$. For further details see the interesting paper [Lin04].

Given $\boldsymbol{p} = \{p_1, \dots, p_n\} \in \mathrm{Conf}_n(\mathbb{C})$, define the set $\mathscr{S}(p) \subseteq \mathbb{R}$ by

$$\mathscr{S}(\boldsymbol{p}) := \{\mathrm{Arg}[-\sqrt{-1}(\overline{p_i} - \overline{p_j})] + 2k\pi\colon \quad k \in \mathbb{Z}\}, \quad \mathrm{Arg}(z) \in]-\pi, \pi].$$

Any number $\tau \in \mathbb{R} \setminus \mathscr{S}(p)$ is said to be an *admissible direction* at $\boldsymbol{p}$.

Introduce the smooth $(2n+1)$-dimensional real manifolds

$$\mathcal{X}_n := \mathrm{Conf}_n(\mathbb{C}) \times \mathbb{R}, \quad \mathcal{X}_n^O := \mathrm{Conf}_n(\mathbb{C}) \times \mathbb{R}, \quad \mathcal{X}_n^Z := \mathrm{Conf}_n^Z(\mathbb{C}) \times \mathbb{R}, \quad \widehat{\mathcal{X}_n} := \widehat{\mathrm{Conf}_n(\mathbb{C})} \times \mathbb{R},$$

Define the *admissibility open subset* $\mathcal{A}_n \subseteq \mathcal{X}_n$ by

$$\mathcal{A}_n := \{(\boldsymbol{p}, \tau) \in \mathcal{X}_n \colon \tau \in \mathbb{R} \setminus \mathscr{S}(\boldsymbol{p})\}.$$

Analogously, define the open subsets $\mathcal{A}_n^O \subseteq \mathcal{X}_n^O$, $\mathcal{A}_n^Z \subseteq \mathcal{X}_n^Z$, and $\widehat{\mathcal{A}_n} \subseteq \widehat{\mathcal{X}_n}$ as the pre-images of $\mathcal{A}_n$ along the projections $\widehat{\mathcal{X}_n} \to \mathcal{X}_n^Z \to \mathcal{X}_n^O \to \mathcal{X}_n$. We have the following commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{A}_n} \arrow[r] \arrow[d, hook] & \mathcal{A}_n^Z \arrow[r] \arrow[d, hook] & \mathcal{A}_n^O \arrow[r] \arrow[d, hook] & \mathcal{A}_n \arrow[d, hook] & \\
\widehat{\mathcal{X}_n} \arrow[r] \arrow[d] & \mathcal{X}_n^Z \arrow[r] \arrow[d] & \mathcal{X}_n^O \arrow[r] \arrow[d] & \mathcal{X}_n \arrow[r] \arrow[d] & \mathbb{R} \\
\widehat{\mathrm{Conf}_n(\mathbb{C})} \arrow[r] & \mathrm{Conf}_n^Z(\mathbb{C}) \arrow[r] & \mathrm{Conf}_n^O(\mathbb{C}) \arrow[r] & \mathrm{Conf}_n(\mathbb{C}) &
\end{tikzcd}$$

Homotopy groups of $\mathcal{A}_n, \mathcal{A}_n^O, \mathcal{A}_n^Z, \widehat{\mathcal{A}_n}$. Consider the subspace $\mathcal{A}_n'$ of admissibility set $\mathcal{A}_n$ defined by

$$\mathcal{A}_n' := \{(\boldsymbol{p}, 0) \in \mathcal{X}_n \colon 0 \text{ is admissible at } \boldsymbol{p}\}.$$

Lemma 5.8. *The subspace $\mathcal{A}_n'$ is a strong deformation retract of $\mathcal{A}_n$.*

Proof. Let $(\boldsymbol{p}, \tau) \in \mathcal{A}_n$. If $\boldsymbol{p} = \{p_1, \dots, p_n\}$, denote by $p^b := \frac{1}{n}\sum_{j=1}^n p_j$ the barycenter of the configuration. Let $F\colon [0,1] \times \mathcal{A}_n \to \mathcal{A}_n$ be defined by a rotation w.r.t. the barycenter

$$F(t, \boldsymbol{p}, \tau) := \left(\left\{e^{\sqrt{-1}t\tau}(p_j - p^b) + p^b \colon \quad j = 1, \dots, n\right\}, \quad \tau(1-t)\right).$$

For all $(\boldsymbol{p},\tau)\in\mathcal{A}_n$, $(\boldsymbol{a},0)\in\mathcal{A}'_n$ and $t\in[0,1]$, we have $F(0,\boldsymbol{p},\tau)=(\boldsymbol{p},\tau)$, $F(1,\boldsymbol{p},\tau)\in\mathcal{A}'_n$, and $F(t,\boldsymbol{a},0)=(\boldsymbol{a},0)$. ☐

Lemma 5.9. *The space $\mathcal{A}'_n$ is contractible.*

Proof. We show that the point $(\{1,\dots,n\},0)$ is a strong deformation retract of $\mathcal{A}'_n$. Given $(\boldsymbol{p},0)\in\mathcal{A}'_n$, with $\boldsymbol{p}=\{p_1,\dots,p_n\}$, without loss of generality we may assume that the p_j's are labelled in 0-lexicographical order. Consider the continuous map $F\colon[0,1]\times\mathcal{A}'_n\to\mathcal{A}'_n$ defined by

$$F(t,\boldsymbol{p})=\begin{cases}\left(\{(1-2t)p_j+2t\mathrm{Re}(p_j)\colon j=1,\dots,n\},\quad 0\right), & 0\leqslant t\leqslant\dfrac{1}{2},\\[2ex] \left(\{2t\mathrm{Re}(p_j)+(2t-1)j\colon j=1,\dots,n\},\quad 0\right), & \dfrac{1}{2}\leqslant t\leqslant 1.\end{cases}$$

The map F defines a strong deformation retraction of $\mathcal{A}'_n$ onto $(\{1,\dots,n\},0)$. ☐

Theorem 5.10. *We have*

$$\begin{aligned} &\pi_i(\mathcal{A}_n)=0, && i=0,1,2,\dots,\\ \pi_0(\mathcal{A}_n^O)=\mathfrak{S}_n,\quad &\pi_i(\mathcal{A}_n^O)=0, && i=1,2,3,\dots,\\ \pi_0(\mathcal{A}_n^Z)=\mathbb{Z},\quad &\pi_i(\mathcal{A}_n^Z)=0, && i=1,2,3,\dots,\\ \pi_0(\widehat{\mathcal{A}_n})=\mathcal{B}_n,\quad &\pi_i(\widehat{\mathcal{A}_n})=0, && i=1,2,3,\dots,\end{aligned}$$

where the homotopy groups are based at an arbitrary point.

Proof. All homotopy groups of $\mathcal{A}_n$ vanish, since $\mathcal{A}_n$ is contractible by Lemmata 5.8, 5.9. Since $\mathcal{A}_n$ is simply connected, we have the homeomorphisms $\mathcal{A}_n^O\cong\mathcal{A}_n\times\mathfrak{S}_n$, $\mathcal{A}_n^Z\cong\mathcal{A}_n\times\mathbb{Z}$, and $\widehat{\mathcal{A}_n}\cong\mathcal{A}_n\times\mathcal{B}_n$. Here $\mathfrak{S}_n$, $\mathbb{Z}$, and $\mathcal{B}_n$ are equipped with the discrete topology. The claim follows. ☐

Relative homotopy groups. Given a triple (A,B,c) of pointed topological spaces, with $c\in B\subseteq A$, the relative homotopy group $\pi_k(A,B,c)$, with $k\geqslant 1$, is the set[6] of homotopy classes of continuous maps $f\colon(\mathbb{D}^k,\mathbb{S}^{k-1},s_0)\to(A,B,c)$. In particular, the set $\pi_1(A,B,c)$ is the set of homotopy classes of paths $f\colon[0,1]\to A$ such that $f(0)\in B$, $f(1)=c$.

Fix a point $\boldsymbol{x}\in\mathcal{A}_n$, and three points $\boldsymbol{x}^o\in\mathcal{A}_n^O,\boldsymbol{x}^z\in\mathcal{A}_n^Z,\hat{\boldsymbol{x}}\in\widehat{\mathcal{A}_n}$ over it.

Theorem 5.11. *We have*

$$\begin{aligned}\pi_1(\mathcal{X}_n,\mathcal{A}_n,\boldsymbol{x})&\cong\pi_1(\mathcal{X}_n^O,\mathcal{A}_n^O,\boldsymbol{x}^o)\cong\pi_1(\mathcal{X}_n^Z,\mathcal{A}_n^Z,\boldsymbol{x}^z)\cong\pi_1(\widehat{\mathcal{X}_n},\widehat{\mathcal{A}_n},\hat{\boldsymbol{x}})\cong\mathcal{B}_n,\\ \pi_k(\mathcal{X}_n,\mathcal{A}_n,\boldsymbol{x})&\cong\pi_k(\mathcal{X}_n^O,\mathcal{A}_n^O,\boldsymbol{x}^o)\cong\pi_k(\mathcal{X}_n^Z,\mathcal{A}_n^Z,\boldsymbol{x}^z)\cong\pi_k(\widehat{\mathcal{X}_n},\widehat{\mathcal{A}_n},\hat{\boldsymbol{x}})\cong 0,\qquad k\geqslant 2.\end{aligned}$$

Proof. With the morphisms of triples of topological spaces

$$(\widehat{\mathcal{X}_n},\widehat{\mathcal{A}_n},\hat{\boldsymbol{x}})\to(\mathcal{X}_n^Z,\mathcal{A}_n^Z,\boldsymbol{x}^z)\to(\mathcal{X}_n^O,\mathcal{A}_n^O,\boldsymbol{x}^o)\to(\mathcal{X}_n,\mathcal{A}_n,\boldsymbol{x}),\tag{5.5}$$

[6]The group structure is well defined only for $k\geqslant 2$.

we can associate the following commutative diagram of relative homotopy groups

$$
\begin{tikzcd}
\dots \arrow[r] & \pi_1(\widehat{\mathcal{A}_n}, \hat{\boldsymbol{x}}) \arrow[r] \arrow[d] & \pi_1(\widehat{\mathcal{X}_n}, \hat{\boldsymbol{x}}) \arrow[r] \arrow[d] & \pi_1(\widehat{\mathcal{X}_n}, \widehat{\mathcal{A}_n}, \hat{\boldsymbol{x}}) \arrow[r] \arrow[d, "\alpha_1"] & \pi_0(\widehat{\mathcal{A}_n}, \hat{\boldsymbol{x}}) \arrow[r] \arrow[d] & \pi_0(\widehat{\mathcal{X}_n}, \hat{\boldsymbol{x}}) \arrow[d] \\
\dots \arrow[r] & \pi_1(\mathcal{A}_n^Z, \boldsymbol{x}^z) \arrow[r] \arrow[d] & \pi_1(\mathcal{X}_n^Z, \boldsymbol{x}^z) \arrow[r] \arrow[d] & \pi_1(\mathcal{X}_n^Z, \mathcal{A}_n^Z, \boldsymbol{x}^z) \arrow[r] \arrow[d, "\alpha_2"] & \pi_0(\mathcal{A}_n^Z, \boldsymbol{x}^z) \arrow[r] \arrow[d] & \pi_0(\mathcal{X}_n^Z, \boldsymbol{x}^z) \arrow[d] \\
\dots \arrow[r] & \pi_1(\mathcal{A}_n^O, \boldsymbol{x}^o) \arrow[r] \arrow[d] & \pi_1(\mathcal{X}_n^O, \boldsymbol{x}^o) \arrow[r] \arrow[d] & \pi_1(\mathcal{X}_n^O, \mathcal{A}_n^O, \boldsymbol{x}^o) \arrow[r] \arrow[d, "\alpha_3"] & \pi_0(\mathcal{A}_n^O, \boldsymbol{x}^o) \arrow[r] \arrow[d] & \pi_0(\mathcal{X}_n^O, \boldsymbol{x}^o) \arrow[d] \\
\dots \arrow[r] & \pi_1(\mathcal{A}_n, \boldsymbol{x}) \arrow[r] & \pi_1(\mathcal{X}_n, \boldsymbol{x}) \arrow[r] & \pi_1(\mathcal{X}_n, \mathcal{A}_n, \boldsymbol{x}) \arrow[r] & \pi_0(\mathcal{A}_n, \boldsymbol{x}) \arrow[r] & \pi_0(\mathcal{X}_n, \boldsymbol{x})
\end{tikzcd}
$$

The raws are the long exact relative homotopy sequences for each triples, and columns are the maps induced by (5.5). The maps $\alpha_1, \alpha_2, \alpha_3$ are bijections: this follows from the unique lifting property of paths for coverings. The claim then follows from Theorem 5.10. $\square$

Monodromy data as functions on $\mathcal{A}_M^Z$. Let M be a flat F-manifold with Euler field E, and denote by M' the set of points $p \in M$ at which the spectrum $\operatorname{spec}(E\circ_p)$ is simple. We have the local biholomorphism

$$\upsilon\colon M' \to \operatorname{Conf}_n(\mathbb{C}), \quad p \mapsto \operatorname{spec}(E\circ_p). \tag{5.6}$$

Consider the pulled-back fiber bundles on M'

$$\mathcal{X}_M := \upsilon^*\mathcal{X}_n, \quad \mathcal{X}_M^O := \upsilon^*\mathcal{X}_n^O, \quad \mathcal{X}_M^Z := \upsilon^*\mathcal{X}_n^Z, \quad \widehat{\mathcal{X}_M} := \upsilon^*\widehat{\mathcal{X}_n},$$

together with their open subsets

$$\mathcal{A}_M := (\upsilon^*)^{-1}\mathcal{A}_n, \quad \mathcal{A}_M^O := (\upsilon^*)^{-1}\mathcal{A}_n^O, \quad \mathcal{A}_M^Z := (\upsilon^*)^{-1}\mathcal{A}_n^Z, \quad \widehat{\mathcal{A}_M} := (\upsilon^*)^{-1}\widehat{\mathcal{A}_n}.$$

Given $p_o \in M'$, the monodromy data of (M', p_o) are well-defined after fixing the choice of normalizations (1)-(6) of Section 5.1.

The choice of (5) only, i.e. an ordering of canonical coordinates at p_o, is equivalent to the choice of a point of $\mathcal{X}_M^O$ over p_o.

The choice of (6) only, i.e. an admissible direction at p_o, is equivalent to the choice of a point of $\mathcal{A}_M$ over p_o.

The choice of both (5) and (6) is equivalent to the choice of a point of $\mathcal{A}_M^O$ over p_o.

If choices of (1),(2),(3),(4) are fixed, however, the 4-tuple (S_1, S_2, Λ, C) is not well-defined as a single-valued function on $\mathcal{A}_M^O$. Indeed, if $(p_o, \boldsymbol{u}_o, \tau)$ is a fixed point of $\mathcal{A}_M^O$, for any $k \in \mathbb{Z}$ there exist paths $\gamma_k\colon [0,1] \to \mathcal{A}_M^O$ such that $\gamma(0) = (p_o, \boldsymbol{u}_o, \tau)$ and $\gamma_k(1) = (p_o, \boldsymbol{u}_o, \tau + 2\pi k)$. Namely γ_k are lifts of loops in the center $Z(\pi_1(\operatorname{Conf}_n^O(\mathbb{C})))$.

Thus, the joint choice of (2),(5),(6) is equivalent to the choice of a point of $\mathcal{A}_M^Z$ over p_o.

Theorems 4.20 and 4.21 can be reformulated as follows.

Theorem 5.12. *Fix a choice of normalizations (1),(3),(4), and a point in $\mathcal{A}_M^Z$. The monodromy data (S_1, S_2, Λ, C) are locally constant functions on $\mathcal{A}_M^Z$.* $\square$

In total we have card $\pi_0(\mathcal{A}_M^Z)$ possible values of the monodromy data at $z=\infty$. Different values at different connected components of $\mathcal{A}_M^Z$, are labelled by paths in $\pi_1(\mathcal{X}_M^Z, \mathcal{A}_M^Z)$. The map (5.6) induces a morphism in homotopy

$$\upsilon_*\colon \pi_1(\mathcal{X}_M^Z, \mathcal{A}_M^Z) \to \pi_1(\mathcal{X}_n^Z, \mathcal{A}_n^Z) \cong \mathcal{B}_n.$$

The paths of settings (I) and (II) of Section 5.4 are representatives of homotopy classes in $\pi_1(\mathcal{X}_M^O, \mathcal{A}_M^O) \cong \pi_1(\mathcal{X}_M^Z, \mathcal{A}_M^Z)$. More precisely, consider the double fibrations

$$\begin{tikzcd}
& \mathcal{A}_M^O \arrow[d] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_2"] & \\
& \mathcal{X}_M^O \arrow[dl, "\rho_1"] \arrow[dr, "\rho_2"'] & \\
M' & & \mathbb{R}
\end{tikzcd}$$

Paths of setting (I) represent classes in $\pi_1(\rho_2^{-1}(\tau), p_2^{-1}(\tau))$ for fixed $\tau \in \mathbb{R}$.
Paths of setting (II) represent classes in $\pi_1(\rho_1^{-1}(p), p_1^{-1}(p))$ for fixed $p \in M'$.
Thus the "duality" mentioned in Section 5.4 reflects the underlying double fibrations above. In both cases (I) and (II), we have induced paths in $\pi_1(\mathcal{X}_M^O, \mathcal{A}_M^O)$.
Following the terminology of [CDG20], given $\tau \in \mathbb{R}$ we call τ-*chamber* of M any connected component of the open set $p_1(p_2^{-1}(\tau))$.

Analytic continuation. Let (M, p_o) be the germ of a semisimple analytic flat F-manifold with Euler vector field. Assume that p_o is tame semisimple. The whole flat F-structure can be analytically continued. The picture described in this section gives an insight on this continuation procedure.

By shrinking M, we can assume that the germ is defined on a simply connected set, sufficiently small so that (5.6) is an embedding. We can thus identify M' with $\upsilon(M') \subseteq \operatorname{Conf}_n(\mathbb{C})$. By fixing a point $\hat{\boldsymbol{u}}_o \in \widehat{\operatorname{Conf}_n(\mathbb{C})}$ over $\upsilon(p_o)$, we have an open embedding of $(M', p_o) \cong (\upsilon(M'), \upsilon(p_o)) \subseteq (\widehat{\operatorname{Conf}_n(\mathbb{C})}, \hat{\boldsymbol{u}}_o)$. In this way, one finds a maximal tame analytic continuation of the initial germ. Notice that the coefficients of the joint system of differential equations (3.22) continue to meromorphic functions of $\boldsymbol{u} \in \widehat{\operatorname{Conf}_n(\mathbb{C})}$: this is the *Painlevé property* of the solution $V^\lambda(\boldsymbol{u})$ of the isomonodromic differential equations (3.25). By fixing choices of normalizations (1),(3),(4), the monodromy data (S_1, S_2, Λ, C) of the system (3.22) can be seen as locally constant functions on the space $\widehat{\mathcal{A}_n}$. This space has countably many connected components in bijection with the braid group $\mathcal{B}_n$. All possible values of (S_1, S_2, Λ, C) are given by the action of the braid group $\mathcal{B}_n$ of Theorem 5.3.

6. Riemann-Hilbert-Birkhoff inverse problem for semisimple flat F-manifolds

6.1. RHB problem $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$ and the Malgrange-Sabbah Theorem.

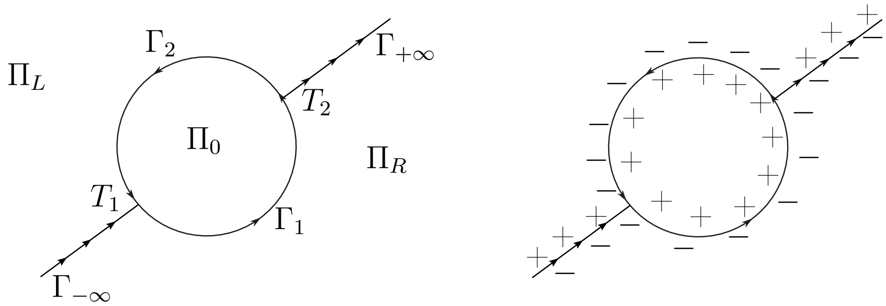

FIGURE 1. Contour Γ, paths $\Gamma_{\pm\infty}, \Gamma_1, \Gamma_2$, domains Π_0, Π_L, Π_R, and $\pm$ sides of Γ.

Admissible data. Denote by $\mathrm{Arg}(z) \in]-\pi, \pi]$ the principal branch of the argument of the complex number z. Let $\boldsymbol{u} \in \mathbb{C}^n$, and set

$$\mathscr{S}(\boldsymbol{u}) := \left\{ \mathrm{Arg}\left(-\sqrt{-1}(\overline{u^i} - \overline{u^j}\right) + 2\pi k \colon k \in \mathbb{Z},\ i, j \text{ are s.t. } u^i \neq u^j \right\}.$$

Any element $\tau \in \mathbb{R} \setminus \mathscr{S}(\boldsymbol{u})$ will be said to be *admissible at* $\boldsymbol{u}$.

Definition 6.1. Let $\boldsymbol{u} \in \mathbb{C}^n$ and τ admissible at $\boldsymbol{u}$. A $(\boldsymbol{u}, \tau)$-*admissible datum* is a 6-tuple $\mathfrak{M} := (B, D, L, S_1, S_2, C)$ of matrices in $M_n(\mathbb{C})$ such that:

(1) the matrix B is diagonal, i.e. $B = B'$,
(2) D is a diagonal matrix of integers,
(3) we have
$$\mathrm{tr}\, B = \mathrm{tr}\, D + \mathrm{tr}\, L. \tag{6.1}$$
(4) the matrices S_1, S_2, C are invertible, with $\det S_1 = \det S_2 = 1$,
(5) $(S_1)_{ii} = (S_2)_{ii} = 1$,
(6) if $i \neq j$, then $(S_1^{-1})_{ij} = 0$ if $\mathrm{Re}\left(e^{\sqrt{-1}(\tau - \pi)}(u^i - u^j) \right) > 0$,
(7) if $i \neq j$, then $(S_2)_{ij} = 0$ if $\mathrm{Re}\left(e^{\sqrt{-1}\tau}(u^i - u^j) \right) > 0$,
(8) we have
$$S_1^{-1} e^{2\pi\sqrt{-1}B} S_2^{-1} = C^{-1} e^{2\pi\sqrt{-1}L} C. \tag{6.2}$$

If $\boldsymbol{u} \in \Delta$, define the partition $\{1, \dots, n\} = \coprod_{r \in J} I_r$ such that for any $r \in J$ we have $\{i, j\} \subseteq I_r$ if and only if $u^i = u^j$. We then require the further vanishing condition

(9) $(S_1^{-1})_{ij} = (S_2)_{ij} = 0$ if $i, j \in I_r$ for some $r \in J$.

Lemma 6.2. *Let $\boldsymbol{u}_o \in \mathbb{C}^n$ and τ admissible at $\boldsymbol{u}_o$. If $\mathfrak{M}$ is $(\boldsymbol{u}_o, \tau)$-admissible, then there exists a sufficiently small neighborhood $\mathcal{V}$ of $\boldsymbol{u}_o$ such*

(1) *τ is admissible at $\boldsymbol{u}$, for all $\boldsymbol{u} \in \mathcal{V}$,*
(2) *$\mathfrak{M}$ is $(\boldsymbol{u}, \tau)$-admissible for all $\boldsymbol{u} \in \mathcal{V}$.* □

Let $\boldsymbol{u} \in \mathbb{C}^n$ and τ admissible at $\boldsymbol{u}$. Consider the complex z-plane with a branch cut from 0 to ∞:

$$\tau - \pi < \arg z < \tau + \pi.$$

Let $r > 0$ and denote by $\Gamma = \Gamma(\tau, r)$ the union of the following oriented paths, see Figure 1:

(1) the half-line $\Gamma_{-\infty}$ defined by $\arg z = \tau \pm \pi$, $|z| > r$, originating from ∞;

(2) the half-line $\Gamma_{+\infty}$ defined by $\arg z = \tau$, $|z| > r$, ending to ∞;
(3) the half-circle Γ_1 defined by $\tau - \pi < \arg z < \tau$, $|z| = r$, counterclockwise oriented;
(4) the half-circle Γ_2 defined by $\tau < \arg z < \tau + \pi$, $|z| = r$, counterclockwise oriented.

The orientations uniquely define the + and - side for each path $\Gamma_{\pm\infty}, \Gamma_1, \Gamma_2$. For $z \in \Gamma_{-\infty}$ we use the symbol $z_\pm$ if $\arg z = \tau \pm \pi$. Set Π_0, Π_L, Π_R to be the components of complement $\mathbb{C} \setminus \Gamma$, and T_1, T_2 to be the two nodes of Γ, as in Figure 1.

Let $\mathfrak{M} := (B, D, L, S_1, S_2, C)$ be a $(\boldsymbol{u}, \tau)$-admissible datum. Define two functions

$$Q(-;\boldsymbol{u}), H(-;\boldsymbol{u}) \colon \Gamma \to GL(n, \mathbb{C}),$$

by

$$Q(z;\boldsymbol{u}) := U(\boldsymbol{u}) z + B \log z, \quad U(\boldsymbol{u}) := \operatorname{diag}(u^1, \dots, u^n),$$

$$H(z;\boldsymbol{u}) := \begin{cases} e^{Q(z_-;\boldsymbol{u})} S_1^{-1} e^{-Q(z_-;\boldsymbol{u})}, & \text{along } \Gamma_{-\infty}, \\ e^{Q(z;\boldsymbol{u})} S_2 e^{-Q(z;\boldsymbol{u})}, & \text{along } \Gamma_{+\infty}, \\ e^{Q(z;\boldsymbol{u})} C^{-1} z^{-L} z^{-D}, & \text{along } \Gamma_1, \\ e^{Q(z;\boldsymbol{u})} S_2^{-1} C^{-1} z^{-L} z^{-D}, & \text{along } \Gamma_2. \end{cases}$$

We denote by $H_{\pm\infty}, H_1, H_2$ the restrictions of H at $\Gamma_{\pm\infty}, \Gamma_1, \Gamma_2$.

Problem 6.3 (Problem $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$)**.** Find an analytic function $G \colon \mathbb{C} \setminus \Gamma \to M_n(\mathbb{C})$ such that

(1) $G|_{\Pi_\nu}$ extends continuously to $\overline{\Pi_\nu}$ for $\nu = 0, L, R$;
(2) the non-tangential limits $G_\pm \colon \Gamma \to M_n(\mathbb{C})$ of G from the - and + sides of Γ exist, and are continuous;
(3) they are related by
$$G_+(z) = G_-(z) H(z;\boldsymbol{u});$$
(4) $G(z)$ tends to the identity matrix I as $z \to \infty$.

Theorem 6.4 ([Cot20c, Section 3])**.** *Let $\boldsymbol{u}_o \in \mathbb{C}^n$. Assume that the pair $(\tau, \mathfrak{M})$ is admissible at each point of a sufficiently small open neighborhood $\mathcal{V}$ of $\boldsymbol{u}_o$. If $\mathcal{P}[\boldsymbol{u}_o, \tau, \mathfrak{M}]$ is solvable, there exists an analytic set $\Theta \subseteq \mathcal{V} \setminus \{\boldsymbol{u}_o\}$ such that $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$ is solvable for all $\boldsymbol{u} \in \mathcal{V} \setminus \Theta$. Moreover, the solution $G(z;\boldsymbol{u})$ is unique and holomorphic w.r.t. $\boldsymbol{u} \in \mathcal{V} \setminus \Theta$.* □

Remark 6.5. In [Cot20c], we showed that Theorem 6.4 is essentially equivalent to a refinement, due to C. Sabbah [Sab18, Th. 4.9], of a previous result of B. Malgrange [Mal83b]. For this reason, we refer to Theorem 6.4 as Malgrange-Sabbah Theorem. The original result of Malgrange concerns the case $\boldsymbol{u}_o \in \mathbb{C}^n \setminus \Delta$. The result of Sabbah concerns the case $\boldsymbol{u}_o \in \Delta$.

6.2. **Construction of semisimple flat F-manifolds via a RHB inverse problem.** Let $\boldsymbol{u}_o \in \mathbb{C}^n$, τ be an admissible direction at $\boldsymbol{u}_o$, and $\mathfrak{M} = (B, D, L, S_1, S_2, C)$ be a $(\boldsymbol{u}_o, \tau)$-admissible datum. Assume that the RHB boundary value problem $\mathcal{P}[\boldsymbol{u}_o, \tau, \mathfrak{M}]$ is solvable. Let $\mathcal{V}$ and Θ as in Malgrange-Sabbah Theorem 6.4: the problem $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$ is well-defined, solvable and with unique solution $G(z, \boldsymbol{u})$, holomorphic w.r.t. $\boldsymbol{u} \in \mathcal{V} \setminus \Theta$. Consider the asymptotic expansions of $G(z, \boldsymbol{u})$ for $z \to 0$ and $z \to \infty$:

$$G(z, \boldsymbol{u}) = \mathbf{1} + z^{-1} F_1(\boldsymbol{u}) + O\left(z^{-2}\right), \quad z \to \infty, \quad z \in \Pi_{L/R},$$
$$G(z, \boldsymbol{u}) = G_0(\boldsymbol{u}) + z G_1(\boldsymbol{u}) + z^2 G_2(\boldsymbol{u}) + O(z^3), \quad z \to 0,$$

with coefficients F_1, G_i's holomorphic w.r.t. $\boldsymbol{u}$. The functions

$$X_{L/R}(z,\boldsymbol{u}) := G(z,\boldsymbol{u})z^B z^{zU}, \qquad z\in\Pi_{L/R},$$
$$X_0(z,\boldsymbol{u}) := G(z,\boldsymbol{u})z^D z^L, \qquad z\in\Pi_0,$$

are easily seen to be solutions of the joint system of differential equations

$$\tfrac{\partial}{\partial u^i}X = \left(zE_i - V_i(\boldsymbol{u})^T\right)X, \qquad V_i(\boldsymbol{u}) := [F_1(\boldsymbol{u})^T, E_i] \equiv -\left(\frac{\partial G_0}{\partial u^i}\cdot G_0^{-1}\right)^T, \tag{6.3}$$

$$\tfrac{\partial}{\partial z}X = \left(U - \tfrac{1}{z}V(\boldsymbol{u})^T\right)X, \qquad V(\boldsymbol{u}) := [F_1(\boldsymbol{u})^T, U] - B, \tag{6.4}$$

see [Cot20c, Section 3.4] for details.

Lemma 6.6. *The off-diagonal entries $(F_1'')^T$ satisfy the Darboux-Egoroff equations (3.26), (3.27), and the homogeneity conditions*

$$\sum_{k=1}^n u^k\partial_k F_1(\boldsymbol{u})^i_j = (b_i - b_j - 1)F_1(\boldsymbol{u})^i_j, \quad B = \mathrm{diag}(b_1,\dots,b_n). \tag{6.5}$$

Proof. The compatibility condition $\partial_i\partial_j = \partial_j\partial_i$ for the joint system (6.3),(6.4) reads

$$[E_j, \partial_i F_1^T] - [E_i, \partial_j F_1^T] + [[E_i, F_1^T],[E_j, F_1^T]] = 0.$$

This coincides with equations (3.26) and (3.27). Let $\kappa\in\mathbb{C}^*$, and set

$$h(z) := z^D z^L C\kappa^{-B}C^{-1}\kappa^{-L}z^{-L}\kappa^{-D}z^{-D}.$$

The piecewise analytic function $\widetilde{G}\colon (\Pi_0\cup\Pi_L\cup\Pi_R)\times(\kappa\mathcal{V}\setminus\kappa\Theta)\to\mathbb{C}$ defined by

$$\widetilde{G}(z;\boldsymbol{u}) := \kappa^{-B}G(\kappa z;\kappa^{-1}\boldsymbol{u})h(z)^{-1}, \qquad z\in\Pi_0,$$
$$\widetilde{G}(z;\boldsymbol{u}) := \kappa^{-B}G(\kappa z;\kappa^{-1}\boldsymbol{u})\kappa^{B}, \qquad z\in\Pi_{L/R},$$

solves the same RHB problem $\mathcal{P}[\boldsymbol{u},\tau,\mathfrak{M}]$ as G. By uniqueness of solution we have $\widetilde{G} = G$. This implies that $F_1(\kappa^{-1}\boldsymbol{u}) = \kappa\cdot\kappa^{-B}F_1(\boldsymbol{u})\kappa^B$, and (6.5) follows. ☐

Define the off-diagonal matrix $\Gamma(\boldsymbol{u})$ by $\Gamma(\boldsymbol{u})^i_j := F_1(\boldsymbol{u})^j_i$.

Corollary 6.7. *For any fixed $\boldsymbol{H}_o\in(\mathbb{C}^*)^n$, there exists a unique $\boldsymbol{H}(\boldsymbol{u}) = (H_1(\boldsymbol{u}),\dots,H_n(\boldsymbol{u}))$, analytic in $\mathcal{V}\setminus\Theta$, satisfying*

$$\partial_j H_i = \Gamma^i_j H_j, \quad i\neq j, \qquad \partial_i H_i = -\sum_{k\neq i}\Gamma^i_k H_k, \tag{6.6}$$

and such that $\boldsymbol{H}(\boldsymbol{u}_o) = \boldsymbol{H}_o$. Moreover, the functions H_i are never vanishing.

Proof. The linear Pfaffian system (6.6) is completely integrable, by Lemma 6.6. This ensures uniqueness and existence of solutions H_i. The non-vanishing of solutions is a standard result, see e.g. [Har20, Ch. 11]. ☐

Lemma 6.8. *Let $G_0(\boldsymbol{u}), H_i(\boldsymbol{u})$ as above. For any $\alpha = 1,\dots,n$, the one form $\varpi_\alpha(\boldsymbol{u}) := \sum_{k=1}^n G_0(\boldsymbol{u})^k_\alpha H_k(\boldsymbol{u})du^k$ is closed.*

Proof. Set $H := \mathrm{diag}(H_1,\dots,H_n)$. We have $\partial_j[G_0^T H]^\alpha_i = (G_0^T)^\alpha_j H_i\Gamma^j_i + (G_0^T)^\alpha_i H_j\Gamma^i_j = \partial_i[G_0^T H]^\alpha_j$. This can be easily seen by invoking equations (6.3), and (6.6). ☐

Lemma 6.9. *Let $G_1(\boldsymbol{u}), H_i(\boldsymbol{u})$ as above. For any $\alpha = 1, \dots, n$, the one form $\varphi_\alpha(\boldsymbol{u}) := \sum_{k=1}^n G_1(\boldsymbol{u})^k_\alpha H_k(\boldsymbol{u}) du^k$ is closed.*

Proof. Firstly, notice that we have $\partial_i G_1 = E_i G_0 + \partial_i G_0 \cdot G_0^{-1} \cdot G_1$. This follows from the fact that X_0 is a solution of the joint system (6.3), (6.4). Furthermore, we have the identity $\varphi_\nu(\boldsymbol{u}) := \sum_\alpha [G_0(\boldsymbol{u})^{-1} G_1(\boldsymbol{u})]^\alpha_\nu \cdot \varpi_\alpha(\boldsymbol{u})$. Hence, we deduce

$$d\varphi_\nu = \sum_{\alpha,i} \partial_i [G_0(\boldsymbol{u})^{-1} G_1(\boldsymbol{u})]^\alpha_\nu \; du^i \wedge \varpi_\alpha = \sum_{\alpha,i,k} (G_0^{-1})^\alpha_i \; (G_0)^i_\nu \; (G_0)^k_\alpha \; H_k \; du^i \wedge du^k = 0. \quad \square$$

Consider a polydisc $D(\boldsymbol{u}_o) \subseteq \mathcal{V} \setminus \Theta$ centered in $\boldsymbol{u}_o$. Define the functions

$$t^\alpha(\boldsymbol{u}) := \int_{\boldsymbol{u}_o}^{\boldsymbol{u}} \varpi_\alpha, \quad F^\nu(\boldsymbol{u}) := \int_{\boldsymbol{u}_o}^{\boldsymbol{u}} \varphi_\nu, \quad \alpha, \nu = 1, \dots, n, \tag{6.7}$$

where $\boldsymbol{u} \in D(\boldsymbol{u}_o)$. By definition, the functions $\boldsymbol{t} = (t^\alpha)_\alpha$ define a system of coordinates on $D(\boldsymbol{u}_o)$, with Jacobian matrix

$$\left(\frac{\partial t^\alpha}{\partial u^i}\right)_{\alpha,i} = G_0(\boldsymbol{u})^T H(\boldsymbol{u}).$$

Theorem 6.10. *The functions $F^\alpha(\boldsymbol{u}(\boldsymbol{t}))$ are solutions of the oriented associativity equations (2.2), and define an analytic semisimple flat F-manifold structure on $D(\boldsymbol{u}_o)$. The coordinates $\boldsymbol{t}$ are flat coordinates, the coordinates $\boldsymbol{u}$ are canonical coordinates.*

Proof. Define the product $\circ$ of vector fields by $\partial_i \circ \partial_j = \partial_i \delta_{ij}$, and the connection ∇^z, with $z \in \mathbb{C}$, defined by $\nabla^z_{\partial_\alpha} \partial_\beta = z \partial_\alpha \circ \partial_\beta$. Introduce the frame of vector fields $f_i := H_i^{-1} \partial_i$, for $i = 1, \dots, n$, and let $(f_i^\flat, \dots, f_n^\flat)$ be its dual frame. Given a column vector $(x_i)_i$ satisfying the joint system of equations (6.3), the one form $\xi := \sum_i x_i f_i^\flat$ is ∇^z-flat. The existence of a fundamental system of solutions X for the joint system (6.3) implies that the product $\circ$ defines a flat F-manifold structure on $D(\boldsymbol{u}_o)$. The functions $F^\alpha(\boldsymbol{u}(\boldsymbol{t}))$ are the potentials, by Corollary 3.2. $\square$

We denote by $\mathcal{F}[\boldsymbol{u}_o, \tau, \mathfrak{M}, \boldsymbol{H}_o]$ the germ (pointed at $\boldsymbol{u}_o$) of the analytic flat F-manifold described in Theorem 6.10. Different choices of the vector $\boldsymbol{H}_o \in (\mathbb{C}^*)^n$ correspond to rescalings of the oriented-associativity potentials F^α's as in Remark 2.7. The unit is given by the sum $e = \sum_i \frac{\partial}{\partial u^i}$. The sum $E = \sum_i u^i \frac{\partial}{\partial u^i}$ defines an Euler vector field. It is not unique: we can modify it by shifts $E \mapsto E - \lambda e$, as in Remark 2.8. If the germ is irreducible, then all other Euler vector fields are of this form by Theorem 2.17.

6.3. **Reconstruction of admissible germs of semisimple flat F-manifold.** In this section we prove that all admissible germs of analytic flat F-manifolds are of the form $\mathcal{F}[\boldsymbol{u}_o, \tau, \mathfrak{M}, \boldsymbol{H}_o]$.

Let (M, p) be an admissible germ of an analytic flat F-manifold. Fix choices of normalizations (1)-(6) of Section 5.1. We then have a well defined system $(\lambda, \mu^\lambda, R, S_1, S_2, \Lambda, C)$ of monodromy data at p, computed w.r.t. the chosen ordering of $\boldsymbol{u}_o = \boldsymbol{u}(p)$, an admissible direction τ, and the normalization $(H_{o,1}, \dots, H_{o,n}) \in (\mathbb{C}^*)^n$ of Lamé coefficients at p.

Lemma 6.11. *The matrix* $\mu^\lambda = \mathrm{diag}(q_1, \dots, q_n) - \lambda \cdot \mathbf{1}$ *has a unique decomposition* $\mu^\lambda = D^\lambda + S^\lambda$ *with*

$$D^\lambda = \mathrm{diag}(d_1, \dots, d_n), \quad d_i \in \mathbb{Z}, \quad i = 1, \dots, n$$
$$S^\lambda = \mathrm{diag}(\rho_1, \dots, \rho_n), \quad \mathrm{Re}(\rho_i) \in [0, 1[, \quad i = 1, \dots, n.$$

We have

$$[D^\lambda, S^\lambda] = 0, \qquad [R, S^\lambda] = 0. \tag{6.8}$$

Proof. The uniqueness of the decomposition, and the first commutation relation of (6.8) are clear. For any pair (i, j), we have $[S^\lambda, R]_{ij} = (\rho_i - \rho_j)R_{ij} = 0$. Indeed, for $i \neq j$ we have two possibilities: if $q_i - q_j \notin \mathbb{Z}_{<0}$, then $R_{ij} = 0$; if $q_i - q_j \in \mathbb{Z}_{<0}$, then $\rho_i - \rho_j = 0$. □

Remark 6.12. It follows that $z^{-\mu^\lambda} z^R = z^{-D^\lambda} z^{R - S^\lambda}$.

Proposition 6.13. *The* 6*-tuple* $\mathfrak{M} = (\Lambda, -D^\lambda, R - S^\lambda, S_1, S_2, C)$ *is* $(\boldsymbol{u}_o, \tau)$*-admissible. The problem* $\mathcal{P}[\boldsymbol{u}_o, \tau, \mathfrak{M}]$ *is solvable.*

Proof. Point (1),(2),(3) of Definition 6.1 directly follow from the definitions and properties of $R, \Lambda, D^\lambda, S^\lambda$. Points (4)-(9) of Definition 6.1 follow from Propositions 4.16 and 4.17. Let

- $\Xi_0(\boldsymbol{t}, z)$ be the fixed solution of the joint system (3.2) in Levelt normal form,
- $X_i(\boldsymbol{u}, z)$, with $i = 1, 2, 3$, be the solutions of the joint system (3.22) uniquely defined by the asymptotics (4.6).

The (unique) solution of the problem $\mathcal{P}[\boldsymbol{u}_o, \tau, \mathfrak{M}]$ is

$$G(z; \boldsymbol{u}_o) = \begin{cases} \left(\widetilde{\Psi}(\boldsymbol{t}_o)\boldsymbol{H}_o^{-1}\right)^T \Xi_0(\boldsymbol{t}_o, z) z^{-R} z^{\mu^\lambda}, & z \in \Pi_0, \\ X_2(\boldsymbol{u}_o, z) e^{-zU_o} z^{-\Lambda}, & z \in \Pi_R, \\ X_3(\boldsymbol{u}_o, z) e^{-zU_o} z^{-\Lambda}, & z \in \Pi_L. \end{cases}$$

Here we set $\boldsymbol{t}_o := \boldsymbol{t}(p)$, $U_o := \mathrm{diag}(u_o^1, \dots, u_o^n)$, and $\boldsymbol{H}_o := \mathrm{diag}(H_{o,1}, \dots, H_{o,n})$. □

Remark 6.14. In [Cot20c] it is proved that solutions of $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$ can be factorized via two auxiliary RHB problems $\mathcal{P}_1[\boldsymbol{u}, \tau, \mathfrak{M}]$ and $\mathcal{P}_2[\boldsymbol{u}, \tau, \mathfrak{M}]$. The problem $\mathcal{P}_1[\boldsymbol{u}, \tau, \mathfrak{M}]$ is shown to admit unique solution $\Psi(z, \boldsymbol{u})$ holomorphically depending on $\boldsymbol{u}$ varying in a neighborhood of $\boldsymbol{u}_o$, see [Cot20c, Th. 3.7]. Given $\Psi(z, \boldsymbol{u})$, the problem $\mathcal{P}_2[\boldsymbol{u}, \tau, \mathfrak{M}]$ is formulated, and it is shown to be locally uniquely solvable, see [Cot20c, proof of Th. 3.13].

If $\boldsymbol{u}_o \in \Delta$, assumption (9) of Definition 6.1 is crucial for the proof of the unique solvability of $\mathcal{P}_1[\boldsymbol{u}, \tau, \mathfrak{M}]$, see [Cot20c, proof of Lem. 3.6]. If (M, p) is not an admissible germ, then the monodromy data are not well defined. Indeed Theorem 4.14 does not hold, solutions $X_i(\boldsymbol{u}_o, z)$, with $i = 1, 2, 3$, are not unique. With each such a triple of solutions we can associated a pair (S_1, S_2) of Stokes matrices. In general these Stokes matrices do not satisfy Proposition 4.17.

Theorem 6.15. *The analytic germ* $\mathcal{F}[\boldsymbol{u}_o, \tau, \mathfrak{M}, \boldsymbol{H}_o]$ *of flat* F*-manifold is isomorphic to the original admissible germ* (M, p)*. They are defined by the same oriented associativity potentials (modulo linear terms). Different choices of* $\boldsymbol{H}_o$ *correspond to rescalings of the potentials.* □

In the light of the construction of Section 6.2, Theorem 6.15 follows from the following crucial result.

Lemma 6.16. *Let one of the following assumptions hold:*
(1) $\boldsymbol{u}_o \in \mathbb{C}^n \setminus \Delta$,
(2) $\boldsymbol{u}_o \in \Delta$ *and* $\delta_i - \delta_j \notin \mathbb{Z} \setminus \{0\}$ *for all* $i, j = 1, \dots, n$.

Let

$$F(\boldsymbol{u}) = F_o + \sum_{k=1}^{\infty} \sum_{\ell_1, \dots, \ell_k = 1}^{n} \frac{1}{k!} F^{(\boldsymbol{\ell})} \prod_{j=1}^{k} \overline{u}_{\ell_j}, \quad \overline{u}_i := u_i - u_{o,i},$$

be a matrix-valued formal power series whose off-diagonal entries F^i_j *are formal solutions of the Darboux-Egoroff system* (3.26), (3.27), (3.28), (3.29), (3.30). *The off-diagonal entries of the coefficients* $F^{(\boldsymbol{\ell})}$ *can be uniquely reconstructed from the off-diagonal entries of* F_o.

Proof. We have to show that the derivatives $\partial_{i_1} \dots \partial_{i_N} F^i_j(\boldsymbol{u}_o)$ can be computed from the only knowledge of the numbers $F^i_j(\boldsymbol{u}_o)$. We proceed by induction on N. Let us start with the case $N = 1$.
Step 1. For i, j, k distinct, by expanding both sides of $\partial_k F^i_j = F^i_k F^k_j$ in power series, and equating the coefficients, one reconstructs the coefficients of $\partial_k F^i_j(\boldsymbol{u}_o)$.
Step 2. From the identities (3.29) and (3.30) for F^i_j, one can compute $\partial_i F^i_j(\boldsymbol{u}_o)$ and $\partial_j F^i_j(\boldsymbol{u}_o)$ provided that $u_{o,i} \neq u_{o,j}$.
Step 3. Assume that $u_{o,i} = u_{o,j}$. By taking the ∂_i-derivative of both sides of (3.29) we obtain

$$(\delta_j - \delta_i - 2)\partial_i F^i_j + (u^j - u^i)\partial_i^2 F^i_j = \sum_{k \neq i,j} (u^k - u^j)[\partial_i F^i_k F^k_j + F^i_k \partial_i F^k_j]. \tag{6.9}$$

By evaluating (6.9) at $\boldsymbol{u} = \boldsymbol{u}_o$ we can compute all the numbers $\partial_i F^i_j(\boldsymbol{u}_o)$, namely

$$\partial_i F^i_j(\boldsymbol{u}_o) = \frac{1}{\delta_j - \delta_i - 2} \sum_{k \neq i,j} (u^k_o - u^j_o) \left[\partial_i F^i_k(\boldsymbol{u}_o) F^k_j(\boldsymbol{u}_o) + F^i_k(\boldsymbol{u}_o) F^k_i(\boldsymbol{u}_o) F^i_j(\boldsymbol{u}_o) \right].$$

Notice that the only terms $\partial_i F^i_k(\boldsymbol{u}_o)$ appearing in this sum are those computed in Step 2.
Step 4. If $u_{o,i} = u_{o,j}$, the numbers $\partial_j F^i_j(\boldsymbol{u}_o)$ can be computed similarly as in Step 3, by invoking equation (3.30):

$$\partial_j F^i_j(\boldsymbol{u}_o) = \frac{1}{\delta_j - \delta_i - 2} \sum_{k \neq i,j} (u^k_o - u^i_o) \left[F^i_j(\boldsymbol{u}_o) F^j_k(\boldsymbol{u}_o) F^k_j(\boldsymbol{u}_o) + F^i_k(\boldsymbol{u}_o) \partial_j F^k_j(\boldsymbol{u}_o) \right].$$

This proves that all the first derivatives $\partial_k F^i_j(\boldsymbol{u}_o)$ can be computed.

Inductive step. Assume to know all the N-th derivatives $\partial_{i_1} \dots \partial_{i_N} F^i_j(\boldsymbol{u}_o)$. We show how to compute the number $\partial_{h_1} \dots \partial_{h_{N+1}} F^i_j(\boldsymbol{u}_o)$ for any $(N+1)$-tuple $(h_1, \dots, h_{N+1})$.
Step 1. Assume that there exists $\ell \in \{1, \dots, N+1\}$ such that $h_\ell \neq i, j$. We have

$$\partial_{h_1} \dots \partial_{h_{N+1}} F^i_j = \partial_{h_1} \dots \partial_{h_{\ell-1}} \partial_{h_{\ell+1}} \dots \partial_{h_{N+1}} [\partial_{h_\ell} F^i_j] = \partial_{h_1} \dots \partial_{h_{\ell-1}} \partial_{h_{\ell+1}} \dots \partial_{h_{N+1}} [F^i_{h_\ell} F^{h_\ell}_j].$$

By evaluation at $\boldsymbol{u} = \boldsymbol{u}_o$, we can compute all the numbers $\partial_{h_1} \dots \partial_{h_{N+1}} F^i_j(\boldsymbol{u}_o)$.

Now we need to compute the mixed derivatives $\partial_i^p\partial_j^{N+1-p}F^i_j(\boldsymbol{u}_o)$, with $0\leqslant p\leqslant N+1$.

Step 2. Assume $p>0$ and $u_{o,i}\neq u_{o,j}$. Take the $\partial_i^{p-1}\partial_j^{N+1-p}$-derivative of both sides of (3.29): by evaluation at $\boldsymbol{u}=\boldsymbol{u}_o$ we can compute the numbers $\partial_i^p\partial_j^{N+1-p}F^i_j(\boldsymbol{u}_o)$.

Step 3. Assume $p>0$ and $u_{o,i}=u_{o,j}$. Take the $\partial_i^{p-1}\partial_j^{N+1-p}$-derivative of both sides of (6.9), to obtain

$$\begin{aligned}(\delta_j-\delta_i-p-1)\partial_i^p\partial_j^{N+1-p}F^i_j+(N+1-p)\partial_i^{p+1}\partial_j^{N-p}F^i_j+(u^j-u^i)\partial_i^{p+1}\partial_j^{N+1-p}F^i_j\\ =\partial_i^{p-1}\partial_j^{N+1-p}\sum_{k\neq i,j}(u^k-u^j)[\partial_iF^i_kF^k_j+F^i_k\partial_iF^k_j].\end{aligned}\tag{6.10}$$

Specialize (6.10) for $p=N+1$: by evaluation at $\boldsymbol{u}=\boldsymbol{u}_o$ of both sides, we can compute the derivative $\partial_i^{N+1}F^i_j(\boldsymbol{u}_o)$.
Specialize (6.10) for $p=N$: by evaluation at $\boldsymbol{u}=\boldsymbol{u}_o$ of both sides, we can compute the derivative $\partial_i^N\partial_jF^i_j(\boldsymbol{u}_o)$.
Repeating this procedure, by decreasing $p\mapsto p-1$ at each step, we can compute all the mixed derivatives $\partial_i^p\partial_j^{N+1-p}F^i_j(\boldsymbol{u}_o)$.

Step 4. Assume $p=0$. The derivative $\partial_j^{N+1}F^i_j(\boldsymbol{u}_o)$ can be computed, as in Steps 2 and 3, by invoking equation (3.30).

This proves that all the $(N+1)$-th derivatives $\partial_{h_1}\dots\partial_{h_{N+1}}F^i_j(\boldsymbol{u}_0)$ can be computed. $\square$

6.4. **Convergence of semisimple admissible formal flat F-manifolds.** We are now ready to prove the following result.

Theorem 6.17. *Let $(H,\boldsymbol{\Phi})$ be an admissible formal semisimple flat F-manifold over $\mathbb{C}$, with Euler field E. The oriented associativity potentials $\boldsymbol{\Phi}=(\Phi^1,\dots,\Phi^n)$ have a non-empty common domain of convergence.*

Proof. Fix one ordering $\boldsymbol{u}_o\in\mathbb{C}^n$ of the eigenvalues of the operator $\mathcal{U}(\boldsymbol{t})$ at $\boldsymbol{t}=0$. We have $n\times n$ matrix-valued (a priori) formal power series in $\boldsymbol{u}$

$$V(\boldsymbol{u})=V_o+\sum_{k=1}^\infty\sum_{\ell_1,\dots,\ell_k=1}^n\frac{1}{k!}V^{(\boldsymbol{\ell})}\prod_{j=1}^k\overline{u}_{\ell_j},\qquad V_i(\boldsymbol{u})=V_{i,o}+\sum_{k=1}^\infty\sum_{\ell_1,\dots,\ell_k=1}^n\frac{1}{k!}V_i^{(\boldsymbol{\ell})}\prod_{j=1}^k\overline{u}_{\ell_j},$$

$$\Psi(\boldsymbol{u})=\Psi_o+\sum_{k=1}^\infty\sum_{\ell_1,\dots,\ell_k=1}^n\frac{1}{k!}\Psi^{(\boldsymbol{\ell})}\prod_{j=1}^k\overline{u}_{\ell_j},\qquad \Gamma(\boldsymbol{u})=\Gamma_o+\sum_{k=1}^\infty\sum_{\ell_1,\dots,\ell_k=1}^n\frac{1}{k!}\Gamma^{(\boldsymbol{\ell})}\prod_{j=1}^k\overline{u}_{\ell_j},$$

where $\overline{u}_i:=u_i-u_{o,i}$ for $i=1,\dots,n$. These power series are well defined by the semisimplicity assumption, and they satisfy properties described in Propositions 3.17 and 3.18.

Set $\boldsymbol{H}_o:=\Psi_o\widetilde{\Psi}_o^{-1}$, where $\widetilde{\Psi}_o:=\left(\left.\frac{\partial u^i}{\partial t^\alpha}\right|_{\boldsymbol{t}=0}\right)_{i,\alpha=1}^n$.

After fixing choices of normalizations of Section 5.1, we can introduce a system of monodromy data $(\lambda,\mu^\lambda,R,S_1,S_2,\Lambda,C)$ for the formal flat F-structure, computed w.r.t. an admissible direction τ at $\boldsymbol{u}_o$, and the normalization $\boldsymbol{H}_o$ of Lamé coefficients at the origin. Proposition 6.13 holds true, with the same proof. We can set the RHB problem $\mathcal{P}[\boldsymbol{u}_o,\tau,\mathfrak{M}]$.

This problem is solvable w.r.t. $\boldsymbol{u}$ on an open neighborhood $\mathcal{V}\setminus\Theta$ of $\boldsymbol{u}_o$, by Theorem 6.4. The unique solution $G(z;\boldsymbol{u})$ is holomorphic in $\boldsymbol{u}\in\mathcal{V}\setminus\Theta$, and with expansion

$$G(z;\boldsymbol{u})=I+\tfrac{1}{z}F_1^{\rm an}(\boldsymbol{u})+O\left(\tfrac{1}{z^2}\right),\qquad z\to\infty,\quad z\in\Pi_{L/R},$$
$$G(z;\boldsymbol{u})=G_0(\boldsymbol{u})+G_1(\boldsymbol{u})z+G_2(\boldsymbol{u})z^2+G_3(\boldsymbol{u})z^3+O(z^4),\qquad z\to 0.$$

Here the superscript “an” stands for *analytic*. As output of Section 6.2, we also obtain a compatible joint system of differential equations (with analytic coefficients in $\boldsymbol{u}$, not just formal) of the form

$$\frac{\partial}{\partial u^i}X=\left(zE_i-V_i(\boldsymbol{u})^T\right)X,\quad \frac{\partial}{\partial z}X=\left(U-\frac{1}{z}V(\boldsymbol{u})^T\right)X,\tag{6.11}$$

where

$$V^{\rm an}(\boldsymbol{u}):=[F_1^{\rm an}(\boldsymbol{u})^T,U]-\Lambda,$$
$$V_i^{\rm an}(\boldsymbol{u}):=[F_1^{\rm an}(\boldsymbol{u})^T,E_i]\equiv-\left(\frac{\partial G_0}{\partial u^i}\cdot G_0^{-1}\right)^T=\left(\frac{\partial}{\partial u^i}(G_0^T)^{-1}\right)\cdot G_0^T.$$

We also have

$$V^{\rm an}(\boldsymbol{u}_o)=V_o,\quad G_0(\boldsymbol{u}_o)=(\Psi_o^{-1})^T,\quad \partial_iG_0=V_i^{\rm an}G_0,\quad i=1,\dots,n.$$

From the equality $[F_1^{\rm an}(\boldsymbol{u}_o)^T,E_i]=V_o=[\Gamma_o,E_i]$ we deduce that $[F_1^{\rm an}(\boldsymbol{u}_o)'']^T=\Gamma_o$. Moreover, by Lemma 6.6, the off diagonal matrix $[F_1^{\rm an}(\boldsymbol{u})'']^T$ solve equations (3.26),(3.27),(3.28),(3.29), (3.30). By Lemma 6.16, we obtain $\Gamma(\boldsymbol{u})=[F_1^{\rm an}(\boldsymbol{u})'']^T$. In particular, $\Gamma(\boldsymbol{u})$ is convergent. It follows that $\Psi(\boldsymbol{u})=(G_0(\boldsymbol{u})^{-1})^T$, $V_i(\boldsymbol{u})=V_i^{\rm an}(\boldsymbol{u})$, and $V(\boldsymbol{u})=V^{\rm an}(\boldsymbol{u})$ are convergent.

The oriented associativity potentials $\Phi^1,\dots,\Phi^n$ can be reconstructed via formulas (6.7). The original formal structure $(H,\boldsymbol{\Phi})$ turns out to be equivalent to the analytic flat F-manifold $\mathcal{F}[\boldsymbol{u}_o,\tau,\mathfrak{M},\boldsymbol{H}_o]$. □

Open question: Does it exist a semisimple and doubly resonant germ of flat F-structure which is purely formal?

A positive answer would imply the optimality of Theorem 6.17. The study of the doubly resonant germs goes beyond the general theory developed in [CDG19].

Remark 6.18. Consider a trivial vector bundle E on $\mathbb{P}^1$, equipped with a meromorphic connection ∇^o with connection matrix Ω given by

$$\Omega=-\left[U_o+\frac{1}{z}\left(\Lambda+[(F_o'')^T,U_o]\right)\right]dz,\quad U_o={\rm diag}(u_o^1,\dots,u_o^n),\quad \Lambda={\rm diag}(\lambda-\delta_1,\dots,\lambda-\delta_n).$$

Malgrange's Theorem [Mal83a, Mal86] asserts that, if $\boldsymbol{u}_o\in\mathbb{C}^n\setminus\Delta$, the connection ∇^o has a germ of universal deformation. Its connection matrix is

$$-d(zU)-([F''(\boldsymbol{u})^T,U]+\Lambda)\frac{dz}{z}-[F''(\boldsymbol{u})^T,dU],\tag{6.12}$$

where $F''(\boldsymbol{u})$ is the unique off-diagonal solution of the Darboux-Egoroff equations of Lemma 6.16. The same statement holds if $\boldsymbol{u}_o\in\Delta$ and $\delta_i-\delta_j\notin\mathbb{Z}\setminus\{0\}$: this is Sabbah's refinement proved in [Sab18]. In both cases the function $F''(\boldsymbol{u})$ is analytic in a neighborhood of $\boldsymbol{u}_o$.
If $\boldsymbol{u}_o\in\Delta$ and $\delta_i-\delta_j\in\mathbb{Z}\setminus\{0\}$ for some $i\neq j$, Lemma 6.16 does not hold, and the

initial datum $(\boldsymbol{u}_o, F_o)$ does not identify a unique (formal) solution $F(\boldsymbol{u})$ of the Darboux-Egoroff equations. The universality of the (formal) deformation (6.12) is lost. This answers a question raised by C. Sabbah in a private communication to the author.

6.5. **On the number of monodromy local moduli.** Consider all n-dimensional germs of homogenous semisimple flat F-manifolds, modulo local isomorphisms.

Theorem 6.19. *The local isomorphism classes of n-dimensional germs of homogeneous semisimple flat F-manifolds generically depend on n^2 parameters. The local isomorphism classes of n-dimensional germs of homogeneous semisimple Frobenius manifolds generically depend on $\frac{1}{2}(n^2-n)$ parameters.*

Proof. We show that germs of flat F-manifolds are identifiable with points of a "stratified space" X, whose generic dimension is n^2.

To parametrize them, we have at least two ways. After fixing $\boldsymbol{u}_o$, and $\lambda_o \in \mathbb{C}$, and the initial value $\boldsymbol{H}_o$ of the Lamé coefficients, one can choose

- the initial datum (Γ_o) of the Darboux-Egoroff equations, and an n-tuple $(\delta_1, \dots, \delta_n)$ of conformal dimensions;
- the monodromy data $(\mu^{\lambda_o}, R, S_1, S_2, \Lambda_o, C)$.

The entries of Γ_o are in total $n^2 - n$. In the generic case, the conformal dimensions $\delta_1, \dots, \delta_n$ do not differ by integers, and we have

$$\dim(\text{generic stratum of } X) = n^2.$$

Let us consider the tuple $(\mu^{\lambda_o}, R, S_1, S_2, \Lambda_o, C)$. For the generic case, the entries of μ^{λ_o} do not differ by integers, so that $R = 0$. The remaining matrices must satisfy equation (4.12). In particular, we have

$$S_1^{-1} e^{2\pi\sqrt{-1}\Lambda_o} S_2^{-1} \in \mathcal{O}(e^{-2\pi\sqrt{-1}\mu^{\lambda_o}}),$$

where $\mathcal{O}(e^{-2\pi\sqrt{-1}\mu^{\lambda_o}})$ denotes the similarity orbit of $e^{-2\pi\sqrt{-1}\mu^{\lambda_o}}$. The codimension of the orbit $\mathcal{O}(e^{-2\pi\sqrt{-1}\mu^{\lambda_o}})$ in $M(n, \mathbb{C})$ equals the dimension of the centralizer

$$\dim\{A \in M(n, \mathbb{C}) \colon \ [A, e^{-2\pi\sqrt{-1}\mu^{\lambda_o}}] = 0\},$$

see [Arn71, §2.4]. Hence, for generic μ^{λ_o} we have

$$\dim \mathcal{O}(e^{-2\pi\sqrt{-1}\mu^{\lambda_o}}) = n^2 - n.$$

Moreover, it is easy to see that if a matrix A admits a LDU-decomposition

$$A = G_1 G_2 G_3, \qquad G_1 \in L_n, \quad G_3 \in U_n, \quad G_2 = \operatorname{diag}(g_1, \dots, g_n),$$

then such a decomposition is unique, see e.g. [HJ85]. From this, it follows that (S_1, S_2) can be used as coordinates on $\mathcal{O}(e^{-2\pi\sqrt{-1}\mu^{\lambda_o}})$. The total number of parameters $(\mu^{\lambda_o}, S_1, S_2)$ equals

$$\dim(\text{generic stratum of } X) = n + n^2 - n = n^2.$$

Alternatively, one can choose (μ^{λ_o}, C) as coordinates on $\mathcal{O}(e^{-2\pi\sqrt{-1}\mu^{\lambda_o}})$, provided that C is defined up to left multiplication by elements of $\mathbb{T} = (\mathbb{C}^*)^n$. The point corresponding to (μ^{λ_o}, C) is

$$C^{-1} e^{-2\pi\sqrt{-1}\mu^{\lambda_o}} C \in \mathcal{O}(e^{-2\pi\sqrt{-1}\mu^{\lambda_o}}).$$

In total, we get $n + (n^2 - n) = n^2$ parameters. At non-generic points, the space X can get additional strata.

Let us also consider the subspace $X_{\text{Frob}} \subseteq X$ of all n-dimensional pointed germs of homogeneous semisimple Frobenius manifolds.

For Frobenius manifolds, one has standard choices $\lambda_o = \frac{d}{2}$ and $H_{o,i} = \eta(\partial_i, \partial_i)^{\frac{1}{2}}$. The initial datum Γ_o of Darboux-Egoroff equations is symmetric, i.e. $\Gamma_o^T = \Gamma_o$, and the conformal charges δ_i are all equal (to $\lambda_o = \frac{d}{2}$). Consequently, the generic stratum of X_{Frob} has dimension

$$\dim(\text{generic stratum of } X_{\text{Frob}}) = n^2 - \frac{n(n-1)}{2} - n = \frac{n(n-1)}{2}.$$

From the point of view of monodromy data, one has to impose some constraints among the coordinates. In the system of monodromy coordinates $(\mu^{\lambda_o}, S_1, S_2)$, one has to impose the η-skew-symmetry of μ^{λ_o} (this implies that all conformal dimensions are equal), and also $S_1^{-1} = S_2^T$, see Remark 4.19. In total we have $n + \frac{1}{2}(n^2 - n)$ constraints. In the system of monodromy coordinates (μ^{λ_o}, C), besides the η-skew-symmetry of μ^{λ_o}, we also invoke equation (4.13). By imposing the triangularity of S, we have a total number of $n + \frac{1}{2}(n^2 - n)$ independent constraints. In summary, we find again

$$\dim(\text{generic stratum of } X_{\text{Frob}}) = n^2 - \frac{n(n-1)}{2} - n = \frac{n(n-1)}{2}.$$

Finally, notice that other choices of $\boldsymbol{H}_o$ do not change the isomorphism class of the germ, and that small perturbations of $\boldsymbol{u}_o$ correspond to transformations in the n-dimensional automorphism group $\mathrm{Aut}(M)_0$, by Proposition 2.15. □

Remark 6.20. We underline that the tuple $(\mu^\lambda, R, S_1, S_2, \Lambda, C)$ of monodromy data actually provides two equivalent systems of "essential parameters" classifying germs. For generic germs, one system is (μ^λ, S_1, S_2), the other is (μ^λ, C). Both have a total of n^2 essential parameters.

7. Applications to LM-CohFTs, F-CohFTs, and open WDVV equations

7.1. **Losev-Manin moduli spaces and LM-CohFTs.** A *n-pointed chain of projective lines* $(C; s_0, s_\infty; s_1, \dots, s_n)$ consists of the following data:

(1) a nodal curve $C = C_1 \cup \dots \cup C_m$ (over $\mathbb{C}$) whose irreducible components C_j are projective lines;
(2) each component C_j is equipped with two marked points $p_j^\pm$, called *poles*;
(3) C_i and C_j intersect only if $|i - j| = 1$;
(4) C_i and C_{i+1} intersect transversally in $p_i^+ = p_{i+1}^-$;
(5) $s_0 = p_1^- \in C_1$ and $s_\infty = p_m^+ \in C_m$ are called *white points*;
(6) $s_1, \dots, s_n \in C \setminus \{p_1^\pm, \dots, p_m^\pm\}$ are called *black points*.

A n-pointed chain of projective lines is *stable* if there is at least one black point on each irreducible components. Notice that black points are allowed to coincide.

Tow n-pointed chains of projective lines $(C; s_0, s_\infty; s_1, \dots, s_n)$ and $(C'; s_0', s_\infty'; s_1', \dots, s_n')$ are *isomorphic* if there exists an isomorphism $\varphi\colon C \to C'$ such that $\varphi(s_j) = s_j'$ for $j = 0, 1, \dots, n, \infty$.

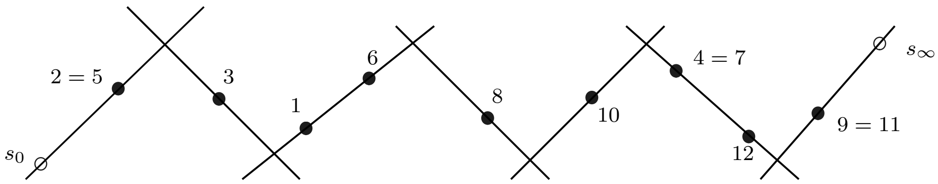

FIGURE 2. A stable 12-pointed chain of projective lines.

The spaces $\overline{L}_n$. The *Losev-Manin moduli space* $\overline{L}_n$, with $n \geqslant 1$, is defined as the fine-moduli space of stable n-pointed chains of projective lines [LM00].

The space $\overline{L}_n$ is a $(n-1)$-dimensional smooth toric varieties (over $\mathbb{C}$): it contains the open dense torus

$$L_n = \{(\mathbb{P}^1; 0, \infty; s_1, \dots, s_n)\}/\text{iso} \cong (\mathbb{C}^*)^n/\mathbb{C}^* \cong (\mathbb{C}^*)^{n-1}.$$

The space $\overline{L}_n$ is the toric variety associated with the convex polytope in $\mathbb{C}^n$ called *permutohedron*, defined as the convex hull of the $\mathfrak{S}_n$-orbit of the point $(1, 2, \dots, n) \in \mathbb{C}^n$, see [LM00]. Such a toric variety can be constructed via iteration of blow-ups of $\mathbb{P}^{n-1}$. As a first step, blow-up n points $p_1, \dots, p_n$ in general position in $\mathbb{P}^{n-1}$. Subsequently, blow-up the strict transforms of the $\frac{1}{2}n(n-1)$ lines passing through the pairs (p_i, p_j) for all $i, j = 1, \dots, n$. Continue this blowing-up procedure up to $(n-3)$-dimensional hyperplanes, see [Kap93, §4.3].

Alternatively, $\overline{L}_n$ is the toric variety defined by the fan formed by the Weyl chambers of the roots system of type A_{n-1}, with $n \geqslant 2$, [BB11].

The group $\mathfrak{S}_2 \times \mathfrak{S}_n$ naturally acts on $\overline{L}_n$ by permuting white and black points, respectively.

The cohomology ring $H^\bullet(\overline{L}_n, \mathbb{Q})$ was studied in [LM00, Man04]. It is algebraic: all odd cohomology groups vanish, and $H^\bullet(\overline{L}_n, \mathbb{Q})$ is isomorphic to the Chow ring $A^\bullet(\overline{L}_n, \mathbb{Q})$, [LM00, Th. 2.7.1]. See also, [BM14] where the groups $H^\bullet(\overline{L}_n, \mathbb{Q})$ are determined as representation of $\mathfrak{S}_2 \times \mathfrak{S}_n$.

Given $n_1, n_2 \geqslant 1$, we have a natural morphism $\overline{L}_{n_1} \times \overline{L}_{n_2} \mapsto \overline{L}_{n_1+n_2}$, defined by concatenation of white points. Furthermore, each boundary divisor of $\overline{L}_n$ is isomorphic to $\overline{L}_{n_1} \times \overline{L}_{n_2}$ with $n_1 + n_2 = n$.

Let $\overline{\mathcal{M}}_{0,n+2}$ be the moduli space of stable $(n+2)$-pointed trees of projective lines. We have a surjective birational morphism $p_n \colon \overline{\mathcal{M}}_{0,n+2} \to \overline{L}_n$ for any choice of two different labels i, j in $(1, ..., n+2)$ (the chosen white points).

Losev-Manin cohomological field theories. Let V_1, V_2 be two complex vector spaces of finite dimensions. A *LM-cohomological field theory* (for short, LM-CohFT), *on the pair* (V_1, V_2), is the datum of polylinear maps

$$\alpha_n \colon V_1^* \otimes V_1 \otimes V_2^{\otimes n} \to H^\bullet(\overline{L}_n, \mathbb{C}), \quad \text{with } n \geqslant 1,$$

such that, for any chosen bases[7] $(v_1, \dots, v_{N_1})$ of V_1 and $(w_1, \dots, w_{N_2})$ of V_2, the following properties are satisfied:

(1) α_n is $\mathfrak{S}_n$-covariant w.r.t. the natural actions of $\mathfrak{S}_n$ on both $V_2^{\otimes n}$ and $H^\bullet(\overline{L}_n, \mathbb{C})$,

[7]In the following paragraphs, if $(e_1, \dots, e_N)$ is a basis of a vector space V, then $(e_1^\vee, \dots, e_N^\vee)$ denotes the dual basis of V^*.

(2) for any partition $I\coprod J=\{1,\dots,n\}$ with $|I|=n_1$ and $|J|=n_2$ we have[8]

$$\mathrm{gl}^*\alpha_n(v_i^\vee\otimes v_h\otimes\bigotimes_{i=1}^n w_{\rho_i})=\alpha_{n_1}(v_i^\vee\otimes v_\mu\otimes\bigotimes_{i\in I} w_{\rho_i})\otimes\alpha_{n_2}(v_\mu^\vee\otimes v_h\otimes\bigotimes_{i\in J} w_{\rho_i}),$$

where $1\leqslant i,h\leqslant N_1$ and $1\leqslant \rho_1,\dots,\rho_n\leqslant N_2$, and $\mathrm{gl}\colon \overline{L}_{n_1}\times\overline{L}_{n_2}\to\overline{L}_{n_1+n_2}$ is the gluing map.

Remark 7.1. The spaces $\overline{L}_n$ and $\overline{\mathcal{M}}_{0,n}$, and their higher genus analogs, are two examples of moduli spaces of *weighted* stable pointed curves constructed in [Has03], corresponding to two different choices of weights. Losev-Manin CohFT's fit in a more general construction developed in [BM09], in the setting of moduli spaces of curves and maps with weighted stability conditions. We borrow the terminology “Losev-Manin CohFT” from [SZ11].

Commutativity equations. Consider two complex vector spaces V_1,V_2 of dimension N_1,N_2 respectively. Fix a basis $(w_1,\dots,w_{N_2})$ of V_2 and let $\boldsymbol{t}:=(t^1,\dots,t^{N_2})$ to be the dual coordinates.

The *Losev-Manin commutativity equation* for $B\in\mathbb{C}[\![\boldsymbol{t}]\!]\otimes\mathrm{End}(V_1)$ is given by

$$dB\wedge dB=0. \tag{7.1}$$

In coordinates $\boldsymbol{t}$, equation (7.1) is equivalent to the commutation relations

$$\left[\frac{\partial B}{\partial t^i},\frac{\partial B}{\partial t^j}\right]=0,\quad i,j=1,\dots,N_2.$$

Fix a basis $(v_1,\dots,v_{N_1})$ of V_1, and let $(v_1^\vee,\dots,v_{N_1}^\vee)$ be the dual basis of V_1^*.

Given a LM-CohFT $(\alpha_n)_{n\geqslant 1}$, define the formal power series $\mathcal{B}^i_j\in\mathbb{C}[\![\boldsymbol{t}]\!]$, with $i,j=1,\dots,N_1$, by

$$\mathcal{B}^i_j(\boldsymbol{t}):=\sum_{m=1}^\infty\sum_{\rho_1,\dots,\rho_m=1}^{N_2}\frac{t^{\rho_1}\dots t^{\rho_m}}{m!}\int_{\overline{L}_m}\alpha_m\left(v_i^\vee\otimes v_j\otimes\bigotimes_{\ell=1}^m w_{\rho_\ell}\right). \tag{7.2}$$

The matrix $\mathcal{B}:=(\mathcal{B}^i_j)_{i,j=1}^{N_1}$ represents an element of $\mathbb{C}[\![\boldsymbol{t}]\!]\otimes\mathrm{End}(V_1)$ in the basis $v_1,\dots,v_{N_1}$ of V_1.

Theorem 7.2. *The matrix $\mathcal{B}$ is a solution of the commutativity equation (7.1). Vice-versa, any solution B of (7.1), such that $B(0)=0$, has the form (7.2) for a unique LM-CohFT $(\alpha_n)_{n\geqslant 1}$.*

Proof. This is an equivalent reformulation of [LM00, Th. 3.3.1, Prop. 3.6.1] and [LM04, Th. 5.1.1]. □

7.2. **F-cohomological field theories.** Let V be a complex vector space of finite dimension N. Denote by $\overline{\mathcal{M}}_{g,n}$ the Deligne-Mumford moduli space of genus g stable curves with n marked points, defined for $g,n\geqslant 0$ in the stable regime $2g-2+n>0$.

An *F-cohomological field theory* (for short F-CohFT) is the datum of

- polylinear maps $c_{g,n+1}\colon V^*\otimes V^{\otimes n}\to H^{\mathrm{ev}}(\overline{\mathcal{M}}_{g,n+1},\mathbb{C})$, for $2g-1+n>0$,
- a distinguished vector $e_1\in V$,

[8]Einstein's summation rule over repeated Greek indices is used.

such that, for any chosen basis $(e_1, \dots, e_N)$ of V, the following properties are satisfied:

(1) $c_{g,n+1}$ is $\mathfrak{S}_n$-covariant w.r.t. the natural actions of $\mathfrak{S}_n$ on both $V^* \otimes V^{\otimes n}$ (permutation of the n copies of V) and on $H^{\mathrm{ev}}(\overline{\mathcal{M}}_{g,n+1}, \mathbb{C})$ (permutation of the last n marked points);
(2) $\pi^* c_{g,n+1}(e^\vee_{\rho_0} \otimes \bigotimes_{i=1}^n e_{\rho_i}) = c_{g,n+2}(e^\vee_{\rho_0} \otimes \bigotimes_{i=1}^n e_{\rho_i} \otimes e_1)$, for $1 \leqslant \rho_1, \dots, \rho_n \leqslant N$, where $\pi \colon \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$ is the map forgetting the last marked point;
(3) $c_{0,3}(e^\vee_\alpha \otimes e_\beta \otimes e_1) = \delta^\alpha_\beta$, for $1 \leqslant \alpha, \beta \leqslant N$;
(4) for any partition $I \coprod J = \{1, \dots, n\}$ with $|I| = n_1$ and $|J| = n_2$, we have[9]

$$\mathrm{gl}^* c_{g_1+g_2, n_1+n_2+1}(e^\vee_{\rho_0} \otimes \bigotimes_{i=1}^n e_{\rho_i}) = c_{g_1, n_1+2}(e^\vee_{\rho_0} \otimes \bigotimes_{i \in I} e_{\rho_i} \otimes e_\mu) \otimes c_{g_2, n_2+1}(e^\vee_\mu \otimes \bigotimes_{j \in J} e_{\rho_j}),$$

for $1 \leqslant \rho_1, \dots, \rho_n \leqslant N$, and $\mathrm{gl} \colon \overline{\mathcal{M}}_{g_1, n_1+2} \times \overline{\mathcal{M}}_{g_2, n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2, n_1+n_2+1}$ is the corresponding gluing map.

The *genus 0 sector* (or *tree-level*) of a given F-CohFT is the datum of the maps $(c_{0,n})_{n \geqslant 2}$ and the distinguished vector $e_1 \in V$ only.

Given a tree-level F-CohFT, fix a basis $(e_1, \dots, e_N)$ of V, and denote by $\boldsymbol{t} := (t^1, \dots, t^N)$ the dual coordinates. Define the formal power series $F^\alpha \in \mathbb{C}[\![\boldsymbol{t}]\!]$, for $\alpha = 1, \dots, N$, by

$$F^\alpha(\boldsymbol{t}) := \sum_{n=2}^\infty \sum_{\rho_1, \dots, \rho_n = 1}^N \frac{t^{\rho_1} \dots t^{\rho_n}}{n!} \int_{\overline{\mathcal{M}}_{0,n+1}} c_{0,n+1}\left(e^\vee_\alpha \otimes \bigotimes_{i=1}^n e_{\rho_i}\right). \tag{7.3}$$

Theorem 7.3. *The functions $F^\alpha(\boldsymbol{t})$ are solution of the oriented associativity equations*

$$\frac{\partial^2 F^\alpha}{\partial t^1 \partial t^\beta} = \delta^\alpha_\beta, \qquad \alpha, \beta = 1, \dots, N, \tag{7.4}$$

$$\frac{\partial^2 F^\alpha}{\partial t^\mu \partial t^\beta} \frac{\partial^2 F^\mu}{\partial t^\gamma \partial t^\delta} = \frac{\partial^2 F^\alpha}{\partial t^\mu \partial t^\gamma} \frac{\partial^2 F^\mu}{\partial t^\beta \partial t^\delta}, \qquad \alpha, \beta, \gamma, \delta = 1, \dots, N, \tag{7.5}$$

and thus define a formal flat F-manifold structure on V with unit e_1.

Vice-versa, any solution $(F^1, \dots, F^N)$ of (7.4)-(7.5), with $F^\alpha(0) = 0$ and $\partial_\beta F^\alpha(0) = 0$ for all $\alpha, \beta = 1, \dots, N$, is of the form (7.3) for a unique tree-level F-CohFT $(c_{0,n})_{n \geqslant 2}$.

Proof. The first part of the statement follows from a simple computation, invoking properties (1)-(4) above. For a proof of the second part of the statement, see Appendix B. ☐

7.3. **From tree-level F-CohFT to LM-CohFT, and vice-versa.** Given a tree-level F-CohFT on V, a LM-CohFT is naturally defined on the pair $(V_1, V_2) = (V, V)$. For any $n \geqslant 1$ define

$$\alpha_n := (p_n)_* \circ c_{0,n+2} \colon V^* \otimes V^{\otimes(n+1)} \to H^\bullet(\overline{L}_n, \mathbb{C}),$$

where $p_n \colon \overline{\mathcal{M}}_{0,n+2} \to \overline{L}_n$ is the surjective birational morphism defined by a choice of two white points.

Proposition 7.4. *The polylinear maps $(\alpha_n)_{n \geqslant 1}$ define a LM-CohFT on (V, V).*

[9]Einstein's summation rule over repeated Greek indices is used.

Proof. The $\mathfrak{S}_n$-covariance of α_n follows from the $\mathfrak{S}_{n+1}$-covariance of $c_{0,n+2}$. For $n_1+n_2=n$, we have the following commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{0,n_1+2}\times\overline{\mathcal{M}}_{0,n_2+2} \arrow[r, "\mathrm{gl}"] \arrow[d, "p_{n_1}\times p_{n_2}"'] & \overline{\mathcal{M}}_{0,n+2} \arrow[d, "p_n"] \\
\overline{L}_{n_1}\times\overline{L}_{n_2} \arrow[r, "\widetilde{\mathrm{gl}}"] & \overline{L}_n
\end{tikzcd}$$

with proper vertical arrows and local complete intersections as horizontal arrows. The gluing property of α_n then follows from the gluing property of $c_{0,n+2}$ and the excess intersection formula [Ful98, Prop. 6.6 and Prop. 17.4.1]. Notice that the excess bundle $\mathbb{E}$ has rank 0 (both gl and $\widetilde{\mathrm{gl}}$ have codimension 1), hence

$$\widetilde{\mathrm{gl}}^*(p_n)_*\,x=(p_{n_1}\times p_{n_2})_*\,\mathrm{gl}^*x,$$

for all $x\in H^{\mathrm{ev}}(\overline{\mathcal{M}}_{0,n+2},\mathbb{C})\cong A^\bullet(\overline{\mathcal{M}}_{0,n+2})_{\mathbb{C}}$. $\square$

Vice-versa, given a LM-CohFT $(\alpha_n)_{n\geqslant 1}$ on (V_1,V_2) we can reconstruct a formal flat F-manifold, provided that

- $\dim V_1=\dim V_2=N$,
- we are given an extra amount of data, namely a *primitive vector*.

Definition 7.5. Let $B\in\mathbb{C}[\![\boldsymbol{t}]\!]\otimes\mathrm{End}(V_1)$ be a solution of commutativity equations (7.1). A vector $h\in V_1$ is *primitive for B* if the vectors

$$\left.\frac{\partial B}{\partial t^1}\right|_{\boldsymbol{t}=0}\cdot h,\quad\dots,\quad\left.\frac{\partial B}{\partial t^N}\right|_{\boldsymbol{t}=0}\cdot h$$

define a basis of V_1. Equivalently, h is primitive if, for any chosen basis $(v_1,\dots,v_N)$ of V_1, we have

$$\det\left(\left.\frac{\partial B^i_\mu}{\partial t^k}\right|_0 h^\mu\right)_{i,k=1}^N\neq 0,\quad\text{where } h=h^\mu v_\mu.$$

If B admits a primitive vector h, we can identify V_1 and V_2 via the isomorphism $w_k\mapsto\left.\frac{\partial B}{\partial t^k}\right|_0 h$, for $k=1,\dots,N$. Under such an identification, $\boldsymbol{t}$ can be thought as coordinates on $V_1\cong V_2$.

Proposition 7.6 ([LM04, Prop. 5.3.3])**.** *If B admits a primitive vector h, then there exists a formal flat F-manifold structure on $V_1\cong V_2$ with flat identity h. The oriented associativity potentials $\boldsymbol{F}=(F^1,\dots,F^N)$ satisfy $B^\alpha_\beta=\frac{\partial F^\alpha}{\partial t^\beta}$.*

Proof. Let $(v_1,\dots,v_N)$ be a basis of V_1 with $v_1=h$. By assumption, we have $\det\left(\left.\frac{\partial B^i_1}{\partial t^k}\right|_0\right)\neq 0$. Hence, up to change of basis $(w_1,\dots,w_N)$ of V_2, we can assume that in the coordinates $\boldsymbol{t}$ we have

$$\frac{\partial B^i_1}{\partial t^k}(\boldsymbol{t})=\delta^i_k\quad\Rightarrow\quad B^i_1(\boldsymbol{t})=t^i+c.$$

Consider the $\mathbb{C}[\![\boldsymbol{t}]\!]\otimes V_1$-valued differential form $dB\wedge d(Bh)$. In the bases $(w_i)_{i=1}^N$ and $(v_i)_{i=1}^N$ chosen as above, it has components

$$[dB\wedge d(Bh)]^i=\left(\frac{\partial B^i_\lambda}{\partial t^\nu}dt^\nu\right)\wedge\left(\delta^\lambda_\mu dt^\mu\right)=\frac{\partial B^i_\mu}{\partial t^\nu}dt^\nu\wedge dt^\mu.$$

On the other hand, $dB\wedge d(Bh)=(dB\wedge dB)h=0$, since h is $\boldsymbol{t}$-independent and B a solution of (7.1). Hence,

$$\frac{\partial B^i_\mu}{\partial t^\nu}-\frac{\partial B^i_\nu}{\partial t^\mu}=0,\quad \mu,\nu=1,\dots,N.$$

This implies the existence of $F^i\in\mathbb{C}[\![\boldsymbol{t}]\!]$ such that $B^i_j=\partial_jF^i$. □

Theorem 7.7. *The following notions are equivalent:*

1. *formal flat F-manifold,*
2. *tree-level F-cohomological field theory,*
3. *LM-cohomological field theory with primitive element.*

Proof. It follows from Theorems 7.2, 7.3 and Propositions 7.4, 7.6. □

Remark 7.8. Black marked points of stable n-pointed chains of projective lines are allowed to coincide. It would be tempting to compare these coincidences of black points with the coalescence phenomenon at irregular singularities of ordinary differential equations studied in [CG18, CDG19]. Any contingent relation deserves further investigations. I thank Yu.I. Manin for pointing out such an analogy in a private communication.

7.4. **Homogeneous F-CohFTs.** A F-CohFT $(c_{g,n+1})_{g,n}$ is said to be *homogeneous* if

1. the vector spaces V and V^* are graded, with homogeneous bases $(e_1,\dots,e_N)$ and $(e_1^\vee,\dots,e_N^\vee)$,
$$\deg e_\alpha=-\deg e^\vee_\alpha=q_\alpha,\quad \alpha=1,\dots,N\qquad \deg e=0;$$
2. there exist $r^1,\dots,r^N,\gamma\in\mathbb{C}$ such that

$$\begin{split}\operatorname{Deg}c_{g,n+1}\left(e^\vee_{\alpha_0}\otimes\bigotimes_{i=1}^n e_{\alpha_i}\right)+\pi_*\,c_{g,n+2}\left(e^\vee_{\alpha_0}\otimes\bigotimes_{i=1}^n e_{\alpha_i}\otimes r^\lambda e_\lambda\right)\\=\left(\sum_{i=1}^n q_{a_i}-q_{\alpha_0}+\gamma g\right)c_{g,n+1}\left(e^\vee_{\alpha_0}\otimes\bigotimes_{i=1}^n e_{\alpha_i}\right),\end{split}\tag{7.6}$$

where $\operatorname{Deg}\colon H^\bullet(\overline{\mathcal{M}}_{g,n})\to H^\bullet(\overline{\mathcal{M}}_{g,n})$ rescale a k-th degree class by a factor $\frac{k}{2}$, and $\pi\colon\overline{\mathcal{M}}_{g,n+2}\to\overline{\mathcal{M}}_{g,n+1}$ is the morphism forgetting the last marked point.

Proposition 7.9. *If a tree-level F-CohFT is homogeneous then the associated formal flat F-manifold with potentials (7.3) is homogeneous, with the Euler vector field*

$$E=\sum_{\alpha=1}^N((1-q_\alpha)t^\alpha+r^\alpha)\frac{\partial}{\partial t^\alpha}.$$

Proof. A simple computations shows that equations (7.6), specialized at $g=0$, imply equations (2.4) for the potentials (7.3). □

The following result follows from Theorem 6.17, and Abel Lemma.

Theorem 7.10. *Let $(H, \boldsymbol{\Phi})$ be a formal flat F-manifold over $\mathbb{C}$, with $\dim_{\mathbb{C}} H = N$. Let $(c_{0,n+1})_{n\geqslant 2}$ and $(\alpha_n)_{n\geqslant 1}$ be the underlying tree-level F-CohFT and LM-CohFT, respectively. If $(H, \boldsymbol{\Phi})$ is admissible, then there exist real positive constants $m, k_1, \dots, k_N \in \mathbb{R}_+$ such that*

$$\left|\int_{\overline{\mathcal{M}}_{0,|\boldsymbol{n}|+1}} c_{0,|\boldsymbol{n}|+1}\left(\Delta_\beta^\vee \otimes \bigotimes_{j=1}^N \Delta_j^{\otimes n_j}\right)\right| \leqslant m\, \boldsymbol{n}! \prod_{j=1}^N k_j^{n_j}, \quad \boldsymbol{n}\in\mathbb{N}^N, \quad \beta = 1,\dots,N,$$

$$\left|\int_{\overline{L}_{|\boldsymbol{n}|}} \alpha_{|\boldsymbol{n}|}\left(\Delta_\beta^\vee \otimes \Delta_\gamma \otimes \bigotimes_{j=1}^N \Delta_j^{\otimes n_j}\right)\right| \leqslant m\, \boldsymbol{n}! \prod_{j=1}^N k_j^{n_j}, \quad \boldsymbol{n}\in\mathbb{N}^N, \quad \beta,\gamma = 1,\dots,N,$$

where we set $\boldsymbol{n}! := \prod_{j=1}^N n_j$, and $|\boldsymbol{n}| := \sum_{j=1}^N n_j$. $\square$

7.5. **Open WDVV equations.** Let k be a commutative $\mathbb{Q}$-algebra. Consider a formal Frobenius manifold over k, with Euler vector field E, defined by the solution $F \in k[\![t^1, \dots, t^n]\!]$ of the WDVV equations:

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^3 F}{\partial t^\nu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^3 F}{\partial t^\nu \partial t^\gamma \partial t^\alpha}, \tag{7.7}$$

$$\frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta} = \text{const.}, \quad \eta = (\eta_{\alpha\beta})_{\alpha,\beta}, \quad \eta^{-1} = (\eta^{\alpha\beta})_{\alpha,\beta}, \tag{7.8}$$

$$E^\nu \frac{\partial F}{\partial t^\nu} = (3-d)F + Q(\boldsymbol{t}), \quad E^\nu = (1-q^\nu)t^\nu + r^\nu, \tag{7.9}$$

where $\alpha, \beta, \gamma \in \{1, \dots, n\}$, $d \in k$ is the conformal dimension (or charge) of the Frobenius manifold, $q^\nu, r^\nu \in k$, and $Q(\boldsymbol{t}) \in k[\boldsymbol{t}]$ is a quadratic polynomial in $\boldsymbol{t}$.

The *open WDVV equations* (OWDVV) are the following overdetermined system of PDEs for $F^o \in k[\![t^1, \dots, t^n, s]\!]$:

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^2 F^o}{\partial t^\nu \partial t^\gamma} + \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta}\frac{\partial^2 F^o}{\partial s \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\gamma \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^2 F^o}{\partial t^\nu \partial t^\alpha} + \frac{\partial^2 F^o}{\partial t^\gamma \partial t^\beta}\frac{\partial^2 F^o}{\partial s \partial t^\alpha}, \tag{7.10}$$

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^2 F^o}{\partial t^\nu \partial s} + \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta}\frac{\partial^2 F^o}{\partial s^2} = \frac{\partial^2 F^o}{\partial s \partial t^\beta}\frac{\partial^2 F^o}{\partial s \partial t^\alpha}, \tag{7.11}$$

$$\frac{\partial^2 F^o}{\partial t^1 \partial t^\alpha} = 0, \qquad \frac{\partial^2 F^o}{\partial t^1 \partial s} = 1, \tag{7.12}$$

$$E^\nu \frac{\partial F^o}{\partial t^\nu} + \left(\frac{1-d}{2}s + r^{n+1}\right)\frac{\partial F^o}{\partial s} = \frac{3-d}{2}F^o + L(\boldsymbol{t}), \tag{7.13}$$

where $\alpha, \beta, \gamma \in \{1, \dots, n\}$, $r^{n+1} \in k$, and $L(\boldsymbol{t}) \in k[\boldsymbol{t}]$ is a linear polynomial in $\boldsymbol{t}$.

The OWDVV equations first appeared in [HS12, Th. 2.7], in the context of open Gromov-Witten theory. These equations subsequently appeared in [PST14, BCT18, BCT19]: although not explicitly mentioned in *loc. cit.*, the OWDVV equations follow from the open Topological Recursion Relations equations, see [PST14, Th. 1.5], [BCT18, Th. 4.1], [BCT19, Lem. 3.6], [Bur18, Sec. 4], [BB19, Sec. 1]. The OWDVV equations play a central role in the general theory of relative quantum cohomology developed in [ST19].

Proposition 7.11 (P. Rossi, [BB19])**.** *The following conditions are equivalent:*

(1) (F, F^o) *is a solution of WDVV and OWDVV equations* (7.7)-(7.13),
(2) $\left(\frac{\partial F}{\partial t^\mu}\eta^{\mu 1}, \dots, \frac{\partial F}{\partial t^\mu}\eta^{\mu n}, F^o\right)$ *is a solution of the oriented associativity equations* (2.1)-(2.2) *in the coordinates* $(\boldsymbol{t}, s)$*, and the corresponding formal flat* F*-manifold is homogenous.*

Proof. The claim follows by a direct check. □

We will refer to the formal flat F-structure of point (2) of Proposition 7.11 as the formal flat F-manifold *underlying the pair* (F, F^o). As a corollary of Theorem 6.17, we deduce the following result.

Theorem 7.12. *Let* $F \in \mathbb{C}[\![\boldsymbol{t}]\!]$, $F^o \in \mathbb{C}[\![\boldsymbol{t}, s]\!]$ *be solutions of the WDVV and OWDVV equations. If the underlying formal flat* F*-manifold is semisimple, and it is not doubly resonant, then both* F *and* F^o *are convergent.* □

Remark 7.13. According to a conjecture of B. Dubrovin, the monodromy data of the quantum cohomology of a smooth projective variety encode information about the derived category $\mathcal{D}^b(X)$, see [Dub98, CDG18, Cot20a, Cot20b]. It would be interesting to look for analog relations starting from the monodromy data, as defined here, of flat F-structures given by relative quantum cohomologies of [ST19]. This will be addressed in a future project of the author.

Appendix A. Proof of Theorem 2.17

Let M be an analytic homogeneous semisimple flat F-manifold, and let $E_1, E_2 \in \Gamma(TM)$ be two Euler vector fields.

Lemma A.1. *We have* $[E_1, E_2] = E_1 - E_2$.

Proof. By Proposition 2.13, we can choose canonical coordinates so that $E_1 = \sum_j u^j \partial_j$ and $E_2 = \sum_j (u^j + c^j)\partial_j$. We have

$$[E_1, E_2] = \sum_j (E_1^h \partial_h E_2^j - E_2^h \partial_h E_1^j)\partial_j = -\sum_j c^j \partial_j = E_1 - E_2. \qquad \square$$

Lemma A.2. *We have* $\nabla E_1 = \nabla E_2$.

Proof. Since ∇ is torsionless, we have $\nabla_{E_1} E_2 = \nabla_{E_2} E_1 + [E_1, E_2]$. For arbitrary vector field X, we have

$$\underbrace{\nabla_X \nabla_{E_1} E_2}_{0} = \underbrace{\nabla_X \nabla_{E_2} E_1}_{0} + \nabla_X [E_1, E_2]. \qquad \square$$

Proof of Theorem 2.17. Introduce the operators $\mathcal{U}_1(X) := E_1 \circ X$ and $\mathcal{U}_2(X) := E_2 \circ X$, and

$$\mu(X) := X - \nabla_X E_1 = X - \nabla_X E_2, \quad X \in \Gamma(TM).$$

Choose canonical coordinates so that $E_1 = \sum_j u^j \partial_j$ and $E_2 = \sum_j (u^j + c^j)\partial_j$. Set $\widetilde{\Psi}^i_\alpha = \frac{\partial u^i}{\partial t^\alpha}$. The matrix $\widetilde{\Psi}$ diagonalizes both $\mathcal{U}_1$ and $\mathcal{U}_2$:

$$U_1 = \widetilde{\Psi}\, \mathcal{U}_1\, \widetilde{\Psi}^{-1} = \operatorname{diag}(u^1, \dots, u^n),$$

$$U_2 = \widetilde{\Psi}\, \mathcal{U}_2\, \widetilde{\Psi}^{-1} = \operatorname{diag}(u^1 + c^1, \dots, u^n + c^n).$$

Set $\widetilde{V} := \widetilde{\Psi}\,\mu\,\widetilde{\Psi}^{-1}$, $\widetilde{V}_i := \partial_i\widetilde{\Psi}\cdot\widetilde{\Psi}^{-1}$, and also introduce an off-diagonal matrix $\widetilde{\Gamma} = (\widetilde{\Gamma}^i_j)$ by

$$\widetilde{\Gamma}^i_j := -(\widetilde{V}_i)^i_j, \quad i \neq j.$$

The matrix $\widetilde{\Gamma}$ is a solution of the Darboux-Tsarev equations. Moreover, we have

$$\widetilde{V} = \widetilde{V}' + [\widetilde{\Gamma}, U_1], \quad \text{and also} \quad \widetilde{V} = \widetilde{V}' + [\widetilde{\Gamma}, U_2], \tag{A.1}$$

where V' denotes the diagonal part of V. This follows from Propositions 3.5 and 3.7.

Let $j \neq k$, and take the (j,k) entry of both equations (A.1). We have

$$\widetilde{V}^j_k(\boldsymbol{u}) = \widetilde{\Gamma}^j_k(\boldsymbol{u})(u^k - u^j) = \widetilde{\Gamma}^j_k(\boldsymbol{u})(u^k - u^j) + \widetilde{\Gamma}^j_k(\boldsymbol{u})(c^k - c^j).$$

Hence, we have

$$\widetilde{\Gamma}^j_k(\boldsymbol{u})(c^k - c^j) = 0, \quad \text{for any } j \neq k.$$

This means that

$$c^j \neq c^k \quad \Longrightarrow \quad \widetilde{\Gamma}^j_k = \widetilde{\Gamma}^k_j \equiv 0.$$

Introduce the partition $\coprod_{r=1}^N I_r = \{1, \dots, n\}$ s.t. $c^i = c^j$ only if i, j are in a same block, i.e. $i, j \in I_r$ for some r.

It follows that $\widetilde{\Gamma}^i_j = 0$ unless $i, j \in I_r$. Take $i, j \in I_r$ and $k \notin I_r$. We have

$$\partial_k\widetilde{\Gamma}^i_j = -\widetilde{\Gamma}^i_j\widetilde{\Gamma}^k_j + \widetilde{\Gamma}^i_j\widetilde{\Gamma}^j_k + \widetilde{\Gamma}^i_k\widetilde{\Gamma}^k_j = 0.$$

We have proved that

- the function $\widetilde{\Gamma}^i_j$ is not identically zero only if the indices i, j are in the same block I_r;
- the function $\widetilde{\Gamma}^i_j$ only depends on coordinates u^k with i, j, k in the same block I_r.

It follows that all the matrices $\widetilde{\Gamma}, \widetilde{V}, \widetilde{V}_i, \widetilde{\Psi}$ admits a direct sum decomposition

$$\widetilde{\Gamma} = \bigoplus_{r=1}^N \widetilde{\Gamma}^{(r)}, \quad \widetilde{V} = \bigoplus_{r=1}^N \widetilde{V}^{(r)}, \quad \widetilde{V}_i = \bigoplus_{r=1}^N \widetilde{V}_i^{(r)}, \quad \widetilde{\Psi} = \bigoplus_{r=1}^N \widetilde{\Psi}^{(r)},$$

and each summand $\widetilde{\Gamma}^{(r)}, \widetilde{V}^{(r)}, \widetilde{V}_i^{(r)}, \widetilde{\Psi}^{(r)}$ only depends on canonical coordinates u^k with $k \in I_r$. The original flat F-manifold M locally decomposes into N corresponding pieces:

$$M \overset{\text{loc.}}{\cong} \bigoplus_{j=1}^N M^{(j)}.$$

The flat F-manifold M is irreducible if and only if $N = 1$. This completes the proof. $\square$

Appendix B. Proof of Theorem 7.3

In order to complete the proof of Theorem 7.3, we need to recall some preliminary known results on the (co)homology groups $H_\bullet(\overline{\mathcal{M}}_{0,n}, \mathbb{C})$ and $H^\bullet(\overline{\mathcal{M}}_{0,n}, \mathbb{C})$.

Graphs. In what follows, a *graph* τ is an ordered family $(V_\tau, H_\tau, \partial_\tau, j_\tau)$ where

- V_τ is a finite set of *vertices*,
- H_τ is a finite set of *half-edges*, equipped with a vertex assignment function $\partial_\tau \colon H_\tau \to V_\tau$, and an involution $j_\tau \colon H_\tau \to H_\tau$.

The set E_τ of 2-cycles of j_τ is the set of *edges* of τ. The set S_τ of fixed points of j_τ is the set of *tails* of τ. The datum of j_τ is thus equivalent to the datum of E_τ and S_τ. For each vertex $v \in V_\tau$ define the set $H_\tau(v)$ of *half-edges attached at* v by

$$H_\tau(v) := \partial_\tau^{-1}(v),$$

the set $E_\tau(v)$ of *edges attached to* v by

$$E_\tau(v) := \{\{f_1, f_2\} \in E_\tau \colon \partial_\tau(f_1) = v \text{ or } \partial_\tau(f_2) = v\},$$

and the set $S_\tau(v)$ of *tails attached to* v by

$$S_\tau(v) := \{f \in S_\tau \colon \partial_\tau f = v\}.$$

We clearly have a partition $H_\tau(v) \cong E_\tau(v) \coprod S_\tau(v)$.

An isomorphism $\tau_1 \to \tau_2$ of graphs is the datum of two bijections $V_{\tau_1} \to V_{\tau_2}$ and $H_{\tau_1} \to H_{\tau_2}$ compatible with ∂ and j.

A graph τ is conveniently identified with its associated topological space $\|\tau\|$. Vertices of τ are identified with $|V_\tau|$ distinct points $\{p(v)\}_{v \in V_\tau}$ on the curve $\mathcal{C} := \{(t, t^2, t^3) \colon t \in \mathbb{R}\} \subseteq \mathbb{R}^3$, an edge $\{f_1, f_2\} \in E_\tau$ is identified with the segment[10] joining the points $p(\partial_\tau(f_1))$ and $p(\partial_\tau(f_2))$, tails at v are identified with a star of $|S_\tau(v)|$ small segments originating from $p(v)$, intersecting neither edges nor other tails at other vertices. The space $\|\tau\|$ is the union of all these vertices and segments, equipped with the topology induced from $\mathbb{R}^3$. The graph τ is a *tree* if $\|\tau\|$ is connected and $H_1(\|\tau\|, \mathbb{Z}) = 0$. A tree with n tails, will be called an *n-tree*.

Dual stable graphs. To each point $[C, (\boldsymbol{x})] \in \overline{\mathcal{M}}_{0,n}$, we can attach a *dual stable graph* τ as follows:

(1) the vertices of τ are in 1-1 correspondence with the irreducible components of C,
(2) each node of C is replaced by an edge connecting the vertices corresponding to the two sides of the node,
(3) for each $i = 1, \dots, n$ attach an *tail* with label i to the vertex corresponding to the irreducible component containing x_i.

The resulting graph τ is always a n-tree satisfying the following *stability condition*: each vertex has valence $|H_\tau(v)|$ at least 3. We say that $[C, (\boldsymbol{x})]$ has *combinatorial type* τ.

Vice-versa, for any stable n-tree τ, there exists a locally closed irreducible subscheme $D(\tau) \subseteq \overline{\mathcal{M}}_{0,n}$ parametrizing curves of combinatorial type τ. The stratum $D(\tau)$ uniquely identifies the isomorphism class of τ, and its codimension equals the number of edges $|E_\tau|$.

For example, the n-tree with one vertex corresponds to the open stratum $\mathcal{M}_{0,n}$. The strata of codimension one are labelled by isomorphism classes of one-edge stable n-trees σ. Each such class can be identified with a *stable unordered 2-partition* of the set $\{1, \dots, n\}$. This consists of a set $\sigma = \{S_1, S_2\}$ such that $\{1, \dots, n\} = S_1 \coprod S_2$, and $|S_i| \geqslant 2$ for $i = 1, 2$. See Figure 3.

Given $(i, j, k, l) \in \{1, \dots, n\}^4$ and a stable unordered 2-partition σ, we write $ij\sigma kl$ if i, j and k, l belong two the two different elements of σ.

[10]By Vandermode determinant any four points $p(v_1), \dots, p(v_4)$ are not coplanar, and they are vertices of a tetrahedron. This argument shows that segments representing edges intersect only at the appropriate vertices.

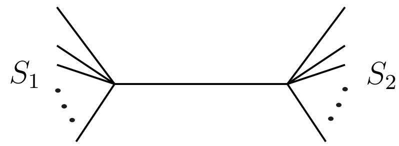

FIGURE 3. Isomorphism classes of one edges n-trees are parametrized by stable unordered 2-partitions $\sigma = \{S_1, S_2\}$ of $\{1, \dots, n\}$.

Keel's Theorem. Introduce commuting indeterminates D_σ, indexed by stable unordered 2-partitions σ of $\{1, \dots, n\}$. Consider the ideal $I_n \subseteq \mathbb{C}[(D_\sigma)_\sigma]$ generated as follows:

(1) for each $(i, j, k, l) \in \{1, \dots, n\}^4$ set

$$R_{ijkl} := \sum_{ij\sigma kl} D_\sigma - \sum_{kj\tau il} D_\tau \in I_n; \tag{B.1}$$

(2) if σ and τ are such that $ij\sigma kl$ and $ij\tau kl$ for some $(i, j, k, l) \in \{1, \dots, n\}^4$, then set

$$D_\sigma D_\tau \in I_n. \tag{B.2}$$

Theorem B.1. *We have an isomorphism of rings*

$$\mathbb{C}[(D_\sigma)_\sigma]/I_n \to H^\bullet(\overline{\mathcal{M}}_{0,n}, \mathbb{C}) \cong A^\bullet(\overline{\mathcal{M}}_{0,n})_\mathbb{C} \tag{B.3}$$

defined by

$$D_\sigma \mapsto \textit{dual of the closed cycle } \overline{D(\sigma)}.$$

In particular, all odd cohomology groups vanish.

Good monomials. Consider a stable n-tree τ. Any edge $e \in E_\tau$ defines a stable unordered 2-partition $\sigma(e)$ of $\{1, \dots, n\}$: by cutting e, we obtain two trees, whose tails (halves of e excluded) form the two parts of $\sigma(e)$.

For each stable n-tree τ, define the monomial $m(\tau) := \prod_{e \in E_\tau} D_{\sigma(e)}$. This is a monomial in $\mathbb{C}[(D_\sigma)_\sigma]$ of degree $|E_\tau|$. Monomials of this form are called *good monomials*.

Under Keel isomorphism (B.3), $m(\tau)$ is the dual of the class $[\overline{D(\tau)}] \in H_\bullet(\overline{\mathcal{M}}_{0,n}, \mathbb{C})$. This follows from the fact that boundary components intersect transversally.

Theorem B.2 ([Man99, Ch.III, §3.6])**.** *Good monomials modulo I_n span $\mathbb{C}[(D_\sigma)_\sigma]/I_n$. Equivalently, the classes $[\overline{D(\tau)}]$ span $H_\bullet(\overline{\mathcal{M}}_{0,n}, \mathbb{C})$.*

Manin's relations in higher codimensions. We need information about linear relations among all good monomials of fixed degree, generalizing Keel's relations (B.1).

Let $n \geqslant 4$, and τ to be an n-tree. Given

(1) $v \in V_\tau$ with valence $|H_\tau(v)| \geqslant 4$,
(2) $(i, j, k, l) \in H_\tau(v)^4$ pairwise distinct half-edges,

set $T := H_\tau(v) \setminus \{i, j, k, l\}$. For any ordered 2-partition $\alpha = (T_1, T_2)$ of T (the case $T_i = \emptyset$ is allowed), we can define two trees $\tau'(\alpha)$ and $\tau''(\alpha)$, with $|E_\tau| + 1$ edges each.

The tree $\tau'(\alpha)$ is defined by replacing the vertex v with a new edge e, at whose vertices we have half-edges $\{i, j\} \cup T_1$ and $\{k, l\} \cup T_2$, respectively.

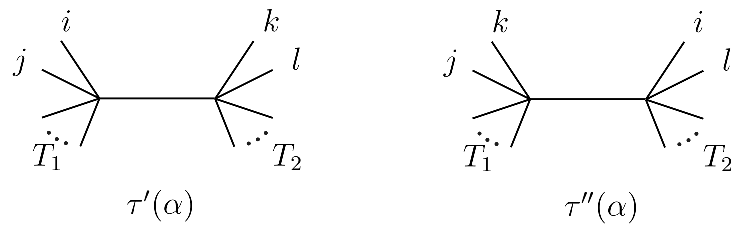

FIGURE 4.

The tree $\tau''(\alpha)$ is defined by replacing the vertex v with a new edge e, at whose vertices we have half-edges $\{k, j\} \cup T_1$ and $\{i, l\} \cup T_2$, respectively. See Figure 4.

For each system (τ, v, i, j, k, l) as above, define the polynomial

$$R(\tau, v, i, j, k, l) := \sum_{\alpha} m(\tau'(\alpha)) - m(\tau''(\alpha)) \in \mathbb{C}[(D_\sigma)_\sigma].$$

Theorem B.3. *We have $R(\tau, v, i, j, k, l) \in I_n$. Moreover, all linear relations modulo I_n between good polynomials of degree $r+1$ are spanned by all the relations $R(\tau, v, i, j, k, l)$ with $|E_\tau| = r$.*

For a proof, see [Man99, Ch. III, Prop. 4.7.1, Th. 4.8].

Proof of Theorem 7.3. We are now able to complete the proof. Given potentials

$$F^\alpha(\boldsymbol{t}) = \sum_{n\geqslant 2} \sum_{\rho_1,\dots,\rho_n=1}^{N} \frac{t^{\rho_1}\dots t^{\rho_n}}{n!} c^\alpha_{\rho_1\dots\rho_n}, \quad c^\alpha_{\rho_1\dots\rho_n} \in \mathbb{C}, \quad \alpha = 1, \dots, N, \tag{B.4}$$

equations (7.4) and (7.5) are

$$c^\alpha_{1\beta} = \delta^\alpha_\beta, \quad c^\alpha_{1\beta\rho_1\dots\rho_n} = 0 \quad \text{for } n > 0, \tag{B.5}$$

$$c^\alpha_{\mu\beta\rho_1\dots\rho_n} c^\mu_{\gamma\delta\tau_1\dots\tau_k} = c^\alpha_{\mu\gamma\rho_1\dots\rho_n} c^\mu_{\beta\delta\tau_1\dots\tau_k}. \tag{B.6}$$

Notice that the lower indices of the coefficients c's can be arbitrarily permuted, i.e. $c^\alpha_{\rho_1\dots\rho_n}$ is uniquely identified by α and the set $\{\rho_1, \dots, \rho_n\}$. We need to prove that the potentials F^α are of the form (7.3) for a unique existing tree-level F-CohFT $(c_{0,n+1})_{n\geqslant 2}$. We first prove the uniqueness, and hence the existence of such an F-CohFT.

Uniqueness. Assume there exists a tree-level F-CohFT $(c_{0,n+1})_{n\geqslant 2}$ such that[11]

$$\int_{\overline{\mathcal{M}}_{0,n+1}} c_{0,n+1}\left(e^\vee_{\rho_1} \otimes \bigotimes_{i=2}^{n+1} e_{\rho_i}\right) = c^{\rho_1}_{\rho_2\dots\rho_n}, \tag{B.7}$$

for any $1 \leqslant \rho_1, \dots, \rho_{n+1} \leqslant N$ and any $n > 1$. We claim that it is then possible to compute the numbers

$$\int_{\overline{D(\tau)}} c_{0,n+1}\left(e^\vee_{\rho_1} \otimes \bigotimes_{i=2}^{n+1} e_{\rho_i}\right), \tag{B.8}$$

[11]For later notational convenience, we slightly changed the labelings: $\alpha \mapsto \rho_1$ and $\rho_i \mapsto \rho_{i+1}$, for $i = 1, \dots, n$.

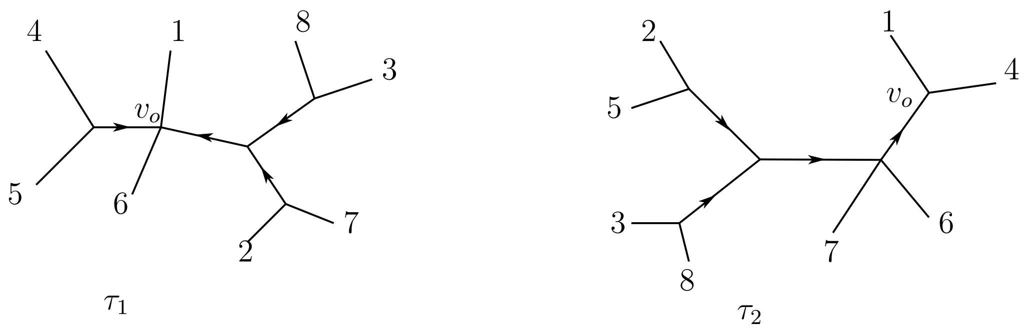

FIGURE 5. Two stable 8-trees τ_1 and τ_2.

for all stable $(n+1)$-trees τ. The homology classes $[\overline{D(\tau)}]$ span $H_\bullet(\overline{\mathcal{M}}_{0,n+1},\mathbb{C})$, by Theorem B.2. Hence the datum of all possible numbers (B.8), for fixed indices $\rho_1,\dots,\rho_{n+1}\in\{1,\dots,N\}$, uniquely defines the cohomology class $c_{0,n+1}\left(e^\vee_{\rho_1}\otimes\bigotimes_{i=2}^{n+1}e_{\rho_i}\right)$ as linear functional on $H_\bullet(\overline{\mathcal{M}}_{0,n+1},\mathbb{C})$. In other words, if we are able to compute all possible numbers (B.8), then any F-CohFT $(c_{0,n+1})_{n\geqslant 2}$ satisfying (B.7) is unique.

Given τ, the number (B.8) can be computed as follows, by iteration of the gluing property (4) of F-CohFT's. Denote by $v_o\in V_\tau$ the vertex of τ such that $1\in S_\tau(v_o)$, i.e. at which the tail 1 is attached. Orient the edges $e\in E_\tau$ in such a way that v_o becomes an "attractor". In this way, at each vertex $v\in V_\tau\setminus\{v_o\}$ there is a single edge with outward orientation, all other edges being with inward orientation. At v_o all edges are inward. Denote by $E^{\rm in}_\tau(v)$ the set of inward edges at v, and by $E^{\rm out}_\tau(v)$ the set of outward edges at v.

Consider now the following monomials attached to τ. Each such monomial is product of coefficients c's in (B.4). In total we have $|V_\tau|$ factors c's, one for each vertex $v\in V_\tau$. An index which is repeated inside the monomial (once upper and once lower) is said to be saturated. The factor corresponding to $v\in V_\tau$ will have a total number of indices (upper and lower) equal to $|H_v|$, i.e. in bijection with half-edges. We will have

(1) a total number of $|E^{\rm out}_\tau(v)|\in\{0,1\}$ upper saturated indices,
(2) and a total number of $|E^{\rm in}_\tau(v)|$ lower saturated indices,
(3) a total number of $|S_\tau(v)|$ lower indices selected from $(\rho_1,\dots,\rho_{n+1})$.

The saturation of indices is dictated by edges: indices labelled by two halves of the same edge are saturated (one is up, the other is down). Non-saturated indices are dictated by the sets $S_\tau(v)$: the factor corresponding to $v\in V_\tau$ will have lower indices ρ_i with $i\in S_\tau(v)$. The vertex v_o is the only vertex whose corresponding factor c has upper index ρ_1.

The number (B.8) equals the sum of all such monomials, over all possible values (ranging in $\{1,\dots,N\}$) of all saturated indices, according to Einstein's summation rule. For example, if $n=7$, and τ_1,τ_2 are the graphs of Figure 5, then

$$\int_{\overline{D(\tau_1)}} c_{0,8}\left(e^\vee_{\rho_1}\otimes\bigotimes_{j=2}^{8}e_{\rho_j}\right)=c^{\rho_1}_{\rho_6\alpha\beta}c^{\alpha}_{\rho_4\rho_5}c^{\beta}_{\gamma\delta}c^{\gamma}_{\rho_3\rho_8}c^{\delta}_{\rho_2\rho_7},$$

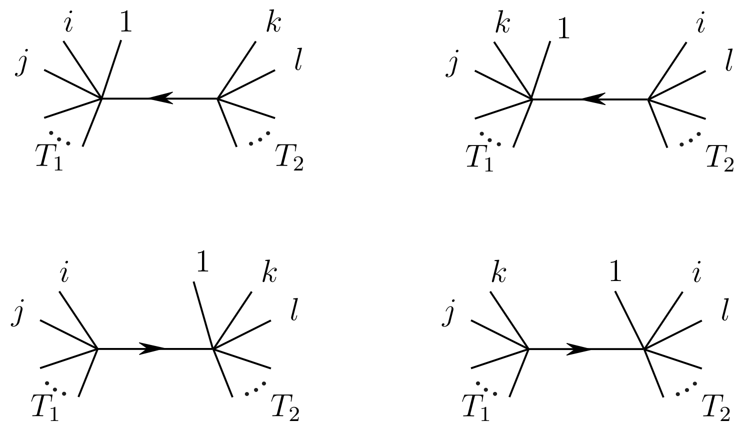

FIGURE 6. On the left (resp. right), we draw the two graphs $\tau'(\alpha)$ (resp. $\tau''(\alpha)$) which contribute to the l.h.s. (resp. r.h.s.) of equation (B.9) in the case $v = v_o$.

$$\int_{\overline{D(\tau_2)}} c_{0,8}\left(e^\vee_{\rho_1}\otimes\bigotimes_{j=2}^{8} e_{\rho_j}\right) = c^{\rho_1}_{\rho_4\alpha}c^{\alpha}_{\rho_7\rho_6\beta}c^{\beta}_{\gamma\delta}c^{\gamma}_{\rho_2\rho_5}c^{\delta}_{\rho_3\rho_8}.$$

This is just an iteration of gluing rule (4) of an F-CohFT, which can be seen as a special instance of computation of (B.8) for a one-edge $(n+1)$-tree. This proves uniqueness.

Existence. In the previous part of the proof, we described an algorithm. For any fixed $\boldsymbol{\rho} = (\rho_1, \dots, \rho_{n+1})$, the algorithm associates with any stable $(n+1)$-tree τ, a complex number $Y_{\boldsymbol{\rho}}(\tau) \in \mathbb{C}$, a polynomial expression in the coefficients c's in (B.4). If we show that all linear relations between the homology classes $[\overline{D(\tau)}]$ are preserved by the map $\tau \mapsto Y_{\boldsymbol{\rho}}(\tau)$, then we would have a well-defined linear functional

$$\tilde{Y}_{\boldsymbol{\rho}} \colon H_\bullet(\overline{\mathcal{M}}_{0,n+1}, \mathbb{C}) \to \mathbb{C}, \qquad \overline{D(\tau)} \mapsto Y_{\boldsymbol{\rho}}(\tau),$$

i.e. a cohomology class. This would lead to a candidate as F-CohFT,

$$c_{0,n+1} \colon V^* \otimes V^{\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{0,n+1}, \mathbb{C}), \qquad \left(e^\vee_{\rho_1}\otimes\bigotimes_{i=2}^{n+1} e_{\rho_i}\right) \mapsto \tilde{Y}_{\boldsymbol{\rho}}.$$

Indeed, the properties (1)-(3) of F-CohFT for $c_{0,n+1}$ would follow from the symmetry of $c^{\rho_1}_{\rho_2\cdots\rho_{n+1}}$ in the lower indices, and equations (B.5), (B.6). Also, the gluing property (4) would follow from the definition of the numbers $Y_{\boldsymbol{\rho}}(\tau)$. This would complete the proof.

By Theorem B.3, it is then sufficient to prove that, for any fixed system (τ, v, i, j, k, l), all the relations $R(\tau, v, i, j, k, l)$ are preserved by $Y_{\boldsymbol{\rho}}$, i.e.

$$\sum_\alpha Y_{\boldsymbol{\rho}}(\tau'(\alpha)) = \sum_\beta Y_{\boldsymbol{\rho}}(\tau''(\beta)). \tag{B.9}$$

The trees τ', τ'' are obtained from τ by replacing the vertex $v \in E_\tau$ with an edge. There are many ways to do this, labelled by 2-partitions of $H_\tau(v)$. They induced a 2-partition of i, j, k, l. We put on the l.h.s. of (B.9) those which split $\{i, j, k, l\}$ in two pieces $\{i, j\} \coprod \{k, l\}$, and we put on the r.h.s. of (B.9) those which split $\{i,j,k,l\}$ in two pieces $\{k,j\}\coprod\{i,l\}$. The remaining partitions do not contribute.

We have in total 2^5 possible cases to consider, according wether v coincides with the marked vertex v_o or not, and wether each of i,j,k,l is an edge or a tail.

Consider for example the case in which $v=v_o$ and each of i,j,k,l is a tail. In the l.h.s. of (B.9) we have the contributions coming from two possible graphs, according to the resulting position of the distinguished tail labeled by 1. Analogously, in the r.h.s. we have the contributions coming from two graphs. See Figure 6.

Equations (B.9) thus reduces to an identity of the form

$$\sum_\alpha\left(c^{\rho_1}_{\rho_i\rho_j\lambda\dots}c^{\lambda}_{\rho_k\rho_l\dots}+c^{\rho_1}_{\rho_k\rho_l\lambda\dots}c^{\lambda}_{\rho_i\rho_j\dots}\right)=\sum_\beta\left(c^{\rho_1}_{\rho_j\rho_k\lambda\dots}c^{\lambda}_{\rho_i\rho_l\dots}+c^{\rho_1}_{\rho_i\rho_l\lambda\dots}c^{\lambda}_{\rho_k\rho_j\dots}\right),$$

where dots stand for all possible partitions of indices, induced by α and β. Both red terms and black terms in this equation cancel, due to equations (B.6).

The reader can check that all other 31 possible cases can be handled similarly. One can always recognize in equation (B.9) a linear combination of identities (B.6), whose left and right sides correspond to the inserted edge in τ' and τ'', respectively.

This completes the proof.

References

[Arn71] V. I. Arnol'd, *On matrices depending on parameters*, Uspekhi Mat. Nauk, 26:2(158) (1971), 101–114; Russian Math. Surveys, 26:2 (1971), 29–43.

[AL13] A. Arsie, P. Lorenzoni, *From the Darboux–Egorov system to bi-flat F-manifolds*, Journal of Geometry and Physics 70 (2013), 98–116.

[AL15] A. Arsie, P. Lorenzoni, *F-manifolds, multi-flat structures and Painlevé transcendents*, , accepted for publication in the Asian Journal of Mathematics, 2015, arXiv:1501.06435v5 , 1–69.

[AL17] A. Arsie, P. Lorenzoni, *Complex reflection groups, logarithmic connections and bi-flat F-manifolds*, Letters in Mathematical Physics 107 (2017), no. 10, 1919–1961.

[ABLR20a] A. Arsie, A. Buryak, P. Lorenzoni, P. Rossi, *Semisimple flat F-manifolds in higher genus*, arXiv:2001.05599 , 1–48.

[ABLR20b] A. Arsie, A. Buryak, P. Lorenzoni, P. Rossi, *Flat F-manifolds, F-CohFTs, and integrable hierarchies*, accepted for publication in Communications in Mathematical Physics. Preprint version: arXiv:2012.05332, 1–34.

[ABLR21] A. Arsie, A. Buryak, P. Lorenzoni, P. Rossi, *Riemannian F-manifolds, bi-flat F-manifolds, and flat pencils of metrics*, arXiv:2104.09380, 1–31.

[BJL79] W. Balser, W.B. Jurkat, and D.A. Lutz, *Birkhoff invariants and Stokes multipliers for meromorphic linear differential equations*, J. Math. Anal. Appl. 71 (1979), 48–94.

[BJL81] W. Balser, W.B. Jurkat , D.A. Lutz, *On the Reduction of Connection Problems for Differential Equations with Irregular Singular Points to ones with only Regular Singularities*, I, SIAM J. Math Anal., Vol 12, No. 5, (1981), 691–721.

[BB19] A. Basalaev, A. Buryak, *Open WDVV Equations and Virasoro Constraints.*, Arnold Math J. 5, 145–186 (2019). https://doi.org/10.1007/s40598-019-00115-w.

[BB11] V. Batyrev and M. Blume, *The functor of toric varieties associated with Weyl chambers and Losev–Manin moduli spaces*, Tohoku Math. J. (2) 63 (2011), no. 4, 581–604.

[BM09] A. Bayer, Yu.I. Manin, *Stability Conditions, Wall-Crossing and Weighted Gromov–Witten Invariants*, Mosc. Math. J. 9 (2009), no. 1, 3–32.

[BM14] J. Bergström, and S. Minabe, *On the cohomology of the Losev–Manin moduli space*. Manuscripta math. 144, 241–252 (2014).

[Bir74] J. Birman, *Braids, links and mapping class groups*, Ann. of Math. Studies, vol. 82, Princeton University Press, Princeton, NJ, 1974.

[Boa01] P. Boalch, *Stokes matrices, Poisson Lie groups and Frobenius manifolds*, Invent. Math. 146, 479–506 (2001).

[Boa02] P. Boalch, *G-bundles, Isomonodromy and Quantum Weyl Groups*, Int. Math. Res. Not. 22 (2002) 1129–1166.

[Bur18] A. Buryak, *Extended r-spin theory and the mirror symmetry for the $A_r - 1$-singularity*, Accepted for publication in the Moscow Mathematical Journal. arXiv:1802.07075v2.

[BCT18] A. Buryak, E. Clader, R.J. Tessler, *Open r-spin theory and the Gelfand–Dickey wave function*, arXiv:1809.02536v1.

[BCT19] A. Buryak, E. Clader, R.J. Tessler, *Closed extended r-spin theory and the Gelfand–Dickey wave function*, Journal of Geometry and Physics 137 (2019), 132–153.

[BR18] A. Buryak, P. Rossi, *Extended r-spin theory in all genera and the discrete KdV hierarchy*, accepted for publication in Advances in Mathematics. Preprint version: arXiv:1806.09825.

[CM20] N. Combe, Yu.I. Manin, *F-manifolds and geometry of information*, Bull. London Math. Soc. 52 (2020) 777–792 doi:10.1112/blms.12411.

[Cot20a] G. Cotti, *Quantum differential equations and helices*, to appear in Geometric Methods in Physics XXXVIII - P. Kielanowski et al. (eds.), Trends in Mathematics, doi.org/10.1007/978-3-030-53305-2_3, 2020, arXiv:1911.11047, 20 pages.

[Cot20b] G. Cotti, *Cyclic stratum of Frobenius manifolds, Borel-Laplace (α, β)-multitransforms, and integral representations of solutions of Quantum Differential Equations*, arXiv:2005.08262, 1–97.

[Cot20c] G. Cotti, *Degenerate Riemann-Hilbert-Birkhoff problems, semisimplicity, and convergence of WDVV-potentials*, arXiv:2011.04498, 1–33.

[CDG19] G. Cotti, B. Dubrovin, and D. Guzzetti, *Isomonodromy deformations at an irregular singularity with coalescing eigenvalues*, Duke Math. J. 168 (2019), no. 6, 967–1108. doi:10.1215/00127094-2018-0059.

[CDG20] G. Cotti, B. Dubrovin, and D. Guzzetti, *Local moduli of semisimple Frobenius coalescent structures*, Symmetry, Integrability and Geometry: Methods and Applications, 16: 040 (2020), doi:10.3842/SIGMA.2020.040.

[CDG18] G. Cotti, B. Dubrovin, D. Guzzetti, *Helix structures in quantum cohomology of Fano varieties*, arXiv:1811.09235, 1–149.

[CG17] G. Cotti and D. Guzzetti, *Analytic Geometry of Semisimple Coalescent Frobenius Structures*, Random Matrices Theory Appl., Vol. 6 (2017), no. 4, 1740004, 36 pp.

[CG18] G. Cotti and D. Guzzetti. *Results on the Extension of Isomonodromy Deformations with a Resonant Irregular Singularity*. Random Matrices Theory Appl., Vol. 07 (2018), no. 4, 1840003, 27 pp.

[DS11] L. David, I.A.B. Strachan, *Dubrovin's duality for F-manifolds with eventual identities*, Advances in Mathematics 226 (2011), no. 5, 4031–4060.

[DKP92] C. De Concini, V.G. Kac, and C. Procesi, *Quantum coadjoint action*, J. Amer. Math. Soc. 5 (1992), no. 1, 151–189.

[DVV91] R. Dijkgraaf, H. Verlinde, E. Verlinde, *Topological strings in $d < 1$*, Nuclear Physics B, Volume 352, Issue 1, 18 March 1991, Pages 59-86.

[Dub92] B. Dubrovin, *Integrable systems in topological field theory*, Nuclear Phys. B 379 (1992), 627–689.

[Dub96] B. Dubrovin, *Geometry of 2D topological field theories*, in Integrable Systems and Quantum Groups (Montecatini Terme, 1993), Lecture Notes in Math., Vol. 1620, Springer, Berlin, 1996, 120–348.

[Dub98] B. Dubrovin, *Geometry and analytic theory of Frobenius manifolds*, Doc. Math. (1998), extra Vol. II, 315– 326, arXiv:math.AG/9807034.

[Dub99] B. Dubrovin, *Painlevé transcendents in two-dimensional topological field theory*, in The Painlevé Property, CRM Ser. Math. Phys., Springer, New York, 1999, 287–412.

[Ful98] W. Fulton, *Intersection theory*, Springer, 1998.

[Gan59] F.R. Gantmacher, *The theory of matrices*, Vols. 1, 2, Chelsea Publishing Co., New York, 1959.

[Get04] E. Getzler, *The jet-space of a Frobenius manifold and higher-genus Gromov–Witten invariants*, Frobenius manifolds, 45–89, Aspects Math., E36, Friedr. Vieweg, Wiesbaden, 2004.

[Guz16] D. Guzzetti, *On Stokes matrices in terms of Connection Coefficients*, Funkcial. Ekvac. 59 (2016), no. 3, 383-433.

[Guz20] D. Guzzetti, *Isomonodromic Laplace Transform with Coalescing Eigenvalues and Confluence of Fuchsian Singularities*, arXiv:2101.03397, 1–55.

[Har20] Y. Haraoka, *Linear Differential Equations in the Complex Domain*, Springer Lecture notes in Mathematics, vol. 2271, 2020, XIII, 396.

[Has03] B. Hasset, *Moduli spaces of weighted pointed stable curves*, Advances in Mathematics, 173, (2003) 316–352.

[Her02] C. Hertling, *Frobenius manifolds and moduli spaces for singularities*, Cambridge Tracts in Mathematics, Vol. 151, Cambridge University Press, Cambridge, 2002.

[HM99] C. Hertling, Yu.I. Manin, *Weak Frobenius manifolds*, Int. Math. Res. Notices 6 (1999)277–286, Preprint math.QA/9810132.

[HS12] A. Horev, J.P. Solomon, *The open Gromov–Witten–Welschinger theory of blowups of the projective plane*, arXiv:1210.4034v1.

[HJ85] R.A. Horn, C.R. Johnson, *Matrix Analysis*, (1985), Cambridge University Press.

[Kal75] S.I. Kaliman, *Holomorphic universal covering of the space of polynomials without multiple roots*, Funkt. Analiz i Priloz. 9, no. 1 (1975), 71 (in Russian). English translation: Funct. Anal. Appl. 9 (1975), no. 1, 67–68.

[Kal77] S.I. Kaliman, *A holomorphic universal covering of the space of polynomials without multiple roots*, Teor. Funkc., Funkcional. Anal. i Prilozen. Vyp. 28, (1977), 25–35 (in Russian).

[Kal93] S.I. Kaliman, *The holomorphic universal covers of the spaces of polynomials without multiple roots*, Selected translations. Selecta Math. Soviet. 12 (1993), no. 4, 395–405.

[KMS20] M. Kato, T. Mano, J. Sekiguchi, *Flat Structure on the Space of Isomonodromic Deformations*, SIGMA 16 (2020), 110, 36 pages.

[Kap93] M.M. Kapranov, *Chow quotients of Grassmannians. I*, Adv. Soviet Math. 16 (1993), 29–110.

[KT08] C. Kassel, V. Turaev, *Braid groups*. Springer, New York, 2008.

[KR90] A.N. Kirillov and N. Reshetikhin, *q-Weyl group and a multiplicative formula for universal R-matrices*, Comm. Math. Phys. 134 (1990), no. 2, 421–431.

[KM94] M. Kontsevich, Yu.I. Manin, *Gromov–Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. 164 (1994), 525–562.

[Lee04] Y.P. Lee, *Quantum K-theory: Foundations*, Duke Math. J. Volume 121, Number 3 (2004), 389-424.

[Lin04] V. Lin, *Configuration spaces of $\mathbb{C}$ and $\mathbb{CP}^1$: some analytic properties*, arXiv:math/0403120, 1–89.

[LRZ15] S.-Q. Liu, Y. Ruan, Y. Zhang. *BCFG Drinfeld-Sokolov hierarchies and FJRW-theory*, Inventiones Mathematicae 201 (2015), no. 2, 711–772.

[Lor14] P. Lorenzoni, *Darboux-Egorov system, bi-flat F-manifolds and Painlevé VI*, IMRN Vol. 2014, No. 12, pp. 3279–3302.

[LPR11] P. Lorenzoni, M. Pedroni, A. Raimondo, *F-manifolds and integrable systems of hydrodynamic type*, Archivum Mathematicum 47 (2011), no. 3, 163–180.

[Los97] A. Losev, *On "Hodge" topological strings at genus zero*, JETP Lett. 65 (1997), no. 5, 386–392.

[Los98] A. Losev, *Hodge strings and elements of K. Saito's theory of primitive form*, in "Topological field theory, primitive forms and related topics", Prog. Math. 160, 305– 335, Boston, Birkhäuser, 1998.

[LM00] A. Losev and Y. Manin, *New moduli spaces of pointed curves and pencils of flat connections*, Michigan Math. J., Volume 48, Issue 1 (2000), 443–472.

[LM04] A. Losev and Yu.I. Manin, *Extended modular operad*, in C. Hertling, M. Marcolli (Eds.) Frobenius Manifolds, Vieweg & Sohn Verlag, Wiesbaden,(2004), pp. 181- 211.

[Lus90] G. Lusztig, *Quantum groups at roots of 1*, Geom. Dedicata 35 (1990), no. 1–3, 89–113.

[Mal83a] B. Malgrange, *Déformations de systèmes différentiels et microdifférentiels.* Séminaire E.N.S. Mathématique et Physique (L. Boutet de Monvel, A. Douady & J.-L. Verdier, eds.), Progress in Math., vol. 37, Birkhäuser, Basel, Boston, 1983, p. 351–379.

[Mal83b] B. Malgrange, *Sur les déformations isomonodromiques, II.* Séminaire E.N.S. Mathématique et Physique (L. Boutet de Monvel, A. Douady & J.-L. Verdier, eds.), Progress in Math., vol. 37, Birkhäuser, Basel, Boston, 1983, p. 427–438.

[Mal86] B. Malgrange, *Deformations of differential systems, II.* J. Ramanujan Math. Soc. **1** (1986), p. 3–15.

[Man99] Yu.I. Manin, *Frobenius manifolds, Quantum Cohomology, and Moduli Spaces*, Amer. Math. Soc., Providence, RI, 1999.

[Man04] Yu.I. Manin, *Moduli stacks* $\overline{L}_{g,S}$, Mosc. Math. J., 4:1 (2004), 181–198.

[Man05] Yu.I. Manin, *F-manifolds with flat structure and Dubrovin's duality* , Adv. Math. 198 (2005), no. 1, 5–26.

[Mer04] S.A. Merkulov S.A., *Operads, deformation theory and F-manifolds*, Frobenius Manifolds. Aspects of Mathematics – Hertling K., Marcolli M. (eds), 2004, vol 36. Vieweg+Teubner Verlag.

[Mer06] S.A. Merkulov, *PROP Profile of Poisson Geometry*, Commun. Math. Phys. 262, 117–135 (2006).

[Pan18] R. Pandharipande, *Cohomological field theory calculations*, Proceedings of the International Congress of Mathematicians (ICM 2018), doi.org/10.1142/9789813272880_0031.

[PST14] R. Pandharipande, J.P. Solomon, R.J. Tessler, *Intersection theory on moduli of disks, open KdV and Virasoro*, arXiv:1409.2191v2.

[Sab07] C. Sabbah, *Isomonodromic deformations and Frobenius manifolds*, Universitext, Springer & EDP Sciences, 2007, (in French: 2002).

[Sab18] C. Sabbah, *Integrable deformations and degenerations of some irregular singularities*, accepted for publication in Publ. RIMS Kyoto Univ., arXiv:1711.08514v3, 35 pages.

[Sab21] C. Sabbah, *A short proof of a theorem of Cotti, Dubrovin and Guzzetti*, preprint, arXiv:2103.16878, 1–11.

[SZ11] S. Shadrin, D. Zvonkine, *A group action on Losev-Manin cohomological field theories*, Annales de l'Institut Fourier, Tome 61, n^o 7 (2011), p. 2719-2743.

[Sib90] Y. Sibuya, *Linear differential equations in the complex domain: problems of analytic continuation*, volume 82 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1990. Translated from the Japanese by the author.

[Soi90] Ya.S. Soibelman, *Algebra of functions on a compact quantum group and its representations*, Leningr. Math. J. 2 (1991), no. 1, 161–178, translated from Algebra Anal. 2 (1990), no.1, 190–212.

[ST19] J.P. Solomon, S.B. Tukachinsky, *Relative quantum cohomology*, arxiv:1906.04795v3.

[Was95] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Pure and Applied Mathematics, Vol. XIV, Interscience Publishers John Wiley & Sons, Inc., New York-LondonSydney, 1965.

[Wit90] E. Witten, *On the structure of the topological phase of two-dimensional gravity*, Nuclear Physics B, Volume 340, Issues 2-3, 20-27 August 1990, 281–332.