

FLAWED GROUPS AND THE TOPOLOGY OF CHARACTER VARIETIES.

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ABSTRACT. A finitely presented group Γ is called *flawed* if $\mathrm{Hom}(\Gamma, G)//G$ deformation retracts onto its subspace $\mathrm{Hom}(\Gamma, K)/K$ for reductive affine algebraic groups G and maximal compact subgroups $K \subset G$. After discussing generalities concerning flawed groups, we show that all finitely generated groups isomorphic to a free product of nilpotent groups are flawed. This unifies and generalizes all previously known classes of flawed groups. We also provide further evidence for the authors' conjecture that RAAGs are flawed. Lastly, we show direct products between finite groups and some flawed group are also flawed. These latter two theorems enlarge the known class of flawed groups.

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1. INTRODUCTION

Let Γ be finitely presented group generated by r elements. Let G be a connected, reductive, affine algebraic group over \mathbb{C} ; a reductive \mathbb{C} -group for short. G acts by conjugation on the set

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of homomorphisms $\mathrm{Hom}(\Gamma, G)$. We may consider $\mathrm{Hom}(\Gamma, G)$ as a subset of G^r by identifying homomorphisms with their evaluations at the generators of Γ . This simultaneously gives $\mathrm{Hom}(\Gamma, G)$ the structure of an affine algebraic set, and an analytic topology over \mathbb{C} (which we assume throughout the paper).

Call a homomorphism in $\mathrm{Hom}(\Gamma, G)$ *polystable* if it has a closed conjugation orbit, and let $\mathrm{Hom}(\Gamma, G)^*$ be the subspace of polystable homomorphisms. The G -character variety of Γ is the quotient space:

$$\mathfrak{X}_\Gamma(G) := \mathrm{Hom}(\Gamma, G)^*/G,$$

with respect to the conjugation action. By [FL14, Theorem 2.1], $\mathfrak{X}_\Gamma(G)$ is homeomorphic (in the analytic topology) to the Geometric Invariant Theory (GIT) quotient $\mathrm{Hom}(\Gamma, G)//G$, and so $\mathfrak{X}_\Gamma(G)$ inherits a natural algebraic structure. It is also homotopic to the non-Hausdorff quotient $\mathrm{Hom}(\Gamma, G)/G$ by [FLR17, Proposition 3.4].

Inspired by [BC01], in [FL09] the authors show that $\mathfrak{X}_\Gamma(G)$ strong deformation retracts to $\mathfrak{X}_\Gamma(K)$ when Γ is a finitely generated free group, and K is a maximal compact in G . Later, in [CFLO16b, CFLO16a] this result was extended to the case when G is *real* reductive. In [FL14] the same theorem is shown to hold when Γ is a finitely generated abelian group. Generalizing the abelian case, in part by combining methods used in [FL09] and [FL14], the result is also shown to be true for all finitely generated nilpotent groups Γ in [Ber15]. On the other hand, whenever Γ is the fundamental group of a closed orientable surface of genus $g \geq 2$ (called a hyperbolic surface group henceforth) such a deformation retraction (indeed, even a homotopy equivalence) is impossible, as follows from [BF11, FGN19].

With these examples in mind, following the suggestions in [FL09], we ask:

Question. What conditions on Γ allow for strong deformation retractions of $\mathfrak{X}_\Gamma(G)$ to $\mathfrak{X}_\Gamma(K)$ for all reductive \mathbb{C} -groups G with maximal compact $K \subset G$?

We call such groups Γ *flawed*.¹ When such a deformation retraction does not exist for any non-abelian G , like hyperbolic surface groups, we call the group Γ *flawless*.

1.1. Summary of Results. In Section 2 we prove a general (necessary and sufficient) criterion for flawedness (Theorem 2.11). Thereafter, we prove our first main theorem (Theorem 4.5). This theorem, although an immediate corollary of the main result in [Ber15] and [FL13, Corollary 4.10] is worth highlighting as it unifies all known cases of flawed groups and brings together the circle of ideas used in [FL09, FL14, Ber15].

Theorem A. *Let Γ be a finite presentable group isomorphic to a free product of nilpotent groups. Then Γ is flawed.*

In particular, $\mathrm{PSL}(2, \mathbb{Z})$ is flawed.

Motivated by work in [PS13], in [FL14] the authors conjectured that all right angled Artin groups (RAAGs) with torsion are flawed.²

As another corollary of Theorem A, we see that free products of cyclic groups (with or without torsion) are flawed, giving further evidence for the conjecture that all RAAGs with torsion are flawed.

¹The name comes from the first two letters of the first named author and the second two letters of the second named author.

²The usual definition of a RAAG does not allow torsion, which is why we say “with torsion” to allow for elements to have finite order. Precisely, these are graph products of cyclic groups.

We then turn our attention to more general RAAGs and prove that in some cases they are flawed. More precisely, our second main theorem (Theorems 5.7 and 5.14) is this:

Theorem B. *Let Γ be a star-shaped RAAG (see Definition 5.2), then Γ is flawed. Moreover, if Γ is a connected RAAG, there is a distinguished irreducible component $\mathfrak{X}_\Gamma^*(G) \subset \mathfrak{X}_\Gamma(G)$ such that, for every reductive \mathbb{C} -group G with maximal compact subgroup K , $\mathfrak{X}_\Gamma^*(G)$ strong deformation retracts to $\mathfrak{X}_\Gamma^*(K) := \mathfrak{X}_\Gamma(K) \cap \mathfrak{X}_\Gamma^*(G)$.*

If there is always such a distinguished irreducible component $\mathfrak{X}_\Gamma^*(G)$ that strong deformation retracts to $\mathfrak{X}_\Gamma^*(K)$ as in Theorem B, then Γ will be called *special flawed* (see Section 3). Also in Section 3, we define the notion of *G-flawed*, allowing for a more nuanced discussion of flawedness (see Theorems 3.6, 3.12 and 3.16).

In the penultimate Section 6, we prove our third main theorem (Theorems 6.3 and 6.2) which gives classes of flawed groups that do not fit into either of the aforementioned classes of flawed groups described in Theorems A and B. Let F_r denote a free group of rank $r \in \mathbb{N}$.

Theorem C. *Let F be a finite group, $\Gamma_1 \cong F \times F_r$ and $\Gamma_2 \cong F \times N$ where N is a finitely generated nilpotent group. Then Γ_1 is flawed, and Γ_2 is special flawed.*

In the final Section 7, we discuss questions and conjectures for further research.

1.2. Philosophy. The guiding philosophy of this paper is that quantifying over the geometric structure defining character varieties (the Lie group G) gives group-theoretic properties about Γ , and such persistent structure should be understood as a general feature in (geometric) group theory. In short, the topology and geometry of the collection of representations of a group Γ is an organizing principle for classifying and distinguishing such groups.

A well-known example of this philosophy is that Kähler groups determine uniform singularity types in character varieties [GM88]. An explicit (recent) application of the concept of flawed groups discussed in this paper is [BHJL21, Proposition 2.4]. This latter result uses the notion of flawed groups to prove that if G -character varieties of orientable surface groups are isomorphic, then the Euler characteristic of the underlying surfaces are equal. Flawedness is used to distinguish open surfaces from closed surfaces (which dimension alone cannot do).

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2. A NUMERICAL CRITERION FOR FLAWED GROUPS

Let Γ be a finitely presentable group. Assume G is a reductive \mathbb{C} -group (a connected reductive affine algebraic group over \mathbb{C}). Then there exists a faithful representation of G and so we may assume $G \subset \mathrm{SL}(n, \mathbb{C})$ for some n . Moreover, we can arrange for a maximal compact subgroup in G , denoted K , such that $K \subset \mathrm{SU}(n)$. As in the introduction, $\mathfrak{X}_\Gamma(G) = \mathrm{Hom}(\Gamma, G) // G$ and $\mathfrak{X}_\Gamma(K) = \mathrm{Hom}(\Gamma, K) / K$ are the corresponding character varieties of Γ . By [FL13, Proposition 4.5], there is a natural inclusion of CW complexes:

$$(2.1) \quad i_G : \mathfrak{X}_\Gamma(K) \rightarrow \mathfrak{X}_\Gamma(G).$$

We adopt the following standard terminology. By a *strong deformation retraction* (SDR) we mean an inclusion $\iota : Y \hookrightarrow X$ of topological spaces such that the identity map id_X is

homotopic (relative to Y) to a retract $r : X \rightarrow Y$ (that is, $r \circ \iota = id_Y$). We also say “ Y is a strong deformation retract of X ,” when such a SDR $\iota : Y \rightarrow X$ exists.

Definition 2.1. We say that Γ is **flawed** if i_G (2.1) is a SDR for all G . We say that Γ is **flawless** if i_G is not a SDR for any non-abelian G .

Let us list a number of examples which are mostly results from previous work.

Example 2.2. All finite groups are flawed. This is an immediate consequence of the following stronger fact in this simple case.

Proposition 2.3. *If Γ is a finite group, then $i_G : \mathfrak{X}_\Gamma(K) \rightarrow \mathfrak{X}_\Gamma(G)$ is a homeomorphism.*

Proof. This is a corollary of many well known facts. For completeness, we provide an elementary proof. Let Γ be finite. Then each $\rho \in \text{Hom}(\Gamma, G)$ has finite image and so is polystable and thus $\mathfrak{X}_\Gamma(G) = \text{Hom}(\Gamma, G)/G$. Also, since the image $\rho(\Gamma)$ is compact it is contained in a maximal compact subgroup of G . Fix a maximal compact K in G . Since all maximal compact subgroups are conjugate, for every $\rho \in \text{Hom}(\Gamma, G)$ there exists $g \in G$ so $g\rho(\Gamma)g^{-1} \subset K$. Thus, i_G is actually surjective, and hence a homeomorphism (since $\mathfrak{X}_\Gamma(K)$ is compact). \square

Example 2.4. Finitely generated free groups [FL09] are flawed. This was the first example of this general phenomenon.

Example 2.5. Finitely generated abelian groups [FL14] are flawed. Later, it was shown that finitely generated nilpotent groups [Ber15] are flawed. Also, virtually nilpotent Kähler groups [BF15] are flawed (this includes finite groups).

Remark 2.6. As shown in the Appendix of [Ber15], there are examples of finitely generated nilpotent Γ and *non-reductive* \mathbb{C} -groups G so that $\mathfrak{X}_\Gamma(G)$ is *not* homotopic to $\mathfrak{X}_\Gamma(K)$ where K is a maximal compact in G (in fact they do not necessarily have the same number of connected components). So working with the class of reductive groups G is necessary in the definition of flawed.

Example 2.7. A hyperbolic surface group Γ (the fundamental group of a closed orientable surface Σ of genus $g \geq 2$) is flawless. This follows from Theorems 3.15 and 3.12 in [FGN19], together with the fact that $\mathfrak{X}_\Gamma(G)$ is homeomorphic to the moduli space of G -Higgs bundles of trivial topological type over a Riemann surface with underlying topological surface Σ (see also Remark 2.8 below).

Remark 2.8. When Γ is the fundamental group of a Kähler manifold Σ , $\mathfrak{X}_\Gamma(G)$ is homeomorphic to $\mathcal{M}_\Sigma(G)$, the moduli space of G -Higgs bundles over Σ with vanishing Chern classes; this is one instance of the so-called non-abelian Hodge correspondence. Likewise, $\mathfrak{X}_\Gamma(K)$ is homeomorphic to the moduli space of flat holomorphic principal G -bundles over Σ ; denote it by $\mathcal{N}_\Sigma(G)$. In general, $\mathcal{M}_\Sigma(G)$ is a *partial* compactification of the cotangent bundle $T^*(\mathcal{N}_\Sigma(G))$, which deformation retracts to $\mathcal{N}_\Sigma(G)$. However, one generally expects the boundary divisors in the partial compactification to change the homotopy type of these moduli spaces. So if this does not happen, one may be justified in saying that Γ has a deficit. This gives another point-of-view about the name “flawed.”

Using Kempf-Ness Theory (see [Nee85, Sch89, KN79]), and Whitehead’s Theorem ([Hat02, Page 346], [Whi49]), allows one to obtain a certain “numerical” criterion for flawedness. The setup is as follows.

Let G be reductive \mathbb{C} -group with maximal compact subgroup K , V an affine variety with a rational action of G , and $V//G := \text{Spec}_{\max}(\mathbb{C}[V]^G)$ the (affine) GIT quotient. By [Kem78, Lemma 1.1], we may assume V is equivariantly embedded as a closed subvariety of a (finite dimensional) \mathbb{C} -vector space \mathbb{V} , via a representation $G \rightarrow \text{GL}(\mathbb{V})$.

Let $\langle \cdot, \cdot \rangle$ be a K -invariant Hermitian form on \mathbb{V} with norm denoted by $\| \cdot \|$. Define, for any $v \in \mathbb{V}$ the mapping $p_v : G \rightarrow \mathbb{R}$ by $g \mapsto \|g \cdot v\|^2$. It is shown in [KN79] that any critical point of p_v is a point where p_v attains its minimum value. Moreover, the orbit $G \cdot v$ is closed and $v \neq 0$ if and only if p_v attains a minimum value.

The *Kempf-Ness* set is the set \mathcal{KN} of critical points $\{v \in V \subset \mathbb{V} \mid (dp_v)_1 = 0\}$, where $\mathbf{1} \in G$ is the identity. Since the Hermitian norm is K -invariant, the Kempf-Ness is stable under the action of K . The following theorem is proved in [Sch89] making reference to [Nee85].

Theorem 2.9 (Schwarz-Neeman). *The composition $\mathcal{KN} \hookrightarrow V \rightarrow V//G$ is proper and induces a homeomorphism $\mathcal{KN}/K \rightarrow V//G$ where $V//G$ has the analytic topology. Moreover, $\mathcal{KN} \hookrightarrow V$ is a K -equivariant strong deformation retraction.*

In our setting, V is $\text{Hom}(\Gamma, G)$ and G acts by conjugation. Choosing r generators for Γ , we first embed:

$$\text{Hom}(\Gamma, G) \subset G^r \subset \mathbb{V},$$

where \mathbb{V} is an affine space where the conjugation action of G extends, and $\text{Hom}(\Gamma, G) \subset \mathbb{V}$ is a closed G -stable subvariety.

Then, following [CFLO16b, Proposition 4.7], the *Kempf-Ness set* of $\text{Hom}(\Gamma, G)$ is:

$$(2.2) \quad \mathcal{KN}_\Gamma := \left\{ (g_1, \dots, g_r) \in \text{Hom}(\Gamma, G) \mid \sum_{i=1}^r [g_i^*, g_i] = 0 \right\},$$

where g^* is the conjugate-transpose of g (defined by a Cartan involution), and $[g, h]$ denotes $gh - hg$ for $g, h \in G \subset \mathbb{V}$. It also follows from this definition that \mathcal{KN}_Γ is K -stable under conjugation, and $\text{Hom}(\Gamma, K) \subset \mathcal{KN}_\Gamma$.

We also need the intermediate spaces: $\mathfrak{Y}_\Gamma(G) := \text{Hom}(\Gamma, G)/K$ (which are also finite CW complexes). From the Schwarz-Neeman Theorem (Theorem 2.9), we conclude:

Theorem 2.10. $\mathfrak{X}_\Gamma(G) \cong \mathcal{KN}_\Gamma/K$ and the natural inclusion $\mathcal{KN}_\Gamma/K \subset \mathfrak{Y}_\Gamma(G)$ is a SDR.

The following result then gives necessary and sufficient conditions for flawedness.

Theorem 2.11. *Let $\eta : \mathfrak{X}_\Gamma(K) \rightarrow \mathfrak{Y}_\Gamma(G)$ be the natural inclusion. Then, the following are equivalent sentences:*

- (1) Γ is flawed.
- (2) η induces isomorphisms $\pi_n(\mathfrak{X}_\Gamma(K)) \cong \pi_n(\mathfrak{Y}_\Gamma(G))$ for all $n \in \mathbb{N}$.
- (3) The inclusion $\mathfrak{X}_\Gamma(K) \subset \mathcal{KN}_\Gamma/K$ induces isomorphisms

$$\pi_n(\mathfrak{X}_\Gamma(K)) \cong \pi_n(\mathcal{KN}_\Gamma/K)$$

for all $n \in \mathbb{N}$.

Proof. This essentially follows from ideas in the proof of [CFLO16b, Theorem 4.10].

We make two technical notes. First, by [FL13, Proposition 4.5] the inclusions i_G, η , and $\mathfrak{X}_\Gamma(K) \subset \mathcal{KN}_\Gamma/K$ can all be taken to be “cellular”, that is, there exist CW structures on these spaces such that the inclusions map onto subcomplexes. Second, when considering

homotopy groups, we always are considering compatible basepoints with respect to the given inclusions (this is relevant since in this generality character varieties may not be connected).

We prove (3) is equivalent to (2). We have the commutative diagram of inclusions:

$$\begin{array}{ccc} \mathfrak{X}_\Gamma(K) & \xrightarrow{\eta} & \mathfrak{Y}_\Gamma(G) \\ \varphi \downarrow & & \parallel \\ \mathcal{KN}_\Gamma/K & \xrightarrow{\iota} & \mathfrak{Y}_\Gamma(G). \end{array}$$

By Theorem 2.10, ι induces isomorphisms on all homotopy groups. Since $\eta = \iota \circ \varphi$, the induced homomorphisms in homotopy are $\eta_* = \iota_* \circ \varphi_*$. Thus, φ induces isomorphisms on all homotopy groups if and only if η does.

Now, we show that (1) and (3) are equivalent. Assume that φ induces an isomorphism for all homotopy groups. Then Whitehead's Theorem ([Hat02, Page 346]) implies that φ is an SDR since φ maps onto a subcomplex. The conclusion that Γ is flawed then follows from the identification $\mathcal{KN}_\Gamma/K \cong \mathfrak{X}_\Gamma(G)$ in Theorem 2.10.

Conversely, if Γ is flawed, then

$$\pi_n(\mathcal{KN}_\Gamma/K) \cong \pi_n(\mathfrak{X}_\Gamma(G)) \cong \pi_n(\mathfrak{X}_\Gamma(K)),$$

for all n . □

Remark 2.12. For a general group Γ (even for a free group) explicitly determining the Kempf–Ness sets appears to be a very difficult task. Criterion (2) above avoids the determination of \mathcal{KN}_Γ , relying only on the topology of the semialgebraic sets $\mathfrak{X}_\Gamma(K)$ and $\mathfrak{Y}_\Gamma(G)$. However, the corresponding homotopy groups are also difficult to compute in general.

There is a stronger condition than being flawed that turns out to be sometimes easier to prove in practice.

Definition 2.13. We will say that Γ is **strongly flawed** if there exists a K -equivariant SDR from $\mathrm{Hom}(\Gamma, G)$ to $\mathrm{Hom}(\Gamma, K)$.

Theorem 2.14. *If Γ is strongly flawed, it is flawed.*

Proof. If Γ is strongly flawed, then there exists a K -equivariant SDR from $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$ and hence $\mathfrak{Y}_\Gamma(G)$ strong deformation retracts onto $\mathfrak{X}_\Gamma(K)$. Therefore, the inclusion $\eta : \mathfrak{X}_\Gamma(K) \rightarrow \mathfrak{Y}_\Gamma(G)$ determines isomorphisms $\pi_n(\mathfrak{X}_\Gamma(K)) \cong \pi_n(\mathfrak{Y}_\Gamma(G))$. Thus, by Theorem 2.11, Γ is flawed. □

Remark 2.15. The above theorem, first used in [FL09], has been used to show just about all known flawed groups are flawed. Indeed, finitely generated free groups [FL09] and finitely generated nilpotent groups [Ber15] (Examples 2.4 and 2.5) are in fact strongly flawed. The proof in [FL14] does not establish the condition of strongly flawed; but from [Ber15], and as a corollary to Theorem 4.5, we now have that finitely generated nilpotent groups are strongly flawed.

Remark 2.16. Although we will not use it here, it is natural to say that Γ is “weakly flawed” if $\pi_n(\mathfrak{X}_\Gamma(G)) \cong \pi_n(\mathfrak{X}_\Gamma(K))$ for all n . The fact that flawed implies weakly flawed is obvious since a SDR between spaces implies those spaces are homotopic and hence weakly homotopic. We know of no examples of weakly flawed groups that are not flawed.

3. BOOTSTRAPPING AND EXTENDING FLAWEDNESS

Consider again the natural inclusion $i_G : \mathfrak{X}_\Gamma(K) \rightarrow \mathfrak{X}_\Gamma(G)$. One often shows that a group Γ is flawed after proving that i_G is a SDR for $G = \mathrm{SL}(n, \mathbb{C})$, for all n , and the general proof for reductive G is usually simply an adaptation of the $\mathrm{SL}(n, \mathbb{C})$ case.

3.1. Bootstrapping Flawedness from the Simple Adjoint Case. In this subsection, and the next one, we prove that to establish flawedness it is sufficient, in some cases, to consider only simple groups G .

Definition 3.1. We will say that Γ is *G-flawed* if $i_G : \mathfrak{X}_\Gamma(K) \rightarrow \mathfrak{X}_\Gamma(G)$ is a strong deformation retraction for a fixed G and any maximal compact $K \subset G$.

We start by considering the case when G is a connected reductive abelian group. It is well known that these are precisely the (affine) algebraic tori which, over \mathbb{C} , are the groups of the form $T \cong (\mathbb{C}^*)^n$ for some $n \in \mathbb{N}$.

Proposition 3.2. *Let Γ be a finitely generated group and T be an algebraic torus. Then Γ is T -flawed.*

Proof. Let $\Gamma_{Ab} := \Gamma/[\Gamma, \Gamma]$ be the abelianization of Γ , with the canonical epimorphism $\pi : \Gamma \rightarrow \Gamma_{Ab}$. Since T is a commutative group, every representation $\rho : \Gamma \rightarrow T$ factors as $\rho = \rho_{Ab} \circ \pi$ for a unique $\rho_{Ab} : \Gamma_{Ab} \rightarrow T$. This shows that the natural inclusion

$$\mathrm{Hom}(\Gamma_{Ab}, T) \subset \mathrm{Hom}(\Gamma, T)$$

is actually an isomorphism of algebraic varieties. Since the conjugation action by T is trivial, we deduce the isomorphism of character varieties:

$$\mathfrak{X}_{\Gamma_{Ab}}(T) \cong \mathfrak{X}_\Gamma(T).$$

Finally, since by [FL14] every finitely generated abelian group Γ_{Ab} is flawed (and naturally $\mathfrak{X}_{\Gamma_{Ab}}(T_K) \cong \mathfrak{X}_\Gamma(T_K)$, for a maximal compact $T_K \subset T$), the same holds for Γ . \square

Lemma 3.3. *Let G and H be reductive \mathbb{C} -groups. If Γ is strongly G -flawed and strongly H -flawed, it is (strongly) $G \times H$ -flawed.*

Proof. Let K_G and K_H be maximal compact subgroups of G and H , respectively. By assumption, there is a K_G -equivariant SDR from $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K_G)$ and a K_H -equivariant SDR from $\mathrm{Hom}(\Gamma, H)$ onto $\mathrm{Hom}(\Gamma, K_H)$. Since products of SDRs are SDRs, the natural identification

$$\mathrm{Hom}(\Gamma, G \times H) = \mathrm{Hom}(\Gamma, G) \times \mathrm{Hom}(\Gamma, H)$$

defines a $(K_G \times K_H)$ -equivariant SDR from $\mathrm{Hom}(\Gamma, G \times H)$ onto $\mathrm{Hom}(\Gamma, K_G \times K_H)$, as wanted (and conjugation by $G \times H$, acts factor-wise). \square

Let G be a reductive \mathbb{C} -group, and F be a finite central subgroup of the maximal compact $K \subset G$. The projection $\pi : G \rightarrow G/F$ induces a morphism of algebraic varieties:

$$\mathrm{Hom}(\Gamma, G) \rightarrow \mathrm{Hom}(\Gamma, G/F).$$

This map is not generally surjective, but it surjects onto a union of path-connected components of $\mathrm{Hom}(\Gamma, G/F)$; see [Gol88, Lemma 2.2] and [Cul86, Thm. 4.1].

Proposition 3.4. *If $\mathrm{Hom}(\Gamma, K/F)$ is a K/F -equivariant SDR of $\mathrm{Hom}(\Gamma, G/F)$, then there exists a K -equivariant SDR from $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$.*

Proof. Denote by $\mathbf{Hom}^*(\Gamma, G/F) \subset \mathbf{Hom}(\Gamma, G/F)$ the union of components so that

$$\pi : \mathbf{Hom}(\Gamma, G) \rightarrow \mathbf{Hom}^*(\Gamma, G/F)$$

is surjective and consider the commutative diagram:

$$(3.1) \quad \begin{array}{ccc} \mathbf{Hom}(\Gamma, K) & \hookrightarrow & \mathbf{Hom}(\Gamma, G) \\ \pi_K \downarrow & & \downarrow \pi \\ \mathbf{Hom}^*(\Gamma, K/F) & \hookrightarrow & \mathbf{Hom}^*(\Gamma, G/F), \end{array}$$

where $\mathbf{Hom}^*(\Gamma, K/F) := \mathbf{Hom}(\Gamma, K/F) \cap \mathbf{Hom}^*(\Gamma, G/F)$, and π_K is the restriction of π to $\mathbf{Hom}(\Gamma, K)$. It is easy to check that, in fact, $\pi_K^{-1}(\mathbf{Hom}^*(\Gamma, K/F)) = \mathbf{Hom}(\Gamma, K)$.

In [LR15, Lemma 3.5] it is shown that the varieties $\mathbf{Hom}(\Gamma, G)$ are locally path-connected, so that π is a covering map (by [Gol88, Lemma 2.2]), in particular a Serre fibration, and thus has the homotopy lifting property for maps from arbitrary CW complexes.

By assumption there is a SDR $\mathbf{Hom}(\Gamma, K/F) \hookrightarrow \mathbf{Hom}(\Gamma, G/F)$. This naturally induces a SDR on path-connected components, so on the bottom of (3.1), we have a strong deformation retraction, which is a homotopy: $H : I \times \mathbf{Hom}^*(\Gamma, G/F) \rightarrow \mathbf{Hom}^*(\Gamma, G/F)$ from the identity to a retract $\mathbf{Hom}^*(\Gamma, G/F) \rightarrow \mathbf{Hom}^*(\Gamma, K/F)$. By pre-composing with $id_I \times \pi$, we get the bottom map in the following diagram:

$$(3.2) \quad \begin{array}{ccc} & & \mathbf{Hom}(\Gamma, G) \\ & \nearrow \tilde{H} & \downarrow \pi \\ I \times \mathbf{Hom}(\Gamma, G) & \longrightarrow & \mathbf{Hom}^*(\Gamma, G/F), \end{array}$$

which can be lifted to the diagonal arrow, yielding a homotopy \tilde{H} which is a (weak) deformation retraction.

It remains to show that this deformation retraction is an SDR, that is, $\tilde{H}(t, \rho) = \rho$ for all t , and all $\rho \in \mathbf{Hom}(\Gamma, K) \subset \mathbf{Hom}(\Gamma, G)$. By commutativity, we have:

$$\tilde{H}(t, \rho) \in \pi^{-1}(H \circ (id_I \times \pi)(t, \rho)) = \pi^{-1}(H(t, \pi(\rho))) = \pi^{-1}(\pi(\rho)).$$

Hence, since the fiber is discrete, $\tilde{H}(0, \rho) = \rho$ for all $\rho \in \mathbf{Hom}(\Gamma, G)$, and continuity, we conclude that $\tilde{H}(t, \rho) = \rho$ for all t and $\rho \in \mathbf{Hom}(\Gamma, K)$, as required.

Finally, since the multiplication action by F commutes with the conjugation action by G (because F is central) and since the conjugation actions of G/F and of G are the same, it is clear that π is K -equivariant and restricts to π_K . By assumption, the bottom homotopy in (3.2) is both K - and K/F -equivariant, so the lifted homotopy is also K -equivariant, by commutativity of the diagram. \square

Remark 3.5. The discreteness of the fiber in the previous argument was essential in showing the SDR lifted to an other SDR. Alternatively, one could use the fact that the inclusion $\mathbf{Hom}(\Gamma, K) \hookrightarrow \mathbf{Hom}(\Gamma, G)$ is cofibration since $\mathbf{Hom}(\Gamma, K)$ is compact (see [Sm66, Sm68]).

Theorem 3.6. *Let G be a reductive group and $\mathbf{Ad}(G)$ be its adjoint group. If Γ is strongly $\mathbf{Ad}(G)$ -flawed, then it is (strongly) G -flawed.*

Proof. Let $DG = [G, G]$ be the derived subgroup of G . Then, DG is semisimple and there is a central algebraic torus T such that the multiplication map $T \times DG \rightarrow G$ is surjective

and has a finite central kernel $F = T \cap DG$

$$F \rightarrow T \times DG \rightarrow G \cong T \times_F DG$$

where F acts by identifying $(t, g) \in T \times DG$ with (tf^{-1}, fg) . This provides another exact sequence of groups:

$$(3.3) \quad F \rightarrow G \xrightarrow{\varphi} T/F \times DG/F$$

where φ is the homomorphism defined by $\varphi([t, g]) := ([t], [g])$ (the notation $[\cdot]$ means the equivalence class under the respective F action), so that $G/F \cong T/F \times DG/F$.

Since T/F is an abelian reductive group, it is again a torus, so by Proposition 3.2, Γ is strongly (T/F) -flawed. Now, since $\text{Ad}(G)$ and DG have the same Lie algebra, we have an isomorphism:

$$\text{Ad}(G) \cong (DG/F)/F_1$$

where F_1 is the finite center of DG/F . By hypothesis, Γ is strongly $\text{Ad}(G)$ -flawed, and so, by Proposition 3.4 (applied to the F_1 quotient) it is strongly (DG/F) -flawed. Now, by Lemma 3.3 Γ is $(T/F \times DG/F)$ -flawed. Finally, again by Proposition 3.4, applied to the isogeny (3.3), Γ is G -flawed. \square

A reductive \mathbb{C} -group is said to be of adjoint type if its center is trivial. Hence, the above theorem allows us to “bootstrap” the property of being flawed from that of being G -flawed for all simple G of adjoint type.

Corollary 3.7. *A finitely presented group Γ is strongly flawed if and only if it is strongly H -flawed for every simple \mathbb{C} -group H of adjoint type.*

Proof. One direction is trivial. Let G be any reductive \mathbb{C} -group. Then $\text{Ad}(G)$ is semisimple, and it is the direct product of its simple factors:

$$\text{Ad}(G) = G_1 \times \cdots \times G_m.$$

Since the center of a product is the product of the centers, every G_j is of adjoint type. So, the result follows from Theorem 3.6 and Proposition 3.4. \square

3.2. Bootstrapping from the Simple and Simply Connected Case. We remark that if we assume that Γ is G -flawed for every simple and simply connected G , this may not be enough to prove that Γ is flawed, since an SDR can be lifted to a covering but not the other way. However, a weaker general result is still possible, which motivates the following definition.

Definition 3.8. We say that Γ is *special flawed* if there exists an irreducible component $\mathfrak{X}_\Gamma^*(G)$ in $\mathfrak{X}_\Gamma(G)$ that SDR onto $\mathfrak{X}_\Gamma^*(K) := \mathfrak{X}_\Gamma^*(G) \cap \mathfrak{X}_\Gamma(K)$.

Example 3.9. One trivial example of special flawedness is when a character variety has an isolated point $[\rho] \in \mathfrak{X}_\Gamma(G)$; in this case, we call ρ *rigid*. This situation occurs, for example, when Γ is a Kazhdan group. It is known that the identity representation of such Γ is rigid, for all G (see [Rap99, Proposition 1] and [Rap13, Theorem 31]), so Kazhdan groups are special flawed.

Since SDRs are continuous, we immediately have:

Proposition 3.10. *If Γ is flawed, it is special flawed.*

In general, for a group Γ and a Lie group G with center Z , $\mathbf{Hom}(\Gamma, Z)$ is a group and acts on $\mathbf{Hom}(\Gamma, G)$ by $(\sigma \cdot \rho)(\gamma) = \sigma(\gamma)\rho(\gamma)$. When Γ is finitely generated, $\mathbf{Hom}(\Gamma, G)$ and $\mathbf{Hom}(\Gamma, Z)$ are naturally subsets of G^r with respect to a set of r generators of Γ . The action of $\mathbf{Hom}(\Gamma, Z)$ is then the restricted action of Z^r on the subspace $\mathbf{Hom}(\Gamma, G) \subset G^r$ given by $(z_1, \dots, z_r) \cdot (g_1, \dots, g_r) = (z_1 g_1, \dots, z_r g_r)$. We can also consider the action of $G \times \mathbf{Hom}(\Gamma, Z)$ on $\mathbf{Hom}(\Gamma, G)$ by combining conjugation and the multiplication action.

Lemma 3.11. *Let G be reductive \mathbb{C} -group with maximal compact subgroup K . Let F be a subgroup of $Z(K)$, the center of K . Let \mathcal{KN}_1 and \mathcal{KN}_2 , respectively, be the Kempf-Ness sets of $\mathbf{Hom}(\Gamma, G)$ with respect to the conjugation action of G , and with respect to the action $G \times \mathbf{Hom}(\Gamma, F)$. Then $\mathcal{KN}_1 = \mathcal{KN}_2$.*

Proof. Without loss of generality, we can assume that K is a subgroup of $\mathrm{SU}(n)$ and that G is a subgroup of $\mathrm{SL}(n, \mathbb{C}) \subset \mathfrak{gl}(n, \mathbb{C}) \cong \mathbb{C}^{n^2}$, and endow $\mathfrak{gl}(n, \mathbb{C})$ with the bilinear form $\langle X, Y \rangle = \mathrm{tr}(X^*Y)$. This form is $K \times Z(K)$ -invariant since

$$(3.4) \quad \mathrm{tr}((zkXk^{-1})^*(zkYk^{-1})) = \mathrm{tr}(kX^*k^{-1}z^{-1}zkYk^{-1}) = \mathrm{tr}(X^*Y),$$

and the complex conjugate transpose g^* of $g \in G$ agrees with the same operation on the vector space $\mathfrak{gl}(n, \mathbb{C})$. If Γ is generated by r elements, we can consider $\mathbf{Hom}(\Gamma, G) \subset G^r \subset \mathbb{C}^{rn^2}$. This allows us to define a $K \times \mathbf{Hom}(\Gamma, Z(K))$ -invariant bilinear form on $\mathbf{Hom}(\Gamma, G)$ by

$$\langle (X_1, \dots, X_r), (Y_1, \dots, Y_r) \rangle = \sum_{j=1}^r \langle X_j, Y_j \rangle.$$

With respect to this form, a representation $\rho \in \mathbf{Hom}(\Gamma, G)$ is minimal with respect to an action of a group H if $\|\rho\| \leq \|h \cdot \rho\|$ for all $h \in H$, and the Kempf-Ness set with respect to the H -action is the collection of minimal vectors.

Clearly the multiplicative action of F^r on G^r commutes with conjugation since $F \subset Z(K)$. Moreover, by (3.4):

$$\|(g, z_1, \dots, z_r) \cdot \rho\| = \|g \cdot \rho\|,$$

for all $(g, z_1, \dots, z_r) \in G \times \mathbf{Hom}(\Gamma, F)$, using the conjugation action of G on the right and the conjugation-translation action of $G \times \mathbf{Hom}(\Gamma, F)$ on the left. Thus, the Kempf-Ness set for both actions are identical. \square

Let us denote by $\mathfrak{X}_\Gamma^0(G)$ the path component of $\mathfrak{X}_\Gamma(G)$ containing the trivial representation $\gamma \mapsto 1 \in G$, for all $\gamma \in \Gamma$. Likewise, denote by $\mathfrak{X}_\Gamma^0(K)$ the path component of $\mathfrak{X}_\Gamma(K)$ containing the trivial representation.

For connected character varieties, we can also bootstrap the property of being flawed from that of G -flawed for all G simple and simply connected.

Theorem 3.12. *If Γ is strongly G -flawed for all simple simply connected \mathbb{C} -groups G , then Γ is (strongly) special flawed.*

Proof. Let G be a connected reductive \mathbb{C} -group. As in the proof of Theorem 3.6, the central isogeny theorem states that $G \cong T \times_{F_1} DG$, with $DG = [G, G]$ semisimple, T a central algebraic torus, and $F_1 := T \cap DG$ a finite central subgroup.

Since DG is semisimple, there exists a collection of simple simply connected \mathbb{C} -groups G_1, \dots, G_n such that $DG \cong (\prod_{i=1}^n G_i)/F_2$, where F_2 is finite central subgroup of $\prod_{i=1}^n G_i$. Putting this together, there is a finite central group $F := F_1 \times F_2 \subset T \times \prod_{i=1}^n G_i$ such that $G \cong (T \times \prod_{i=1}^n G_i)/F$.

Let $\mathrm{Hom}^0(\Gamma, G)$ be the component of $\mathrm{Hom}(\Gamma, G)$ that contains the trivial representation and let $\mathrm{Hom}'(\Gamma, F)$ be the subgroup of $\mathrm{Hom}(\Gamma, F)$ mapping $\mathrm{Hom}^0(\Gamma, T \times \prod_{i=1}^n G_i)$ to itself. Then, by [Sik15, Proposition 5], we have:

$$\begin{aligned} \mathrm{Hom}^0(\Gamma, G) &\cong \mathrm{Hom}^0(\Gamma, T \times \prod_{i=1}^n G_i) / \mathrm{Hom}'(\Gamma, F) \\ &\cong \left(\mathrm{Hom}^0(\Gamma, T) \times \prod_{i=1}^n \mathrm{Hom}^0(\Gamma, G_i) \right) / \mathrm{Hom}'(\Gamma, F). \end{aligned}$$

By Lemma 3.11, the Kempf-Ness sets for the conjugation action of G and that of the mixed action $G \times \mathrm{Hom}'(\Gamma, F)$ are the same.

Let K_i be maximal compact subgroups of G_i , and $T_{\mathbb{R}}$ a maximal compact in T . By assumption, for each index i , there is a K_i -equivariant SDR the Kempf-Ness set of $\mathrm{Hom}^0(\Gamma, G_i)$ onto $\mathrm{Hom}^0(\Gamma, K_i)$. Thus, since products of strong deformation retracts are strong deformation retracts, and using Proposition 3.2 for the torus case, there is a $(T_{\mathbb{R}} \times \prod_{i=1}^n K_i)$ -equivariant SDR from the Kempf-Ness set of

$$\mathrm{Hom}^0(\Gamma, T) \times \prod_{i=1}^n \mathrm{Hom}^0(\Gamma, G_i)$$

to $\mathrm{Hom}^0(\Gamma, T_{\mathbb{R}}) \times \prod_{i=1}^n \mathrm{Hom}^0(\Gamma, K_i)$.

Note that the conjugation action of $(T_{\mathbb{R}} \times \prod_{i=1}^n K_i)$ and that of K are the same since the conjugation action of F is trivial (it is central). Therefore, by Lemma 3.11, there is a K -equivariant SDR from the Kempf-Ness set of $\mathrm{Hom}^0(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$. \square

Remark 3.13. By [LR15, Proposition 4.2] the above theorem can be improved in some cases. In particular, when Γ is “exponent-canceling” (e.g. free groups, free abelian groups, surface groups, RAAGs) the covering $T \times \prod_{i=1}^n G_i \rightarrow G$ with deck group F induces a surjective map $\mathfrak{X}_{\Gamma}(T \times \prod_{i=1}^n G_i) \rightarrow \mathfrak{X}_{\Gamma}(G)$ whose quotient by $\mathrm{Hom}(\Gamma, F)$ induces an isomorphism. In these cases, the conclusion of Theorem 3.12 can be improved from special flawed to flawed.

3.3. Extending Flawedness to the Real Case . We now extend the theory to allow for real groups G and real character varieties. We obtain, in particular, a very general criterion for when a flawed group Γ will be also flawed in the more general real context.

We call a Lie group G *real K -reductive* if the following conditions hold:

- (1) K is a maximal compact subgroup of G ;
- (2) G is a real algebraic subgroup of $\mathbf{G}(\mathbb{R})$; the \mathbb{R} -points of a reductive \mathbb{C} -group \mathbf{G} ;
- (3) G is Zariski dense in \mathbf{G} .

Any real K -reductive Lie group has a faithful representation since every reductive \mathbb{C} -group does. Therefore, we may consider any such group as a subgroup of $\mathrm{SL}(n, \mathbb{C})$ for appropriate n .

Given a self-map α of a set X , we will use the notation $\mathrm{Fix}_{\alpha}(X) := \{x \in X \mid \alpha(x) = x\}$ to denote the fix-point set.

Let \mathfrak{g} denote the Lie algebra of G , and $\mathfrak{g}^{\mathbb{C}}$ the Lie algebra of \mathbf{G} . We will fix a Cartan involution $\theta : \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$ which restricts to a Cartan involution

$$(3.5) \quad \theta : \mathfrak{g} \rightarrow \mathfrak{g}.$$

The map θ is defined as $\theta := \sigma\tau$, where σ, τ are commuting involutions of $\mathfrak{g}^{\mathbb{C}}$, such that $\mathfrak{g} = \text{Fix}_{\sigma}(\mathfrak{g}^{\mathbb{C}})$ and $\mathfrak{k}' := \text{Fix}_{\tau}(\mathfrak{g}^{\mathbb{C}})$ is the compact real form of $\mathfrak{g}^{\mathbb{C}}$. Thus, \mathfrak{k}' is the Lie algebra of a maximal compact subgroup of \mathbf{G} . We call σ a *real structure*.

Using θ we obtain a Cartan decomposition of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where

$$\mathfrak{k} = \mathfrak{g} \cap \mathfrak{k}', \quad \mathfrak{p} = \mathfrak{g} \cap i\mathfrak{k}'$$

and $\theta|_{\mathfrak{k}} = 1$ and $\theta|_{\mathfrak{p}} = -1$. Furthermore, \mathfrak{k} is the Lie algebra of a maximal compact subgroup K of G . Then $K = K' \cap G$, where K' is a maximal compact subgroup of \mathbf{G} , with Lie algebra $\mathfrak{k}' = \mathfrak{k} \oplus i\mathfrak{p}$. Also, \mathfrak{k} and \mathfrak{p} satisfy $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$ is:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$$

with $\theta|_{\mathfrak{k}^{\mathbb{C}}} = 1$ and $\theta|_{\mathfrak{p}^{\mathbb{C}}} = -1$.

The Cartan involution (3.5) lifts to a Lie group involution $\Theta : G \rightarrow G$ whose differential is θ and such that $K = \text{Fix}_{\Theta}(G)$. Θ is also the composition of two commuting involutions T and S , where T corresponds to τ , and S corresponds to σ .³

For a finitely generated group Γ , there is an inclusion $\text{Hom}(\Gamma, K) \hookrightarrow \text{Hom}(\Gamma, G)$, and so there is a natural map $i_G : \mathfrak{X}_{\Gamma}(K) \rightarrow \mathfrak{X}_{\Gamma}(G)$. The map i_G is injective by observing that [FL13, Remark 4.7] applies to this setting by [CFLO16b, Section 3.2].⁴

Definition 3.14. We will say that Γ is *G-flawed* if $\mathfrak{X}_{\Gamma}(G)$ strong deformation retracts onto $i_G(\mathfrak{X}_{\Gamma}(K))$ for a fixed G and any maximal compact $K \subset G$.

In every known case of a flawed group, it was first showed that Γ is *G-flawed* for all reductive \mathbb{C} -groups and then that Γ was *G-flawed* for all *real* reductive G .

Definition 3.15. If Γ is *G-flawed* for all *real* reductive G , we will say that Γ is *real flawed*.

We conjecture that this is a general phenomenon. In particular, we conjecture that if Γ is flawed, then it is real flawed.

The following theorem, giving evidence to the aforementioned conjecture, *does* account for all known cases where flawed groups turn out to be real flawed.

Theorem 3.16. *If Γ is real flawed, it is flawed. Conversely, if Γ is strongly flawed and the SDR commutes with a real structure on G , then Γ is real flawed.*

Proof. Since every reductive \mathbb{C} -group is a real reductive K -group, Γ is flawed if it is real flawed by definition.

Now suppose that Γ is strongly flawed. Let G be a real reductive K -group that is a subgroup of the real points of \mathbf{G} . Let $\Theta = ST$ be the Cartan involution on \mathbf{G} such that $G = \text{Fix}_S(\mathbf{G})$, $K = \text{Fix}_{\Theta}(\mathbf{G})$, and $K' = \text{Fix}_T(\mathbf{G})$. Note that K' is a maximal compact subgroup of \mathbf{G} such that $K = K' \cap G$.

The action of S , T , and Θ on \mathbf{G} extends to an action on $\text{Hom}(\Gamma, \mathbf{G})$ by post-composition of homomorphisms.

³For an appropriate linear representation of \mathbf{G} , we can arrange for S to be complex conjugation and for T to be complex conjugation composed with inverse-transpose.

⁴In further generality, a theorem of Cartan says that any connected real Lie group G is diffeomorphic to $K \times \mathbb{R}^n$ where K is a maximal compact subgroup of G . The argument in [FL13, Remark 4.7] can be adapted to this setting too.

Since Γ is strongly flawed, there exists a K' -equivariant SDR,

$$\Phi : \mathrm{Hom}(\Gamma, \mathbf{G}) \times [0, 1] \rightarrow \mathrm{Hom}(\Gamma, \mathbf{G}),$$

from $\mathrm{Hom}(\Gamma, \mathbf{G})$ onto $\mathrm{Hom}(\Gamma, K')$. By assumption, Φ is S -equivariant. In other words, $S(\Phi(\rho, t)) = \Phi(S(\rho), t)$. Therefore, Φ restricts to the fix-point set of S . Since Φ is K' -equivariant and $K \subset K'$, we conclude that Φ is K -equivariant too. Therefore, since $\mathrm{Fix}_S(\mathrm{Hom}(\Gamma, \mathbf{G})) = \mathrm{Hom}(\Gamma, G)$ and $\mathrm{Fix}_S(K') = K$, Φ restricts to a K -equivariant SDR from $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$. Therefore, Γ is (strongly) real flawed. \square

4. FREE PRODUCTS OF NILPOTENT GROUPS ARE FLAWED

In this section, we prove our first main theorem, as discussed in the introduction (Theorem A). We start with a generalization of Theorem 2.14 to free products, that may have independent interest.

Theorem 4.1. *A free product of strongly flawed groups is strongly flawed (hence flawed). More concretely, let $\Gamma_1, \dots, \Gamma_m$ be finitely generated groups and $\Gamma_1 * \dots * \Gamma_m$ be their free product. If there is a K -equivariant SDR from $\mathrm{Hom}(\Gamma_i, G)$ to $\mathrm{Hom}(\Gamma_i, K)$ for all $1 \leq i \leq m$, then $\Gamma_1 * \dots * \Gamma_m$ is strongly flawed.*

Proof. We prove in [FL13, Corollary 4.10] that the hypothesis of this theorem implies that $\Gamma_1 * \dots * \Gamma_m$ is flawed. Here, we provide a short proof of the stronger result. Given K -equivariant strong deformation retracts from $\mathrm{Hom}(\Gamma_i, G)$ to $\mathrm{Hom}(\Gamma_i, K)$ for each i , we immediately conclude that there is a K^m -equivariant SDR from $\mathrm{Hom}(\Gamma, G) \cong \prod_{i=1}^m \mathrm{Hom}(\Gamma_i, G)$ to $\mathrm{Hom}(\Gamma, K) \cong \prod_{i=1}^m \mathrm{Hom}(\Gamma_i, K)$ since the product of equivariant SDR's is an SDR that is equivariant with respect to the product of acting groups. Since the action by K is contained diagonally in the product action of K^m , we conclude there is a K -equivariant SDR from $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$, as required. \square

Remark 4.2. The proof of Theorem 4.1, can be directly adapted to the real reductive case as done in [CFLO16b, CFLO16a]. In fact, this follows from Theorem 3.16 with the observation that the SDR commutes with a real structure on G .

Corollary 4.3. *The class of strongly flawed groups is closed under free product.*

Now, we come to the main result on free products of nilpotent groups. The lower central series of a group Γ is defined inductively by $\Gamma_1 := \Gamma$, and $\Gamma_{i+1} := [\Gamma, \Gamma_i]$ for $i > 1$. A group Γ is *nilpotent* if the lower central series terminates to the trivial group.

Example 4.4. The Heisenberg group

$$H(\mathbb{Z}) := \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \mid x, y, z \in \mathbb{Z} \right\}$$

admits the presentation:

$$\langle a, b, c \mid [a, c] = [b, c] = 1, [a, b] = c \rangle.$$

Hence, it is a nilpotent group.

Theorem 4.5. *Let Γ be isomorphic to a free product of finitely many nilpotent groups, each of which is finitely generated. Then Γ is strongly real flawed. In particular, if G is a real reductive K -group, then $\mathfrak{X}_\Gamma(G)$ strong deformation retracts onto $\mathfrak{X}_\Gamma(K)$.*

Proof. As with earlier arguments, we assume that K is a subgroup of $U(n)$, G is a subgroup of $GL(n, \mathbb{C})$, g^* is the complex conjugate transpose of $g \in G$, and $\rho \in \text{Hom}(\Gamma, G)$ is represented by an r -tuple of elements in G since Γ is finitely generated.

Let $\Gamma = \Gamma_1 * \cdots * \Gamma_m$ with every Γ_i finitely generated and nilpotent. In Bergeron's paper, [Ber15, Theorem 1] it is shown that there exists a K -equivariant strong deformation retraction of $\text{Hom}(\Gamma_i, G)$ to $\text{Hom}(\Gamma_i, K)$ when Γ is a finitely generated nilpotent group. The main idea of the proof in [Ber15] is this:

- (1) For a real reductive group G acting on a real algebraic variety V , there is also a *Kempf-Ness* set $\mathcal{KN}_V \subset V$ such that there exists a K -equivariant strong deformation retraction from V to \mathcal{KN}_V ; see [RS90]. This idea was first used to study the topology of representation spaces in [FL09] and its generalizations [CFLO16b, CFLO16a].
- (2) The important observation in [Ber15] is that for Γ a nilpotent group one can take $\mathcal{KN}_{\text{Hom}(\Gamma, G)}$ to be $\mathcal{N}_\Gamma :=$

$$\{\rho \in \text{Hom}(\Gamma, G) \mid \rho(\Gamma) \text{ consists of normal elements}\},$$

where $g \in G$ is normal if and only if $gg^* = g^*g$. In [CFLO16b, Proposition 4.7/4.8] the Kempf-Ness set is described generally for $\text{Hom}(\Gamma, G)$ where Γ is any finitely generated group. From this description it is clear that $\mathcal{N}_\Gamma \subset \mathcal{KN}_{\text{Hom}(\Gamma, G)}$ for any finitely generated Γ . However, it takes more work to show these sets are equal if Γ is nilpotent.

- (3) Therefore, one has a K -equivariant strong deformation retraction from $\text{Hom}(\Gamma, G)$ to the set \mathcal{N}_Γ .
- (4) Lastly, following [PS13], as done in [FL14, Section 4] for the case where Γ is abelian, one shows there is a K -equivariant strong deformation retraction from \mathcal{N}_Γ onto $\text{Hom}(\Gamma, K)$ by applying the scaling SDR \mathbb{C}^* to S^1 to the eigenvalues of the components of ρ . Although it is fairly clear that this makes sense when Γ is abelian, it takes more work to show this SDR applies in the nilpotent case.

Thus, Theorem 4.5 now follows from Theorem 4.1 and Remark 4.2. \square

Corollary 4.6. *A finitely generated group isomorphic to a free product of nilpotent groups is real flawed.*

Corollary 4.7. *The modular group $\text{PSL}(2, \mathbb{Z})$ is flawed.*

Proof. $\text{PSL}(2, \mathbb{Z})$ is isomorphic to the free product of \mathbb{Z}_2 and \mathbb{Z}_3 . Since the free product of finitely many finite cyclic groups is an example of a free product of nilpotent groups, we conclude that the modular group is flawed by the above corollary. \square

Remark 4.8. Theorem 4.5 includes finitely generated groups that are free groups, abelian groups, nilpotent groups, and free products of cyclic groups. Hence this theorem unifies all prior known cases of (non-finite) flawed groups. Also, it gives further evidence that RAAGs with torsion are flawed since free products of cyclic groups are RAAGs with torsion but were not before now known to be flawed.

There is a class of groups that includes both RAAGs and free products of nilpotent groups, namely, the *graph product of nilpotent groups*. Let $Q = (V, E)$ be a finite graph and $\{G_v \mid v \in V\}$ a collection of finitely generated nilpotent groups. The graph product of the G_v 's with respect to the graph Q is defined as F/N where F is the free product of all the G_v 's and N is the normal subgroup generated by subgroups of the form $[G_u, G_v]$ whenever there is an edge joining u and v .

Conjecture 4.9. *If Γ is a finitely presentable group that is isomorphic to a graph product of nilpotent groups, then Γ is flawed.*

The next example emphasizes that the above conjecture still does not unify all known cases and conjectures about flawed groups.

Example 4.10. Let F be a finite group that is not nilpotent. Since F is finite it is flawed. If it were a free product $A * B$ with both A, B non-trivial then it would be infinite. Hence, since it is not nilpotent, it is not in the class of groups isomorphic to a graph product of nilpotent groups.

Remark 4.11. The first theorem in [BDD⁺20] states that if a group Γ acts on a simplicial tree T without inversions and with trivial edge stabilizers, and is generated by the vertex stabilizers Γ_v , then there is a subset \mathcal{O} of the vertices of T intersecting each Γ -orbit in one vertex such that G is isomorphic to a free product $*_{v \in \mathcal{O}} G_v$. From this point-of-view, Theorem 4.5 says that finitely presented groups acting in this way on trees with nilpotent stabilizers are flawed.

5. RIGHT ANGLED ARTIN GROUPS

As in the previous section, let G be a reductive \mathbb{C} -group, and let $K \subset G$ be a maximal compact subgroup. We continue with the assumption (without loss of generality) that $K \subset \mathrm{SU}(n)$ and $G \subset \mathrm{SL}(n, \mathbb{C})$.

A *Right Angled Artin Group* (RAAG) is a finitely presented group having only commutator relations: $ab = ba$. Associated to any RAAG Γ is a graph Q whose vertices correspond to generators of Γ and whose edges correspond to relations in Γ . Conversely, given a finite graph Q , there exists a RAAG Γ_Q whose generators correspond to the vertices of Q and whose commutator relations correspond to the edges of Q . A RAAG *with torsion* is a finitely presented group in which all relations are either commutators or torsion relators ($a^n = 1$); these are exactly finite graph products of cyclic groups.

Free products of finitely many cyclic groups (no edges in Q) and finitely generated abelian groups (Q is a complete graph) are both extremal examples of RAAGs (with torsion). Since both these classes of groups are flawed, in [FL14] we conjectured that RAAGs (with torsion) are flawed. Theorem 4.6 gives further evidence of this conjecture. We now summarize a strategy to prove that RAAGs with torsion are flawed (in the outline Γ is a RAAG with torsion). Define the *elliptic* elements in G to be

$$(5.1) \quad G_K := \bigcup_{g \in G} gKg^{-1}$$

(see [Kos73]) and let G_{ss} denote the set of semisimple elements in G .

- (1) A weak deformation retraction between a space X and a subspace A is a continuous family of mappings $F_t : X \rightarrow X$, $t \in [0, 1]$, such that F_0 is the identity on X , $F_1(X) \subset A$, and $F_t(A) \subset A$ for all t . Define

$$\mathrm{Hom}(\Gamma, G_{ss}) := \{\rho \in \mathrm{Hom}(\Gamma, G) \mid \rho(\gamma_1), \dots, \rho(\gamma_r) \in G_{ss}\}.$$

Using ideas from [PS13], [FL14, Lemma 4.15] proves there exists a G -equivariant weak deformation retraction from $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, G_{ss})$ that fixes K during the retraction.

(2) Define

$$\mathrm{Hom}(\Gamma, G_K) := \{\rho \in \mathrm{Hom}(\Gamma, G) \mid \rho(\gamma_1), \dots, \rho(\gamma_r) \in G_K\}.$$

Using the previous step [FL14, Theorem 4.16] proves that when Γ is a RAAG with torsion, $\mathfrak{X}_\Gamma(G)$ strongly deformation retracts onto $\mathrm{Hom}(\Gamma, G_K) // G$ and fixes the subspace $\mathfrak{X}_\Gamma(K)$ for all time.

(3) By Theorem 2.11, it remains to prove that there exists a K -equivariant weak retraction from $\mathrm{Hom}(\Gamma, G_K)$ to $\mathrm{Hom}(\Gamma, K)$ when Γ is a RAAG (with torsion). This step remains an open problem.

The above steps provide proof for:

Theorem 5.1. [FL14, Theorem 4.18] *Let Γ be a RAAG with torsion, G be a reductive \mathbb{C} -group and let K be a maximal compact subgroup of G . If there exists a K -equivariant weak retraction from $\mathrm{Hom}(\Gamma, G_K)$ to $\mathrm{Hom}(\Gamma, K)$ for all G , then Γ is flawed.*

We now illustrate that this last step holds for RAAGs we call *star shaped*.

5.1. Star Shaped RAAGs are Flawed. As described above, given a graph $Q = (V, E)$ with vertex set $V = \{1, \dots, r\}$ and edge set E (consisting of cardinality 2 subsets of V), we define the *RAAG of Q* as the finitely presented group:

$$\Gamma_Q := \langle a_1, \dots, a_r \mid [a_i, a_j] = 1 \text{ iff } \{i, j\} \in E \rangle.$$

Definition 5.2. Let (V, E) be a star graph, that is $V = \{0, 1, \dots, r\}$, and the distinguished vertex $0 \in V$ is connected to every other vertex and there are no further edges (in particular, such a graph is connected). The RAAG associated to this star graph will be called the *star shaped RAAG* of rank $r + 1$. It has the presentation:

$$\Gamma_\star := \langle a_0, \dots, a_r \mid [a_0, a_i] = 1 \ \forall i = 1, \dots, r \rangle.$$

As before, fix K and T , maximal compact and maximal torus, respectively, of the reductive \mathbb{C} -group G . Without loss of generality, consider the Cartan involution on $\mathrm{SL}(n, \mathbb{C})$, given by inverse conjugate transpose, and its restriction to $G \subset \mathrm{SL}(n, \mathbb{C})$ so that $K = \mathrm{Fix}_\Theta G \subset \mathrm{SU}(n)$.

The torus T can be decomposed into its compact and positive parts

$$T = T_K A,$$

where $T_K = T \cap K = \mathrm{Fix}_\Theta T$ is a maximal torus of K , and A is a ‘‘positive real torus’’ (e.g. when $G = \mathrm{GL}(n, \mathbb{C})$, T the diagonal torus, A consists of diagonal matrices with real positive entries, written as exponentials).

Let us recall the KAK decomposition on a reductive group (see [Kna02]). Define $*$: $G \rightarrow G$ by $g^* := \Theta(g)^{-1}$, so that $k^* = k^{-1}$ for $k \in K$.

Proposition 5.3. *Let G be a reductive \mathbb{C} -group. Then, every element $g \in G$ may be written as $g = kah^*$ for some $k, h \in K$ and $a \in A$. Moreover, the restricted exponential $\exp : \mathfrak{a} \rightarrow A$ (where \mathfrak{a} is the Lie algebra of A) is a diffeomorphism and the element $a \in A$ is unique up to conjugation by the Weyl group W .*

We need the following result.

Proposition 5.4. *If $g \in K$ and it commutes with $ke^x h^* \in G$ ($k, h \in K$) then g commutes with $ke^{tx} h^*$, for every $t \in \mathbb{R}$.*

For the proof we use two lemmata.

Lemma 5.5. *If $k \in K$ commutes with $e^x \in A$ then it commutes with e^{tx} for every $t \in \mathbb{R}$.*

Proof. Noting that $k^* = k^{-1}$, and that e^x (a positive definite matrix in a given linear representation) has a unique t power, for $t \in \mathbb{R}$, we see that the commuting hypothesis $ke^xk^* = e^{kxk^*} = e^x$ is equivalent to $ke^{tx}k^* = e^{tx}$ for every $t \in \mathbb{R}$. \square

Lemma 5.6. *If $k \in K$, $e^x \in A$ and $e^{-x}ke^x \in K$, then k commutes with e^x .*

Proof. If $e^{-x}ke^x \in K$, then $1 = (e^{-x}ke^x)(e^{-x}ke^x)^* = e^{-x}ke^{2x}k^*e^{-x}$ which implies $ke^{2x}k = e^{2x}$. So by the previous lemma, this means that k commutes with e^x . \square

We now prove Proposition 5.4.

Proof. We start with $g, h, k \in K$, and assume $gke^xh^* = ke^xh^*g$. On “passing” g from left to right, we write:

$$gke^xh^* = kg_1e^xh^* = ke^xg_2h^* = ke^xh^*g_3,$$

with $g_1 = k^*gk \in K$, $g_2 = e^{-x}g_1e^x$, and $g_3 = hg_2h^* = g$ by the hypothesis that g commutes with ke^xh^* . Then, $g_3 \in K$ which implies $g_2 = h^*g_3h \in K$. By Lemma 5.6, both g_2 and $g_1 = e^xg_2e^{-x} \in K$ implies g_2 commutes with e^x . So, $g_1 = g_2$. Then,

$$gke^{tx}h^* = kg_1e^{tx}h^* = ke^{tx}g_1h^* = ke^{tx}h^*g_3 = ke^{tx}h^*g,$$

as wanted, for all $t \in \mathbb{R}$. \square

Now denote by:

$$\text{Hom}(\Gamma, G_K)$$

the representations of Γ where the image of the generators lie in the elliptic elements $G_K = \cup_{g \in G} gKg^{-1} \subset G$. It is clear that $\text{Hom}(\Gamma, K) \subset \text{Hom}(\Gamma, G_K)$ and using the embedding $\mathfrak{X}_\Gamma(K) \hookrightarrow \mathfrak{X}_\Gamma(G)$ it is not hard to see that $\mathfrak{X}_\Gamma(K)$ maps into $\text{Hom}(\Gamma, G_K) // G$.

Theorem 5.7. *Let Γ_\star be a star shaped RAAG. Then, $\text{Hom}(\Gamma_\star, K) / K$ is a strong deformation retract of $\text{Hom}(\Gamma_\star, G_K) // G$.*

Proof. Write $\rho \in \text{Hom}(\Gamma_\star, G_K)$ as

$$\rho = (A_0, A_1, \dots, A_r) \in G_K^{r+1}, \quad A_i := \rho(a_i).$$

Denote by

$$\text{Hom}_0(\Gamma_\star, G_K) \subset \text{Hom}(\Gamma_\star, G_K)$$

the subset of representations with $A_0 \in K$. Since, in every G -orbit there is a G -conjugate of A_0 which is already in K , we find that there is a natural identification between the orbit spaces:

$$\text{Hom}(\Gamma_\star, G_K) // G \cong \text{Hom}_0(\Gamma_\star, G_K) / K.$$

Now, write the KAK decompositions of the A_i 's:

$$A_i = k_i e^{x_i} h_i^*, \quad i = 1, \dots, r.$$

Then, letting $A_i(t) := k_i e^{tx_i} h_i^*$, there is a homotopy:

$$\begin{aligned} F : [0, 1] \times \text{Hom}_0(\Gamma_\star, G_K) &\rightarrow \text{Hom}_0(\Gamma_\star, G_K) \\ (t, \dots, A_i, \dots) &\mapsto (\dots, A_i(t), \dots), \end{aligned}$$

with $A_0 \in K$ kept fixed for all t . This is a homotopy of $\text{Hom}_0(\Gamma_\star, G_K)$ since Proposition 5.4 shows that the commutation relations $[A_i(t), A_0] = 1$ are satisfied for all $t \in [0, 1]$. Note

that, even though the KAK decomposition is not unique, F is well defined, since for every t and $A_i \in G$, the element $A_i(t)$ is the same regardless of the initial choices (x_i, k_i, h_i) for $A_i = A_i(0)$. It is also easy to see that F is indeed continuous, since any sequence $A_i^{(n)}$ ($n \in \mathbb{N}$) converging to A_i will give a sequence $A_i^{(n)}(t)$ converging to $A_i(t)$ for all t and $i = 1, \dots, r$. Since this homotopy is clearly K -equivariant (even $K \times K$ -equivariant), and $\text{Hom}(\Gamma_\star, K)/K$ is kept fixed, we have determined a SDR from $\text{Hom}_0(\Gamma_\star, G_K)/K$ to $\text{Hom}(\Gamma_\star, K)/K$, as required. \square

Corollary 5.8. *Star shaped RAAGs are flawed.*

Proof. Theorem 5.1 and Theorem 5.7 give the result. \square

Remark 5.9. (1) Star shaped RAAGs are cartesian products of free groups with \mathbb{Z} . Consequently, they are not in the class of groups isomorphic to free products of nilpotent groups. (2) The above proof can be easily generalized to allow a_0 to have torsion.

5.2. Connected RAAGs are Special Flawed. As above, G is a reductive group \mathbb{C} -group, with a Cartan involution $\Theta : G \rightarrow G$, and $K = \text{Fix}_\Theta(G)$ is a maximal compact subgroup. Fix also a maximal torus $T \subset G$ with maximal compact $T_K = \text{Fix}_\Theta(T)$.

Again, without loss of generality, we assume $G \subset \text{SL}(n, \mathbb{C})$, $K = G \cap \text{SU}(n)$, $T = \Delta_n \cap G$, where Δ_n is the diagonal torus of $\text{SL}(n, \mathbb{C})$, and the Cartan involution is given by $\Theta(g) = (g^{-1})^*$, where $*$ is conjugate transpose. We make the following general definitions.

Definition 5.10. An element $g \in G$ is called *normal* if $g^*g = gg^*$. It is called *semisimple* if it is diagonalizable (there is $h \in G$ such that $hgh^{-1} \in T$), and is called *elliptic* if it is semisimple and all its eigenvalues have norm 1 (this agrees with (5.1)). Finally, g is called *unitary* if $g \in K$.

It is clear that unitary elements are both normal and elliptic. The fact that the converse is also valid, will be crucial later on.

Lemma 5.11. *If $g \in G$ is normal with non-repeating eigenvalues and $hg = gh$ for some $h \in G$, then h is normal too.*

Proof. Since $hg = gh$ and g is diagonalizable, there exists $k \in \text{SU}(n)$ so $khk^{-1} = t$ and $kgk^{-1} = d$ where t is upper-triangular and d is diagonal. Thus, $td = dt$. Note that $(dt)_{ij} = d_{ii}t_{ij}$ and $(td)_{ij} = t_{ij}d_{jj}$ and so we conclude that

$$0 = (dt - td)_{ij} = t_{ij}(d_{ii} - d_{jj})$$

which in turn gives that $t_{ij} = 0$ if and only if $i \neq j$ since $d_{ii} = d_{jj}$ if and only if $i = j$. Thus, t is diagonal and hence normal which implies h is normal since it is unitarily diagonalizable. \square

Lemma 5.12. *Let $N, E \subset G$ be the subsets of normal and elliptic elements, respectively. Then $N \cap E = K$.*

Proof. The inclusion $K \subset N \cap E$ is clear. Conversely, let $g \in N \cap E$. Since g is normal, it is well known that g is unitarily diagonalizable, that is there is $k \in K$ such that $kgk^{-1} \in T$. Now, note that the unitary torus T_K consists of the elements of T with eigenvalues of norm 1. Since kgk^{-1} has the same eigenvalues as g , and g is elliptic, $kgk^{-1} \in T_K \subset K$. Therefore, $g \in K$. \square

Corollary 5.13. *Let $g \in G$ be elliptic and $h \in G$ be normal with non-repeating eigenvalues. If $gh = hg$, then $g \in K$.*

Proof. By Lemma 5.11, since $gh = hg$ and $h \in N$ has non-repeating eigenvalues, g is also normal. So, g is normal and elliptic. Thus, by the previous Lemma, $g \in K$. \square

Now, let Γ be a finitely generated group, with a fixed collection of generators $\{\gamma_1, \dots, \gamma_r\}$. The evaluation map gives an embedding:

$$\mathbf{Hom}(\Gamma, G) \hookrightarrow G^r.$$

Let $\mathcal{KN}_\Gamma \subset \mathbf{Hom}(\Gamma, G)$ be the Kempf-Ness set and consider the *normal Kempf-Ness subset*:

$$\mathcal{N}_\Gamma := \{\rho(\gamma_i) \in G \text{ is normal } \forall i = 1, \dots, r\}.$$

By the general Kempf-Ness-Neeman-Schwarz theory described earlier, we have:

$$\mathcal{N}_\Gamma \subset \mathcal{KN}_\Gamma,$$

and the inclusion is K -equivariant. Let us say that the marked group $(\Gamma, \{\gamma_1, \dots, \gamma_r\})$ is *normal* if $\mathcal{N}_\Gamma = \mathcal{KN}_\Gamma$.

Now, let Γ_Q be a RAAG. Such a group has a natural marking coming from a graph $Q = (V, E)$, whose set of vertices is precisely $V = \{\gamma_1, \dots, \gamma_r\}$:

$$\Gamma_Q := \langle \gamma_1, \dots, \gamma_r \mid [\gamma_i, \gamma_j] = 1 \text{ iff } \gamma_i \gamma_j \text{ is an edge of } Q \rangle$$

We say that Γ_Q is connected if Q is connected. When the marking is understood, we just write Γ instead of $(\Gamma, \{\gamma_1, \dots, \gamma_r\})$.

Finally, define the subset of elliptic representations of $(\Gamma, \{\gamma_1, \dots, \gamma_r\})$:

$$\mathbf{Hom}^e(\Gamma, G) := \{\rho(\gamma_i) \in G \text{ is elliptic } \forall i = 1, \dots, r\}.$$

and the subset:

$$\mathbf{Hom}^{de}(\Gamma, G) := \{\rho(\gamma_i) \text{ is elliptic and has distinct eigenvalues } \forall i = 1, \dots, r\},$$

which is G -invariant. Note that $\mathbf{Hom}^e(\Gamma, G)$ is just the set $\mathbf{Hom}(\Gamma, G_K)$ from earlier.

Theorem 5.14. *Connected RAAGs are special flawed.*

Proof. We know that the GIT quotient, can be interpreted as the polystable quotient, and also as the Kempf-Ness (symplectic) quotient:

$$\mathbf{Hom}(\Gamma, G) // G \cong \mathbf{Hom}^{ps}(\Gamma, G) / G \cong \mathcal{KN}_\Gamma / K,$$

and for every G -invariant subset $Y \subset \mathbf{Hom}^{ps}(\Gamma, G)$ we can define the Kempf-Ness set of Y as

$$\mathcal{KN}_\Gamma^Y := \mathcal{KN}_\Gamma \cap Y = \left\{ (A_1, \dots, A_r) \in Y \mid \sum_{i=1}^r [A_i^*, A_i] = 0 \right\},$$

and we get the identification:

$$(5.2) \quad Y // G \cong \mathcal{KN}_\Gamma^Y / K,$$

as topological (Hausdorff) spaces. In particular, \mathcal{KN}_Γ^Y is always a closed subset of Y .

Now, let $\Gamma = \Gamma_Q$ be a connected RAAG. Then, there is a SDR from $\mathbf{Hom}(\Gamma, G) // G$ to $\mathbf{Hom}^e(\Gamma, G) // G$ by Theorem 5.1.

Consider the subset $Y := \mathbf{Hom}^{de}(\Gamma, G)$. Then,

$$Y \subset \bar{Y} \subset \mathbf{Hom}^e(\Gamma, G)$$

which is dense in \bar{Y} , the closure of Y in $\mathbf{Hom}^e(\Gamma, G)$. Note that $\bar{Y} // G$ contains the identity representation. In fact, commutation relations do not impose any restriction on eigenvalues,

so the distinct eigenvalues condition is the complement of equality conditions on representations, which form a Zariski closed set in every irreducible component of $\mathrm{Hom}(\Gamma, G)$ ⁵.

Now, we have from Equation (5.2):

$$\mathcal{KN}_{\Gamma}^{\bar{Y}}/K \cong \bar{Y} // G.$$

The following lemma finishes the proof by showing $\mathcal{KN}_{\Gamma}^{\bar{Y}}/K$ is $\mathfrak{X}_{\Gamma}^*(K)$ for some irreducible component $\mathfrak{X}_{\Gamma}^*(G)$. \square

Lemma 5.15. *If $Y := \mathrm{Hom}^{de}(\Gamma, G) \subset \bar{Y}$, then $\mathcal{KN}_{\Gamma}^Y = \mathrm{Hom}^{de}(\Gamma, K)$, and the closure of \mathcal{KN}_{Γ}^Y equals $\mathcal{KN}_{\Gamma}^{\bar{Y}}$. Moreover, $\mathcal{KN}_{\Gamma}^{\bar{Y}}/K = \mathfrak{X}_{\Gamma}^*(K)$ for some irreducible component $\mathfrak{X}_{\Gamma}^*(G)$.*

Proof. We can G -conjugate the first element, $\rho(\gamma_1)$, to be in K . So, there is an identification:

$$\mathrm{Hom}^e(\Gamma, G) // G = \mathrm{Hom}_1(\Gamma, G) / K,$$

where $\mathrm{Hom}_1(\Gamma, G)$ are the representations such that $\rho(\gamma_1) \in K$. So, we also have:

$$\mathrm{Hom}^{de}(\Gamma, G) // G \cong \mathrm{Hom}_1^{de}(\Gamma, G) / K.$$

Now, let $\rho \in \mathrm{Hom}_1^{de}(\Gamma, G)$, so that $\rho(\gamma_1) \in K$ with distinct eigenvalues. For any other vertex, say γ_2 , that is adjacent to γ_1 (i.e., $\rho(\gamma_1)$ and $\rho(\gamma_2)$ commute) we get that $\rho(\gamma_2)$ is elliptic and commutes with the normal (in fact unitary $\rho(\gamma_1)$). So, by Corollary 5.13, we get that $\rho(\gamma_2) \in K$ (and has also distinct eigenvalues). Since Γ is a connected RAAG, we proceed in the same way, and get that all $\rho(\gamma_i)$ are unitary.

This means that the Kempf-Ness set of $\mathrm{Hom}^{de}(\Gamma, G)$ consists of unitary representations; that is, $\mathcal{KN}_{\Gamma}^Y = \mathrm{Hom}^{de}(\Gamma, K)$.

Finally, since $\mathrm{Hom}^{de}(\Gamma, G) \subset \bar{Y}$ is dense (and G -invariant), then $\mathcal{KN}_{\Gamma}^Y \subset \mathcal{KN}_{\Gamma}^{\bar{Y}}$ is also dense. And since \mathcal{KN}_{Γ}^Y is a closed subset of $\mathrm{Hom}^e(\Gamma, G)$, we get that the closure of \mathcal{KN}_{Γ}^Y equals $\mathcal{KN}_{\Gamma}^{\bar{Y}}$. But any sequence of matrices g_n that verify $[g_n^*, g_n] = 0$ will have a limit that is normal. Since we are in $\mathrm{Hom}^e(\Gamma, G)$ the limit will be unitary, by Corollary 5.13. Hence, $\mathcal{KN}_{\Gamma}^{\bar{Y}}$ consists of unitary representations and includes the identity representation. Thus, $\mathcal{KN}_{\Gamma}^{\bar{Y}}/K$ is $\mathfrak{X}_{\Gamma}^*(K)$ where $\mathfrak{X}_{\Gamma}^*(G)$ is the component of abelian representations (henceforth called the *abelian component*). \square

As in Subsection 5.1, let $Q = (V, E)$ be a graph and Γ_Q be the associated RAAG. Γ_Q will be called a *tree* if Q is a tree, that is, a connected graph without cycles. By removing a *leaf* $v \in V$ (that is, v is a vertex with valence 1) from a tree Q , we get another tree with $n - 1$ vertices. The following lemma was suggested to us by M. Bergeron.

Lemma 5.16. *Let G be a simply-connected reductive \mathbb{C} -group and Γ a tree RAAG. Then $\mathrm{Hom}(\Gamma, G)$ and $\mathfrak{X}_{\Gamma}(G)$ are connected.*

Proof. First, since G is connected, $\mathfrak{X}_{\Gamma}(G)$ is connected if $\mathrm{Hom}(\Gamma, G)$ is connected. So we prove the latter. By [PS13], there exists a G -equivariant weak deformation retraction from $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, G_{ss})$. So it suffices to show that there is a path in $\mathrm{Hom}(\Gamma, G)$ from any point in $\mathrm{Hom}(\Gamma, G_{ss})$ to the identity representation.

Let $\Gamma = \Gamma_Q$ be a RAAG with generators $\{a_1, \dots, a_r\}$, and $Q = (V, E)$ be a tree with vertices labeled by the integers $\{1, \dots, r\}$. There is an edge between i and j , that is $\{i, j\} \in E$, if and only if $[a_i, a_j] = 1$ is a relation in Γ .

⁵Note that some components may not intersect with Y at all, see the next section for examples.

Let $\rho \in \text{Hom}(\Gamma, G_{ss})$, with $A_i := \rho(a_i) \in G_{ss}$. Then, if $\{i, j\} \in E$, A_i and A_j are in each other's centralizers, so that $A_i \in C_G(A_j)$ and $A_j \in C_G(A_i)$. Since G is simply-connected, centralizers of semisimple elements are connected by [SS04, Theorem 3.9]. Hence, $C_G(A_i)$ is connected, and contains the identity of G , for all i .

We can relabel the vertices so that $1 \in V$ is a leaf of Q , $2 \in V$ is a leaf of the tree obtained by removing 1, and so on. Let $\sigma(i) \in V$ be the unique vertex connected to $i \in V$ by an edge after removing vertices $1, \dots, i-1$. Observe that $\sigma(i) > i$ for all $i = 1, \dots, r$.

Let $\gamma_1(t)$, $t \in [0, 1]$, be the path in $C_G(A_{\sigma(1)})$ from A_1 to the identity $I \in G$. From this, we construct a path $\rho_1(t)$ in $\text{Hom}(\Gamma, G)$ from $\rho = (A_1, \dots, A_r) = \rho_1(0)$ to $\rho_1(1) = (I, A_2, \dots, A_r)$. Now, we repeat the process with a path $\gamma_2(t)$ from A_2 to the identity in $C_G(A_{\sigma(2)})$, obtaining a path from (I, A_2, \dots, A_r) to (I, I, A_3, \dots, A_r) . Thus, in a finite number of steps we obtain a path from ρ to $(I, \dots, I) \in \text{Hom}(\Gamma, G)$, since in each step we preserve the relations in Γ . This concludes the proof. \square

Remark 5.17. Let Γ be a tree. From Theorem 5.14 we know that Γ is special flawed. Therefore, Γ is G -flawed whenever $\mathfrak{X}_\Gamma(G)$ is irreducible. From Lemma 5.16, we know that $\mathfrak{X}_\Gamma(G)$ is connected whenever G is simply-connected. It is natural to then think, from Theorem 3.12 and Remark 3.13, that trees are G -flawed whenever DG is simply-connected. However, the examples in the next subsection show character varieties of trees Γ with simply-connected G which are *not* irreducible. So we cannot conclude that trees are G -flawed, even for simply-connected G , from Lemma 5.16 alone.

Example 5.18. In [Ber15] it is shown that if $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}_4$ then the Kempf-Ness set of $\text{Hom}(\Gamma, G)$ is not equal to \mathcal{N}_Γ . This shows that the strategy used above will not work to prove that solvable, virtually abelian, virtually nilpotent, nor supersolvable groups are flawed (although we expect that they are).

5.3. Examples: Simple Non-Abelian Non-Free RAAGs. In this section we consider the simplest non-free, non-abelian RAAG, which is a star shaped RAAG on 3 vertices. For simplicity, we consider G to be a simply connected semisimple \mathbb{C} -group.

Let \angle be a graph with vertices $\{a, b, c\}$ and edges $\{a, b\}$ and $\{b, c\}$. The corresponding RAAG to \angle admits a presentation:

$$\Gamma_\angle := \langle a, b, c \mid [a, b] = 1, [b, c] = 1 \rangle.$$

For any $\rho \in \text{Hom}(\Gamma_\angle, G)$, letting $B := \rho(b)$, we have that $\rho(a) := A$ and $\rho(c) := C$ are in the centralizer of B , denoted $Z_G(B)$. Conversely, for any $A, C \in Z_G(B)$, by letting $\rho(a) := A$, $\rho(b) := B$, and $\rho(c) := C$, we define a G -representation of Γ_\angle . Thus,

$$(5.3) \quad \text{Hom}(\Gamma_\angle, G) = \{(A, B, C) \in G^3 \mid A, C \in Z_G(B)\}.$$

Define a map $\pi_b : \text{Hom}(\Gamma_\angle, G) \rightarrow G$ by $\pi_b(\rho) = \rho(b)$.

Lemma 5.19. π_b is a G -equivariant epimorphism.

Proof. Since G is an algebraic group $\text{Hom}(\Gamma_\angle, G)$ is a subvariety of G^3 . Thus, π_b is the restriction (to an algebraic set) of the projection $G^3 \rightarrow G$, and hence is an algebraic map. Since for every $B \in G$, there exists $\rho : \Gamma_\angle \rightarrow G$ defined by $\rho(a) = I = \rho(c)$ and $\rho(b) = B$, we see that π_b is surjective (and hence an epimorphism). The map π_b is G -equivariant with respect to conjugation since $\pi_b(g\rho g^{-1}) = g\rho(b)g^{-1} = g\pi_b(\rho)g^{-1}$. \square

By G -equivariance we have a map $\mathrm{Hom}(\Gamma_{\angle}, G)/G \rightarrow G/\mathrm{Ad}(G)$, which restricts to a map $\mathrm{Hom}(\Gamma_{\angle}, G)^*/G \rightarrow G/\mathrm{Ad}(G)$. By post-composing with the projection, $G/\mathrm{Ad}(G) \rightarrow G//G$ we obtain a map

$$\pi_{b,G} : \mathfrak{X}_{\angle}(G) \rightarrow G//\mathrm{Ad}(G).$$

Lemma 5.20. $\pi_{b,G}$ is continuous and onto, and defines a family over $G//\mathrm{Ad}(G) \cong \mathbb{C}^r$ where $r = \mathrm{Rank}(G)$.

Proof. Semisimple elements in G have closed conjugation orbits. Since for every semisimple $B \in G$, there exists $\rho : \Gamma_{\angle} \rightarrow G$ defined by $\rho(a) = I = \rho(c)$ and $\rho(b) = B$, we see that $\pi_{b,G}$ is surjective. Since π_b is continuous and G -equivariant, the induced map $\mathrm{Hom}(\Gamma_{\angle}, G)/G \rightarrow G/\mathrm{Ad}(G)$ is continuous with respect to the quotient topology. Consequently, the restriction of domain to $\mathrm{Hom}(\Gamma_{\angle}, G)^*/G \rightarrow G/\mathrm{Ad}(G)$ is continuous. Finally, the quotient map $G/\mathrm{Ad}(G) \rightarrow G//G$ is continuous and so $\pi_{b,G}$ is continuous since composition of continuous maps is continuous. We note that by [Ste65] that $G//\mathrm{Ad}(G) \cong \mathbb{C}^r$ where r is the rank of G (since G is simply connected). \square

Now define $\mathrm{Hom}_B(\Gamma_{\angle}, G) := \pi_b^{-1}(B)$ for $B \in G$. Thus, we have subvarieties $\pi_{b,G}^{-1}([B]) := \mathfrak{X}_{\angle}^B(G)$ which are isomorphic to $\mathrm{Hom}_B(\Gamma_{\angle}, G)//G$. We will see examples where these fibers are isomorphic (up to finite quotients) to free group or free abelian group character varieties. We handle various cases of the fibers through a series of lemmata.

Equation (5.3) and the above definitions give:

Lemma 5.21. For every $B \in G$, $\mathfrak{X}_{\angle}^B(G) \cong (Z_G(B) \times Z_G(B))//G$.

Now we consider some special cases.

Lemma 5.22. If $B \in Z(G)$, then $\mathfrak{X}_{\angle}^B(G)$ is an irreducible component of $\mathfrak{X}_{\angle}(G)$. It is isomorphic to $\mathfrak{X}_{\mathbb{F}_2}(G)$ where \mathbb{F}_2 is a free group of rank 2.

Proof. In the special case $B \in Z(G)$, we get $Z_G(B) = G$, so the previous lemma shows that

$$\mathfrak{X}_{\angle}^B(G) \cong G^2//G \cong \mathrm{Hom}(\mathbb{F}_2, G)//G = \mathfrak{X}_{\mathbb{F}_2}(G),$$

as wanted. \square

The following proposition of Springer and Steinberg will be useful in our analysis. Recall that an element in $g \in G$ is *regular* if the dimension of its centralizer $Z_G(g)$ is minimal among all centralizers. This minimal dimension is just the rank of G ; that is, the dimension of a maximal torus. In fact, for a sufficient general semisimple element g (such elements are contained in a maximal torus T) it is true that $Z_G(g) = T$.

The following proposition follows from [SS04, Pages 206, 221] and [Ste65, Rem. 2.10].

Proposition 5.23. Let G be a simply connected semisimple \mathbb{C} -group. If B is regular, then $Z_G(B)$ is a maximal torus. If $A \in Z_G(B)$ and A, B are semisimple, then A, B are contained in the same maximal torus.

Let $G_{reg} := \{g \in G \mid g \text{ is regular}\}$.

Proposition 5.24. Let G be a simply connected semisimple \mathbb{C} -group. If $B \in G_{reg}$, then $\mathfrak{X}_{\angle}^B(G)$ is a subvariety of $\mathfrak{X}_{\angle}(G)$ isomorphic to $\mathfrak{X}_{\mathbb{Z}^3}(G)$. Moreover, the closure of

$$\bigcup_{B \in G_{reg}} \mathfrak{X}_{\angle}^B(G)$$

is a subvariety of $\mathfrak{X}_{\angle}(G)$ isomorphic to $\mathfrak{X}_{\mathbb{Z}^3}(G)$.

Proof. If B is regular then A, B, C commute by Proposition 5.23, so they define an element in $\mathrm{Hom}(\mathbb{Z}^2, T)$, and conversely. Moreover, if the triple (A, B, C) defines a polystable representation then it consists of semisimple elements, which suffices to prove the first statement. We now obtain a family of subvarieties $\mathrm{Hom}(\mathbb{Z}^2, Z_G(B)) \hookrightarrow \mathrm{Hom}(\Gamma_\angle, G)$ parametrized by G_{reg} . This family consists of triples (A, B, C) of semisimple elements which can be simultaneously conjugated to T and B is regular. The closure, thus consists of such triples that can be simultaneously conjugated to T (without further restriction on B). \square

Corollary 5.25. *If $G = \mathrm{SL}(n, \mathbb{C})$ or $\mathrm{Sp}(n, \mathbb{C})$ and $B \in G_{reg}$ and semisimple, then $\mathfrak{X}_\angle^B(G) \cong \mathfrak{X}_{\mathbb{Z}^2}(G)$ and is therefore irreducible.*

Proof. In these cases $Z_G(B)$ is a maximal torus T . We thus have $\mathfrak{X}_\angle^B(G) \cong \mathfrak{X}_{\mathbb{Z}^2}(Z_G(B)) \cong T^2/W \cong \mathfrak{X}_{\mathbb{Z}^2}(G)$. From [FL14], $\mathrm{Hom}(\mathbb{Z}^2, G)//G$ is irreducible if and only if G is simply connected. \square

Remark 5.26. (1) More generally, the above corollary holds whenever $Z_G(B)$ is a maximal torus, which is true when B is regular and G is simple and simply connected, by [Ste65]. (2) In [Sik14, FL14] it is shown that, for $G = \mathrm{SL}(n, \mathbb{C})$ and $G = \mathrm{Sp}(n, \mathbb{C})$ there is also an isomorphism $\mathfrak{X}_{\mathbb{Z}^r}(G) \cong T^r/W$, for every $r \in \mathbb{N}$ where W is the Weyl group of G , acting diagonally. See also [FS17].

Proposition 5.27. *Let $G = \mathrm{SL}(n, \mathbb{C})$, $B \in G$ and $A, C \in Z_G(B)$. If B is not diagonalizable but is non-derogatory, then the triple (A, B, C) corresponding to a point in $\mathrm{Hom}(\Gamma_\angle, G)$ is not polystable.*

Proof. B is non-derogatory if and only if its minimal polynomial is equal to its characteristic polynomial (this is equivalent to the eigenvalues of the Jordan blocks of B being distinct). By [HJ13, Theorem 3.2.4.2], we conclude that A and C are polynomials in B . Since B is upper-triangularizable it follows that all three are upper-triangularizable and hence (A, B, C) is not polystable since B is non-diagonalizable. \square

We now look at a couple special cases.

5.3.1. $G = \mathrm{SL}(2, \mathbb{C})$. We consider three cases for B :

- (1) $B = \pm I := \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
- (2) B is conjugate to $J_\pm := \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$, or
- (3) B is regular (conjugate to a diagonal matrix with distinct eigenvalues).

In Case (1), by Lemma (5.22) $\mathfrak{X}_\angle^B(G)$ is isomorphic to $\mathfrak{X}_{\mathbb{F}_2}(G)$ which is in this case

$$\mathrm{SL}_2(\mathbb{C})^2 // \mathrm{SL}(2, \mathbb{C}) \cong \mathbb{C}^3$$

(see [Vog89]). $\mathfrak{X}_{\mathbb{F}_2}(\mathrm{SL}(2, \mathbb{C}))$ strong deformation retracts to $\mathrm{SU}(2)^2 / \mathrm{SU}(2)$ which is homeomorphic to a closed 3-ball (see [FL09]).

In Case (2), $\mathfrak{X}_\angle^B(G)$ is empty. Indeed, by Proposition 5.27 all triples (A, B, C) with $B = \pm J$ such that B commutes with both A and C are simultaneously upper-triangular; so the corresponding representation will not be polystable.

In Case (3), since B is regular, by combining (5.24) with Remark (5.26), we get $\mathfrak{X}_\angle^B(G) \cong T^2/W$ where $W \cong \mathbb{Z}_2$ is the Weyl group corresponding to $T \cong \mathbb{C}^*$. This space strong

deformation retracts to

$$\mathrm{Hom}(\mathbb{Z}^2, \mathrm{SU}(2))/\mathrm{SU}(2) \cong (S^1)^2/\mathbb{Z}_2 \cong S^2;$$

see [CFLO16b, Page 20]. By Proposition (5.24) we have:

$$\overline{\bigcup_{B \in G_{reg}} \mathfrak{X}_{\mathbb{Z}}^B(G)} \cong \mathfrak{X}_{\mathbb{Z}^3}(G) \cong T^3/W.$$

This space strong deformation retracts to

$$\mathrm{Hom}(\mathbb{Z}^3, \mathrm{SU}(2))/\mathrm{SU}(2) \cong (S^1)^3/\mathbb{Z}_2,$$

which is a 3-dimensional orbifold; see [CFLO16b, Page 23] for details and a visualization.

Summarizing, for $G = \mathrm{SL}(2, \mathbb{C})$, $\mathfrak{X}_{\mathbb{Z}}(G)$ consists of exactly three irreducible (Zariski closed) components, two corresponding to Case (1) and isomorphic to a free group character variety $\mathfrak{X}_{F_2}(G) \cong \mathbb{C}^3$, and the third, corresponding to Case (3), is isomorphic to a free abelian group character variety $\mathfrak{X}_{\mathbb{Z}^3}(G) \cong T^3/W$ (also 3 dimensional). See Figure 5.1 for a schematic drawing of this example. We can see explicitly that each component strong deformation retracts to the corresponding $\mathrm{SU}(2)$ -character variety and the union of those retracts is exactly $\mathfrak{X}_{\mathbb{Z}}(\mathrm{SU}(2))$.

Note that the singular locus of each component $\mathfrak{X}_{\mathbb{Z}}^{\pm I}(G)$, where A and C commute (and $B = \pm I$), is exactly where they intersect the other irreducible component (see [FL12]).

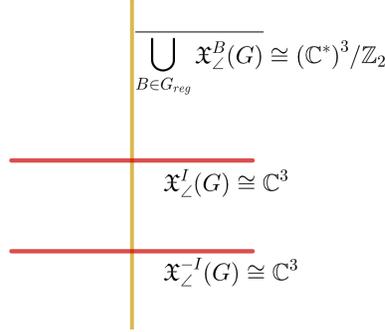


FIGURE 5.1. $\mathfrak{X}_{\mathbb{Z}}(\mathrm{SL}(2, \mathbb{C}))$

5.3.2. $G = \mathrm{SL}(3, \mathbb{C})$. Now there are four cases to consider for the “node” b :

- (1) B is central, so that $B \in \{I, \omega I, \omega^2 I\}$ where I is the identity matrix, and ω is a third root of unity. By Lemma (5.22), $\mathfrak{X}_{\mathbb{Z}}^B(G) \cong \mathfrak{X}_{F_2}(G)$, where F_2 is a free group of rank 2, a branched double cover of \mathbb{C}^8 (see [Law06, Law07]) homotopic to S^8 (see [FL09]).
- (2) B is upper triangularizable but not diagonalizable (and hence has repeated eigenvalues). Writing B in Jordan form, one concludes that the commutation of B with A and C implies that either A, C are both upper-triangular too (Proposition 5.27), or that they are both simultaneously upper-triangularizable with B (an easy calculation). Either way the triple (A, B, C) does not correspond to a polystable G -representation. Hence $\mathfrak{X}_{\mathbb{Z}}^B(G)$ is empty (by definition).
- (3) B is regular (diagonalizable with distinct eigenvalues), which implies

$$\mathfrak{X}_{\mathbb{Z}}^B(G) \cong \mathfrak{X}_{\mathbb{Z} \times \mathbb{Z}}(G) \cong T^2/W,$$

by Proposition (5.24) where $T \cong (\mathbb{C}^*)^2$ is a maximal torus and the Weyl group is the symmetric group on 3 letters $W \cong S_3$. So, this is a 4-dimensional orbifold.

- (4) B is diagonalizable with two repeated eigenvalues, but not central. This last case is new (compared to the case $\mathrm{SL}(2, \mathbb{C})$) and so we detail it below.

Denote $B_\lambda := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$ where $\lambda^3 \neq 1$. If $A, C \in Z_G(B)$ and (A, B, C) is polystable,

then A, C are both of the form $\begin{pmatrix} X & \mathbf{0}^\dagger \\ \mathbf{0} & \mathrm{Det}(X)^{-1} \end{pmatrix}$ where $X \in \mathrm{GL}(2, \mathbb{C})$, \dagger signifies transpose, and $\mathbf{0} = (0, 0)$. Hence, $\mathfrak{X}_Z^B(G) \cong \mathfrak{X}_{F_2}(\mathrm{GL}(2, \mathbb{C}))$, which is a variety of dimension 5.

By varying B over regular elements (a 2 dimensional variety) in Case (3), and over B_λ in Case (4), and taking their closures as before, we obtain 2 irreducible components:

$$\overline{\bigcup_{B \in G_{reg}} \mathfrak{X}_Z^B(G)}, \quad \text{and} \quad \overline{\bigcup_{B_\lambda} \mathfrak{X}_Z^{B_\lambda}(G)},$$

both of dimension 6, and both fibered by $\mathrm{SL}(3, \mathbb{C})$ -character varieties of either \mathbb{Z}^2 or F_2 . Figure 5.2 is a schematic drawing of this example. Note that the singular locus of each $\mathfrak{X}_Z^{B_\lambda}(G)$ (the blue curve) intersects the abelian locus T^3/S_3 (yellow curve), so the diagram is slightly misleading in that there is a continuum of such intersections.

As with $\mathrm{SL}(2, \mathbb{C})$, each of these cases corresponds to a character variety known to strong deformation retract as required, and the SDR restricts to the intersections, providing a SDR on the whole space.

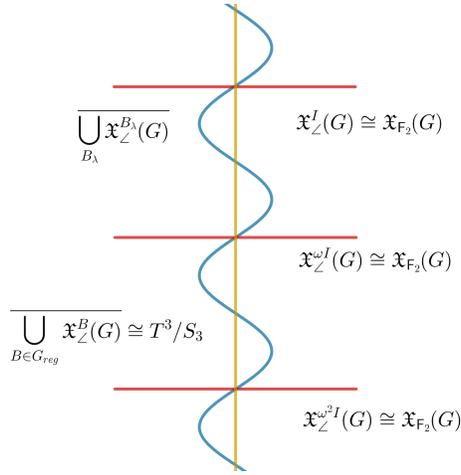


FIGURE 5.2. $\mathfrak{X}_Z(\mathrm{SL}(3, \mathbb{C}))$

Notice that the abelian component in both the above examples intersects every other component. It would be interesting, given Theorem 5.14, to determine if this is a general phenomenon for connected RAAGs.

6. DIRECT PRODUCTS WITH FINITE GROUPS

In this last section, we consider some classes of flawed groups which have a finite group F as a direct (cartesian) factor. We consider products $\Gamma = F \times G$ where G is either free or

nilpotent. This extends the class of flawed groups further as finite-by-nilpotent and finite-by-free groups are not, in general, free products of nilpotent groups as the next example shows.

Example 6.1. Let A and B be groups of order greater than 3 (possibly infinite). By [Ser03, Proposition 4] the free product $A * B$ contains a free group of rank at least 2. Let Γ be a direct product of an finitely generated abelian groups A with a finite non-nilpotent group F . Then Γ is finite by nilpotent but not nilpotent. If Γ was a free product of two or more non-trivial nilpotent groups, then it would contain a free group of rank at least 2 by the above reference; which it does not.

6.1. Finite by Free Groups are Flawed. Let Γ be isomorphic to a product $F \times F_r$, where F is a finite group and F_r is a free group of rank r .

Theorem 6.2. *Let F be a finite group. Then $\Gamma \cong F_r \times F$ is flawed.*

Proof. Consider a presentation of Γ of the form:

$$\Gamma = \langle a_1, \dots, a_r, b_1 \cdots b_s \mid [a_i, b_j] = 1, R_k(b) \rangle,$$

where $R_k(b)$ denote relations only among the b_j 's (the a_i 's are free generators of the F_r factor).

Let $\rho : \Gamma \rightarrow G$ be a polystable representation to a reductive \mathbb{C} -group, and denote $A_i := \rho(a_i)$ and $B_j := \rho(b_j)$. First we note that all B_j 's are elliptic. Indeed, being of finite order implies that all their eigenvalues are complex numbers of unit norm. Moreover, since the group they generate is compact, and all such maximal compact are conjugated, there is a $g \in G$ that simultaneously conjugates all B_j into our fixed maximal compact $K \subset G$.

The proof now proceeds as in the proof of Theorem 5.7: Denote by

$$\mathrm{Hom}_0(\Gamma, G) \subset \mathrm{Hom}^{ps}(\Gamma, G)$$

the subset of polystable representations with all $B_i \in K$. Since, in every G -orbit there is $g \in G$ so so that $g \cdot B_j \in K$, there is a natural identification between the orbit spaces:

$$\mathrm{Hom}^{ps}(\Gamma, G) // G \cong \mathrm{Hom}_0(\Gamma, G) / K.$$

Now, as before, write the KAK decompositions: $A_i = k_i e^{x_i} h_i^*$, $i = 1, \dots, r$, let $A_i(t) := k_i e^{tx_i} h_i^*$, and define a homotopy:

$$\begin{aligned} H : [0, 1] \times \mathrm{Hom}_0(\Gamma, G) &\rightarrow \mathrm{Hom}_0(\Gamma, G) \\ (t, A_i, B_j) &\mapsto (A_i(t), B_j) \end{aligned}$$

(so $B_j \in K$ are kept fixed for all t). As before, H is well defined and continuous since for every t and $A_i \in G$, the element $A_i(t)$ is the same regardless of the initial choices (x_i, k_i, h_i) for $A_i = A_i(0)$. Moreover, Proposition 5.4 shows that the commutation relations $[A_i(t), B_j] = 1$ are satisfied for all $t \in [0, 1]$. Since this homotopy is K -equivariant, and $\mathrm{Hom}(\Gamma, K) / K$ is kept fixed, we have determined a SDR from $\mathrm{Hom}_0(\Gamma, G) / K$ to $\mathrm{Hom}(\Gamma, K) / K$, and so a SDR $\mathrm{Hom}(\Gamma, K) / K \hookrightarrow \mathrm{Hom}^{ps}(\Gamma, G) // G$ as wanted. \square

6.2. Finite by Nilpotent Groups are Special Flawed.

Theorem 6.3. *If Γ is a direct product of a nilpotent group with a finite group, then Γ is special flawed.*

Proof. Let $\Gamma = N \times F$, where N is nilpotent and F is finite, and let a_1, \dots, a_r be generators of N and b_1, \dots, b_s be generators of F . For a representation $\rho : \Gamma \rightarrow G$ write $A_i = \rho(a_i)$ and $B_i = \rho(b_i)$, so that:

$$[A_i, A_j] = [A_i, B_j] = 1$$

Now, let $\rho : \Gamma \rightarrow G$ be a polystable representation. Then the $A_i = \rho(a_i)$ generate a reductive nilpotent group. Hence, by Bergeron's result [Ber15], there is some $g \in G$, such that $g \cdot A = (gA_1g^{-1}, \dots, gA_rg^{-1})$ is an r -tuple of normal matrices.

Again, let $Y := \text{Hom}^{de}(\Gamma, G) \subset \overline{Y} \subset \text{Hom}(\Gamma, G)$ be the subset of representations which have all A_i with non-repeating eigenvalues. Then, the Kempf-Ness set of Y is given by:

$$\mathcal{KN}_\Gamma^Y = \{(A_1, \dots, A_r, B_1, \dots, B_s) \in Y \mid A_j \text{ are normal and } B_j \text{ are unitary}\}.$$

Indeed, $\mathcal{KN}_\Gamma^Y \subset \mathcal{KN}_\Gamma$ consists of matrices with minimum Frobenius norm in each G -orbit, normal matrices have the minimum norm in their respective G -orbits and, since B_j 's are elliptic, by Corollary 5.13, they are in fact unitary.

Now, by the same argument as before, we get that $\mathcal{KN}_\Gamma^{\overline{Y}}$ is the closure of \mathcal{KN}_Γ^Y , which means that:

$$\mathcal{KN}_\Gamma^{\overline{Y}} = \{(A_1, \dots, A_r, B_1, \dots, B_s) \in \overline{Y} \mid A_j \text{ are normal and } B_j \text{ are unitary}\}.$$

Finally, using the eigenvalue scaling map [FL14], we can show that there is a K -equivariant SDR from $\mathcal{KN}_\Gamma^{\overline{Y}}$ to $\overline{\text{Hom}^{de}(\Gamma, K)}$, inducing an SDR from $\mathcal{KN}_\Gamma^{\overline{Y}}/K$ to $\overline{\text{Hom}^{de}(\Gamma, K)}/K$ as wanted. \square

7. QUESTIONS AND CONJECTURES

In this final section we list some questions and conjectures for further research:

- (1) From the work in [Muñ09, MMM15] it is clear that torus knots are $\text{SL}(2, \mathbb{C})$ -flawed. We will call a group a *generalized torus knot group* if it can be presented as

$$\langle a_1, \dots, a_r \mid a_i^{n_i} = a_j^{n_j} \text{ for all } i, j \rangle$$

for positive integers n_1, \dots, n_r . When $r = 2$ these are torus knot groups. We conjecture that generalized torus knot groups are flawed.

- (2) We know that closed hyperbolic surface groups are flawless and that was shown using Higgs bundle theory. Given the work in [FGN19], we conjecture that all non-abelian Kähler groups are flawless.
- (3) A group is *supersolvable* if it admits an invariant normal series where all the factors are cyclic groups. Finitely generated supersolvable groups generalize finitely generated nilpotent groups. Given results in [BS53, SS04], we conjecture that finitely generated supersolvable groups are flawed.
- (4) We have shown free products of nilpotent groups are flawed (Theorem A). Are free products of nilpotent groups amalgamated over abelian groups also flawed?
- (5) Thinking more like a geometric group theorist, if two groups are commensurable, and one is flawed, is the other also flawed?

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