THE $*$-MARKOV EQUATION FOR LAURENT POLYNOMIALS

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On the Occasion of the 70th Birthday of Sabir Gusein-Zade

Abstract. We consider the $*$-Markov equation for the symmetric Laurent polynomials in three variables with integer coefficients, which appears as an equivariant analog of the classical Markov equation for integers. We study how the properties of the Markov equation and its solutions are reflected in the properties of the $*$-Markov equation and its solutions.

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1. Introduction

1.1. Markov equation.

1.1.1. The *Markov equation* is the Diophantine equation

$$a^2+b^2+c^2-abc=0 \quad a,b,c\in\mathbb{Z}, \tag{1.1}$$

with initial solution $(3,3,3)$. If a triple (a,b,c) is a solution, then a permutation of the triple is a solution. One may also change the sign of two of the three coordinates of a solution. The braid group $\mathcal{B}_3$ acts on the set of solutions,

$$\begin{aligned}\tau_1 &: (a,b,c)\mapsto(-a,c,b-ac),\\ \tau_2 &: (a,b,c)\mapsto(b,a-bc,-c).\end{aligned} \tag{1.2}$$

The classical Markov theorem says that all nonzero solutions of the Markov equation can be obtained from the initial solution $(3,3,3)$ by these operations, see [Mar79, Mar80]. This group of symmetries of the equation is called the Markov group. A solution with positive coordinates is called a Markov triple, the positive coordinates are called the Markov numbers.

The Markov equation is traditionally studied in the form

$$a^2+b^2+c^2-3abc=0, \quad a,b,c\in\mathbb{Z}. \tag{1.3}$$

Equations (1.1) and (1.3) are equivalent. A triple $(a,b,c)\in\mathbb{Z}^3$ is a solution of (1.3) if and only if $(3a,3b,3c)$ is a solution of (1.1).

The equation was introduced by A.A. Markov in [Mar79, Mar80] in the analysis of minimal values of indefinite binary quadratic forms and was studied in hundreds of papers, see for example the book [Ai13] and references therein.

1.2. Motivation from exceptional collections and Stokes matrices.

1.2.1. Our motivation came from the works by A. Rudakov [Ru89] on full exceptional collections in derived categories and by B. Dubrovin [Du96, Du98, Du99] on Frobenius manifolds and isomonodromic deformations.

In 1989 A. Rudakov studied the full exceptional collections in the derived category $\mathcal{D}^b(\mathbb{P}^2)$ of the projective plane $\mathbb{P}^2$. These are triples (E_1, E_2, E_3) of objects in $\mathcal{D}^b(\mathbb{P}^2)$ generating $\mathcal{D}^b(\mathbb{P}^2)$ and such that the matrix of Euler characteristics $(\chi(E_i^* \otimes E_j))$ has the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$. Rudakov observed that the triple (a, b, c) is a solution of the Markov equation. The braid group $\mathcal{B}_3$ naturally acts on the set of full exceptional collections and the induced action on the set of matrices of Euler characteristics coincides with the action of the braid group on the set of solutions of the Markov equation.

In the 90's Dubrovin considered the isomonodromic deformations of the quantum differential equation of the projective plane $\mathbb{P}^2$, see [Du99]. This is a system of three first order linear ordinary differential equations with two singular points: one regular point at the origin and one irregular at the infinity. Dubrovin observed that the Stokes matrix S of a Stokes basis of the space of solutions at the infinity is of the form $S = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, where (a, b, c) is a solution of the Markov equation. The braid group $\mathcal{B}_3$ naturally acts on the set of Stokes bases, and the induced action on the set of Stokes matrices coincides with the action of the braid group on the set of solutions of the Markov equation.

These two observations allowed to Dubrovin to conclude that the Stokes bases of the isomonodromic deformations of the quantum differential equation of $\mathbb{P}^2$ correspond to the full exceptional collections in the derived category $\mathcal{D}^b(\mathbb{P}^2)$ and more generally to conjecture that the derived category of an algebraic variety is responsible for the monodromy data of its quantum differential equation, see [Du98, CDG18].

1.2.2. Recently in [TV19] V. Tarasov and the second author considered the equivariant quantum differential equation for $\mathbb{P}^2$ with respect to the torus $T = (\mathbb{C}^\times)^3$ action on $\mathbb{P}^2$. That equivariant quantum differential equation is a system of three first order linear ordinary differential equations depending on three equivariant parameters $\boldsymbol{z} = (z_1, z_2, z_3)$. The system has two singular points: one regular point at the origin and one irregular at the infinity. It turns out that the Stokes matrix S of a Stokes basis of the space of solutions at the infinity is of the form $S(\boldsymbol{z}) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, where a, b, c are symmetric Laurent polynomials in the equivariant parameters $\boldsymbol{z}$ with integer coefficients. In [CV20] we observed that the Stokes bases correspond to T-full exceptional collections in the equivariant derived category $\mathcal{D}^b_T(\mathbb{P}^2)$. If (E_1, E_2, E_3) is a T-full exceptional collection, then the equivariant Euler characteristic $\chi_T(E_i^* \otimes E_j)$ is an element of the representation ring of the torus, that is, a Laurent polynomial in the equivariant parameters with integer coefficients. It turns out that if a T-full exceptional collection (E_1, E_2, E_3) corresponds to a Stokes basis, then the corresponding Stokes matrix equals the matrix $(\chi_T(E_i^*\otimes E_j))$ of equivariant Euler characteristics. Moreover the three symmetric Laurent polynomials (a,b,c), appearing in this construction, satisfy the equation

$$aa^*+bb^*+cc^*-ab^*c=3-\frac{z_1^3+z_2^3+z_3^3}{z_1z_2z_3}, \tag{1.4}$$

where $f^*(z_1,z_2,z_3):=f(1/z_1,1/z_2,1/z_3)$ for any Laurent polynomial, see [CV20, Formula (3.20)]. If $z_1=z_2=z_3=1$, then the right-hand side of (1.4) equals zero, the equivariant Euler characteristics $\chi_T(E_i^*\otimes E_j)$ become the non-equivariant Euler characteristics $\chi(E_i^*\otimes E_j)$, and the triple of symmetric Laurent polynomials (a,b,c) evaluated at $z_1=z_2=z_3=1$ becomes a solution of the Markov equation (1.1).

We call equation (1.4) for symmetric Laurent polynomials with integer coefficients the ∗*-Markov equation*.

The transition from the Markov equation to the ∗-Markov equation provides us with a deformation of the Markov numbers by replacing Markov numbers with symmetric Laurent polynomials, which recover the numbers after the evaluation at $z_1=z_2=z_3=1$.

The goal of this paper is to observe how the properties of the Markov equation and its solutions are reflected in the properties of the ∗-Markov equation and its solutions.

1.2.3. There are interesting instances of the transition from the Diophantine Markov equation (1.1) to an equation of the form

$$a(t)^2+b(t)^2+c(t)^2-a(t)b(t)c(t)=R(t),$$

where $a(t),b(t),c(t)$ are unknown functions in some variables t and $R(t)$ is a given function. Such deformations among other subjects are related to hyperbolic geometry and cluster algebras, see the fundamental papers [CP07, FG07].

The difference between deformations of this type and the ∗-Markov equation is that equation (1.4) includes the ∗-operation dictated by the equivariant K-theoretic setting. It is an interesting problem to find relations of the ∗-Markov equation to hyperbolic geometry and cluster algebras.

1.3. ∗-Markov equation and ∗-Markov group.

1.3.1. It is convenient to use the elementary symmetric functions (s_1,s_2,s_3),

$$s_1=z_1+z_2+z_3,\quad s_2=z_1z_2+z_1z_3+z_2z_3,\quad s_3=z_1z_2z_3,$$

change variables (a,b,c) to (a,b^*,c), and reformulate equation (1.4) in a more symmetric form

$$aa^*+bb^*+cc^*-abc=\frac{3s_1s_2-s_1^3}{s_3}. \tag{1.5}$$

The problem is to find Laurent polynomials $a,b,c\in\mathbb{Z}[s_1,s_2,s_3^{\pm1}]$ satisfying equation (1.5). The equation has the initial solution

$$I=\left(z_1+z_2+z_3,\frac{z_1+z_2+z_3}{z_1z_2z_3},z_1+z_2+z_3\right)=\left(s_1,\frac{s_1}{s_3},s_1\right), \tag{1.6}$$

whose evaluation at $s_1=s_2=3, s_3=1$ is the initial solution (3,3,3) of the Markov equation.

From now on we call equation (1.5) the ∗-Markov equation.

1.3.2. The group Γ_M of symmetries of the $*$-Markov equation is called the $*$-Markov group. It consists of permutations of variables, changes of sign of two of the three variables, the braid group $\mathcal{B}_3$ transformations

$$\begin{aligned} \tau_1 &\colon (a,b,c) \mapsto (-a^*, \quad c^*, \quad b^* - ac), \\ \tau_2 &\colon (a,b,c) \mapsto (b^*, \quad a^* - bc, \quad -c^*), \end{aligned} \tag{1.7}$$

and the new transformations

$$\mu_{i,j} \colon (a,b,c) \mapsto (s_3^i a, s_3^{-i-j} b, s_3^j c), \quad i,j \in \mathbb{Z}.$$

We have an obvious epimorphism of the $*$-Markov group onto the Markov group. This fact and the Markov theorem imply that for any Markov triple of numbers there exists a triple of Laurent polynomials, solving the $*$-Markov equation, obtained from the initial solution I by transformations of the $*$-Markov group, whose evaluation at $s_1 = s_2 = 3, s_3 = 1$ gives the Markov triple.

It is an open question if any solution of the $*$-Markov equation can be obtained from the initial solution I by a transformation of the $*$-Markov group. In analogy with the Markov equation we may expect that all solutions lie in $\Gamma_M I$.

1.4. **Solutions in the orbit of the initial solution.**

1.4.1. As the first topic of this paper we study $\Gamma_M I$, the set of solutions of the $*$-Markov equation obtained from the initial solution I by transformations of the $*$-Markov group.

In the interpretation of solutions of the $*$-Markov equation as matrices $(\chi_T(E_i^* \otimes E_j))$ of equivariant Euler characteristics for T-full exceptional collections, the set $\Gamma_M I$ corresponds to the set of matrices $(\chi_T(E_i^* \otimes E_j))$ for the T-full exceptional collections in $\mathcal{D}^b_T(\mathbb{P}^2)$ lying in the braid group orbit of the so-called Beilinson T-full exceptional collection, see [CV20, Be78].

Several first elements of $\Gamma_M I$ different from I are

$$(s_1^*, s_1^2 - s_2, s_1^*), \tag{1.8}$$
$$(s_2^*, s_1^2 s_2 - s_1 s_3 - s_2^2, (s_1^2 - s_2)^*), \tag{1.9}$$
$$(s_1^*, s_1^3 s_2 - 2 s_1^2 s_3 - s_1 s_2^2 + s_2 s_3, (s_1^2 s_2 - s_1 s_3 - s_2^2)^*), \tag{1.10}$$
$$((s_1^2 s_2 - s_1 s_3 - s_2^2)^*, s_1^2 s_2^3 - s_1^3 s_2 s_3 - s_2 s_3^2 + s_1^2 s_3^2 - s_2^4, (s_2^2 - s_1 s_3)^*). \tag{1.11}$$

Evaluated at $s_1 = s_2 = 3, s_3 = 1$ they represent the Markov triples (3,6,3), (3,15,6), (3,39,15), (15,87,6), respectively.

Random application of generators of the $*$-Markov group to the initial solution I will produce the Laurent polynomial solutions of the $*$-Markov equation, but they will not be polynomial. We make them close to being polynomial as follows.

We look for solutions of the $*$-Markov equation in the form (f_1^*, f_2, f_3^*). We say that a solution (f_1^*, f_2, f_3^*) is a *reduced polynomial solution* if each of f_1, f_2, f_3 is a nonconstant polynomial in s_1, s_2, s_3 not divisible by s_3.

For example, all triples in (1.8)-(1.11) are reduced polynomial solutions.

Theorem 1.1. *Let (a,b,c) be a Markov triple, $0 < a < b$, $0 < c < b$, $6 \leqslant b$. Then there exists a unique reduced polynomial solution $(f_1^*, f_2, f_3^*) \in \Gamma_M I$ representing (a,b,c).*

See Theorem 6.4.

1.4.2. Let $f(s_1, s_2, s_3)$ be a polynomial. We consider two degrees of f: the *homogeneous degree* $d := \deg f$ with respect to weights $(1,1,1)$ and the *quasi-homogeneous degree* $q := \mathrm{Deg} f$ with respect to weights $(1,2,3)$. For example, $\deg(s_1^{a_1}s_2^{a_2}s_3^{a_3}) = a_1+a_2+a_3$, $\mathrm{Deg}(s_1^{a_1}s_2^{a_2}s_3^{a_3}) = a_1+2a_2+3a_3$.

Let $f(s_1, s_2, s_3)$ be a polynomial of homogeneous degree d not divisible by s_3, then

$$g(s_1, s_2, s_3) := s_3^d\, f\Big(\frac{s_2}{s_3}, \frac{s_1}{s_3}, \frac{1}{s_3}\Big) \tag{1.12}$$

is a polynomial of homogeneous degree d not divisible by s_3. If additionally $f(s_1, s_2, s_3)$ is a quasi-homogeneous polynomial of quasi-homogeneous degree q, then $g(s_1, s_2, s_3)$ is a quasi-homogeneous polynomial of quasi-homogeneous degree $3d - q$.

The polynomial g is denoted by $\mu(f)$.

1.4.3. The assignment to a quasi-homogeneous polynomial f its bi-degree vector (d, q) could be seen as a "de-quatization" in the following sense. Let

$$s_1 = c_1 e^{\alpha+\beta}, \qquad s_2 = c_2 e^{\alpha+2\beta}, \qquad s_3 = c_3 e^{\alpha+3\beta},$$

where α, β are real parameters which tend to $+\infty$ and c_1, c_2, c_3 are fixed generic real numbers. If $f(s_1, s_2, s_3)$ is a quasi-homogeneous polynomial of bi-degree (d, q), then

$$\ln f(c_1 e^{\alpha+\beta}, c_1 e^{\alpha+\beta}, c_1 e^{\alpha+\beta})$$

has *leading term* $d\alpha + q\beta$ independent of the choice of c_1, c_2, c_3, which may be considered as a vector (d, q).

1.4.4. We say that a polynomial $P(s_1, s_2, s_3)$ is a *$*$-Markov polynomial* if there exists a Markov triple (a, b, c), $0 < a < b$, $0 < c < b$, $6 \leqslant b$, with reduced polynomial presentation $(f_1^*, f_2, f_3^*) \in \Gamma_M I$, such that $P = f_2$. The polynomial s_2 will also be called a *$*$-Markov polynomial*.

We say that a polynomial $Q(s_1, s_2, s_3)$ is a *dual $*$-Markov polynomial* if Q is not divisible by s_3 and $\mu(Q)$ is a $*$-Markov polynomial.

For example, $s_1^2 - s_2$, $s_2(s_1^2 - s_2) - s_3 s_1$ are $*$-Markov polynomials, since they appear as the middle terms in the reduced polynomial presentations in (1.8) and (1.9) and $s_2^2 - s_1 s_3$, $s_1(s_2^2 - s_1 s_3) - s_3 s_2$ are the corresponding dual $*$-Markov polynomials.

Theorem 1.2. *Let $(f_1^*, f_2, f_3^*) \in \Gamma_M I$ be the reduced polynomial presentation of a Markov triple (a, b, c), $0 < a < b$, $0 < c < b$, $6 \leqslant b$. Then each of f_1, f_3 is either a $*$-Markov polynomial or a dual $*$-Markov polynomial. If f_1, f_2, f_3 have bi-degree vectors $(d_1, q_1), (d_2, q_2), (d_3, q_3)$, then*

$$(d_2, q_2) = (d_1, q_1) + (d_3, q_3).$$

Theorem 1.3. *Let $f(s_1, s_2, s_3)$ be a $*$-Markov polynomial or a dual $*$-Markov polynomial of bi-degree (d, q). Then $f(s_1, s_2, s_3)$ is a quasi-homogeneous polynomial with respect to weights $(1, 2, 3)$. Moreover, $|2q - 3d| = 1$ if d is odd and $|2q - 3d| = 2$ if d is even.*

See Theorems 6.4, 7.2, and examples (1.8)-(1.11).

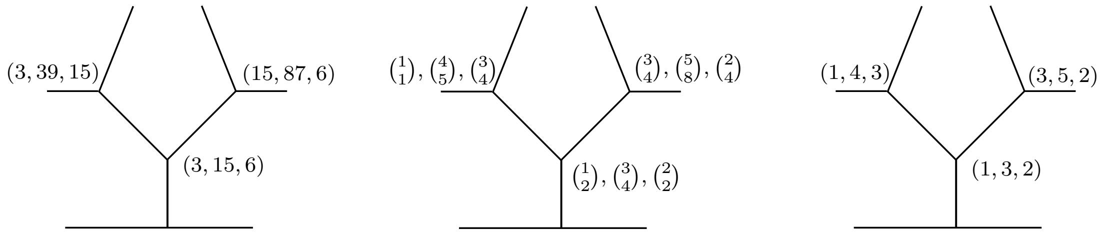

Figure 1.

1.4.5. It is convenient to put Markov triples at the vertices of the infinite binary planar tree as in Figure 1 and obtain what is called the Markov tree. Similarly, we may put at the vertices the triples of polynomials (f_1, f_2, f_3), such that the triples (f_1^*, f_2, f_3^*) are the reduced polynomial presentations of the corresponding Markov triples. In that way we would put in Figure 1 the triple (f_1, f_2, f_3) shown in (1.9) instead of (3,15,6), the triple (f_1, f_2, f_3) shown in (1.10) instead of (3,39,15), the triple (f_1, f_2, f_3) shown in (1.11) instead of (15,87,6). Or we may put at the vertices the triples $(d_1, q_1), (d_2, q_2), (d_3, q_3)$ of bi-degree vectors of the triples (f_1, f_2, f_3), or the triples (d_1, d_2, d_3) of degrees, see Figure 1, or we may even put at the vertices the triples of Newton polytopes of the polynomials (f_1, f_2, f_3), see Sections 7.7 and 7.8.

These decorated trees have interesting interrelations consisting of "quatizations" and "dequantizations", see short discussion in Section 7.5.

A compelling problem is to study asymptotics of these decorations along the infinite paths going from the root of the tree to infinity, see remarks in Sections 7.8 and 7.9.

1.4.6. It is well known that the Markov triples of the left branch of the Markov tree are composed of the odd Fibonacci numbers, multiplied by 3. These triples have the form $(3, 3\varphi_{2n+1}, 3\varphi_{2n-1})$, where φ_{2n+1}, φ_{2n-1} are odd Fibonacci numbers. We describe the reduced polynomial presentations $(g_{n-1}^*, F_{2n+1}, F_{2n-1}^*)$ of these Markov triples, where $g_{n-1} = s_2$ if n is even, $g_{n-1} = s_1$ if n is odd, and F_{2n+1}, F_{2n-1} are polynomial in s_1, s_2, s_3, called the odd $*$-Fibonacci polynomials.

The first of them are

$$\begin{aligned}
F_3(\boldsymbol{s}) &= s_1^2 - s_2, \\
F_5(\boldsymbol{s}) &= s_1^2 s_2 - s_1 s_3 - s_2^2, \\
F_7(\boldsymbol{s}) &= s_1^3 s_2 - 2 s_1^2 s_3 - s_1 s_2^2 + s_2 s_3.
\end{aligned}$$

We describe the recurrence relations for the odd $*$-Fibonacci polynomials, explicit formulas for them, their Newton polytopes, the Binet formula, the Cassini identity, describe the continued fractions for F_{2n+3}/F_{2n+1} and the limit of this ratio as $n \to \infty$.

1.4.7. It is well known that the Markov triples of the right branch of the Markov tree are composed of the odd Pell numbers, multiplied by 3. We describe the reduced polynomial presentations of these Markov triples in terms of the polynomials, which we call the odd $*$-Pell polynomials. We develop the properties of the odd $*$-Pell polynomials, which are analogous to properties of odd Pell numbers and to properties of the odd $*$-Fibonacci polynomials.

1.4.8. The q-deformations of Fibonacci and Pell numbers is an active subject related to several branches of combinatorics and number theory, see, e.g., [Ca74, An86, Pa06, MO20] and references therein. It would be interesting to determine if these numerous q-deformations of Fibonacci and Pell numbers could be obtained by specifications of our $*$-deformation depending on the three parameters s_1, s_2, s_3.

1.5. **$*$-Analogs of the Dubrovin Poisson structure.** In [Du96] Dubrovin considered $\mathbb{C}^3$ with coordinates (a, b, c), the braid group $\mathcal{B}_3$ action (1.2), and introduced a Poisson structure on $\mathbb{C}^3$,

$$\{a, b\}_H = 2c - ab, \qquad \{b, c\}_H = 2a - bc, \qquad \{c, a\}_H = 2b - ac.$$

which is braid group invariant and has the polynomial $a^2+b^2+c^2-abc$ as a Casimir element[1].

The second topic of this paper is a construction of a $*$-analog of the Dubrovin Poisson structure. Our Poisson structure is defined on $\mathbb{C}^6$, is anti-invariant with respect to the braid group $\mathcal{B}_3$ action (1.7), is invariant with respect to the involution

$$(a, a^*, b, b^*, c, c^*) \mapsto (a^*, a, b^*, b, c^*, c),$$

has the polynomials

$$aa^* + bb^* + cc^* - abc,$$
$$aa^* + bb^* + cc^* - a^*b^*c^*,$$

as Casimir elements, and is log-canonical, see Section 11. Here the word anti-invariant means that the Poisson structure is multiplied by -1 under the action of generators of the braid group. Recall also that a Poisson structure on a space with coordinates $x_1, \dots, x_n$ is log-canonical if $\{x_i, x_j\} = a_{ij}x_ix_j$ for all i, j, where $a_{i,j}$ are constants. Our log-canonical Poisson structure has $a_{i,j} = \pm 1, 0$.

1.5.1. The space $\mathbb{C}^3$ considered by Dubrovin is actually identified with the group U_3 of unipotent upper triangular matrices (the Stokes matrices of three dimensional Frobenius manifolds). Standing on such an identification, M. Ugaglia generalized the construction of Dubrovin's Poisson structure to all groups U_n, see [Ug99] for the explicit equations. Remarkably enough, the same braid invariant Poisson structure on U_n was found independently also in [Bo01, Bo04] from two completely different perspectives. Let $B_\pm$ be the groups of upper and lower triangular $n \times n$ matrices. In [Bo01], P. Boalch proved that U_n is the stable locus of a Poisson involution of the Poisson-Lie group $B_+ * B_-$, and that the standard Poisson structure of $B_+ * B_-$ induces the braid invariant Poisson structure on U_n. The construction in [Bo04], is based on the identification of the group U_n with the space of Gram matrices $(\chi(E_i, E_j))_{i,j}$ for exceptional collections $(E_1, \dots, E_n)$ in triangulated categories[2]. A. Bondal discovered a symplectic groupoid whose space of objects is U_n: the existence of a braid invariant Poisson structure on U_n is then deduced from the general theory of symplectic groupoids. The quantization of the Poisson structure on U_n is also known as Nelson-Regge algebra in 2+1 quantum gravity [NR89, NRZ90], and as Fock-Rosly bracket in Chern-Simons theory [FR97]. Furthermore, L. Chekhov and M. Mazzocco generalized the construction of

[1]Recall that a function f is a Casimir element for a Poisson structure $\{\ ,\ \}$ if $\{f, g\} = 0$ for any g.

[2]Notice that the two identification of U_n as Stokes matrices or Gram matrices of the χ-pairing should coincide, at least for quantum cohomologies, according to a conjecture of Dubrovin, see [Du98, CDG18].

the Dubrovin Poisson structures to the space of bilinear forms with block-upper-triangular Gram matrix, they also extensively studied the related Poisson algebras, their quantization and affinization, see [CM11, CM13]. See very interesting short paper [CF00] by L.O. Chekhov and V.V. Fock.

It would be interesting to see the $*$-analogs of these considerations.

1.6. **$*$-Analogs of the Horowitz theorem.** In [Ho75], R.D. Horowitz proved the following result, characterizing the Markov group as a subgroup of the group of ring automorphisms of $\mathbb{Z}[a, b, c]$.

Theorem 1.4 ([Ho75, Theorem 2])**.** *The group of ring automorphisms of $\mathbb{Z}[a, b, c]$, which preserve the polynomial*

$$H = a^2 + b^2 + c^2 - abc,$$

is isomorphic to the Markov group.

As the third topic of this paper we develop $*$-analogs of the Horowitz theorem, see Section 12 and Appendix A.

1.7. **Exposition of material.** In Section 2 we introduce the $*$-Markov equation and evaluation morphism. The $*$-Markov group and its subgroups, in particular, the important $*$-Viète subgroup, are defined in Section 3. The Markov and extended Markov trees are introduced in Section 4. In Section 5 we introduce the notion of a distinguished representative of a Markov triple and show that the $*$-Viète group acts freely and transitively on the set of distinguished representatives.

In Section 6 we introduce the notion of an admissible triple of Laurent polynomials and the notion of a reduced polynomial presentation of a Markov triple. One of the main theorems of the paper, Theorem 6.4 says that a Markov triple has a unique reduced polynomial presentation. We also introduce the notion of a $*$-Markov polynomial.

In Section 7 six decorated infinite planar binary trees are defined. They are the $*$-Markov polynomial tree, 2-vector tree, matrix tree, deviation tree, Markov tree, Euclid tree. We discuss the interrelations between the trees. An interesting problem is to study the asymptotics of the decorations along the infinite paths from the root of the tree to infinity.

In Sections 8 and 9 we introduce the odd $*$-Fibonacci and odd $*$-Pell polynomials and discuss their properties.

In Section 10 we construct actions of the $*$-Markov group on the spaces $\mathbb{C}^6$ and $\mathbb{C}^5$ and a map $F : \mathbb{C}^6 \to \mathbb{C}^5$ commuting with the actions. Using these objects we construct equivariant Poisson structures on $\mathbb{C}^6$ and $\mathbb{C}^5$ in Section 11.

In Section 12 we establish $*$-analogs of the Horowitz theorem on $\mathbb{C}^6$ and $\mathbb{C}^5$. In Appendix A we discuss more analogs of the Horowitz theorem.

In Appendix B we discuss briefly the $*$-equations for $\mathbb{P}^3$ and associated Poisson structures on $\mathbb{C}^{12}$.

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2. $*$-Markov equation

2.1. $*$-Involution. Denote $\boldsymbol{z} = (z_1, z_2, z_3)$, $\boldsymbol{s} = (s_1, s_2, s_3)$. Let $\mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_3}$ be the ring of symmetric Laurent polynomials in $\boldsymbol{z}$ with integer coefficients. We define an isomorphism $\mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_3} \cong \mathbb{Z}[s_1, s_2, s_3^{\pm 1}]$ by sending

$$(z_1 + z_2 + z_3,\ z_1 z_2 + z_1 z_3 + z_2 z_3,\ z_1 z_2 z_3) \quad \to \quad (s_1, s_2, s_3).$$

Define the involution

$$(-)^* \colon \mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_3} \to \mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_3}, \quad f \mapsto f^*,$$

where

$$f^*(\boldsymbol{z}) := f\left(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}\right), \quad f \in \mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_3}.$$

This induces a $*$-involution on $\mathbb{Z}[s_1, s_2, s_3^{\pm 1}]$.

$$f^*(s_1, s_2, s_3) = f\left(\frac{s_2}{s_3}, \frac{s_1}{s_3}, \frac{1}{s_3}\right).$$

Denote $\boldsymbol{s}_o := (3, 3, 1)$. Define the evaluation morphisms

$$\begin{aligned} &\mathrm{ev}_{\boldsymbol{s}_o} \colon \mathbb{Z}[s_1, s_2, s_3^{\pm 1}] \to \mathbb{Z}, && f(\boldsymbol{s}) \mapsto f(\boldsymbol{s}_o), \\ &\mathrm{Ev}_{\boldsymbol{s}_o} \colon \left(\mathbb{Z}[s_1, s_2, s_3^{\pm 1}]\right)^3 \to \mathbb{Z}^3, && (a, b, c) \mapsto (a(\boldsymbol{s}_o), b(\boldsymbol{s}_o), c(\boldsymbol{s}_o)). \end{aligned}$$

The evaluation morphism corresponds to the evaluation of a Laurent polynomial $f(z_1, z_2, z_3)$ at $z_1 = z_2 = z_3 = 1$.

2.2. Evaluation morphism. The $*$-*Markov equation* is the equation

$$aa^* + bb^* + cc^* - abc = \frac{3s_1 s_2 - s_1^3}{s_3}, \tag{2.1}$$

where $a, b, c \in \mathbb{Z}[s_1, s_2, s_3^{\pm 1}]$. The solution

$$I = \left(s_1, \frac{s_1}{s_3}, s_1\right) \tag{2.2}$$

is call the *initial solution.*

We have

$$\mathrm{ev}_{\boldsymbol{s}_o}\left(\frac{3s_1 s_2 - s_1^3}{s_3}\right) = 0.$$

Proposition 2.1. *If $f = (f_1, f_2, f_3)$ is a solution of the $*$-Markov equation (2.1), then $\mathrm{Ev}_{\boldsymbol{s}_o}(f)$ is a solution of the Markov equation (1.1).* □

For example, the evaluation of the initial solution I gives the triple (3,3,3).

Remark 2.2. The $*$-Markov equation (2.1) can be studied by looking for solutions (a,b,c) in $\mathbb{A}^3$, where $\mathbb{A}$ is a ring more general than $\mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_3} \cong \mathbb{Z}[s_1, s_2, s_3^{\pm 1}]$.

For instance, if we look for solutions of the form $(\alpha s_1, \beta \frac{s_1}{s_3}, \gamma s_1)$, where $\alpha,\beta,\gamma \in \mathbb{C}$, then

$$\alpha^2+\beta^2+\gamma^2 = 3, \qquad \alpha\beta\gamma = 1.$$

This curve has infinitely many algebraic points, for example

$$\alpha = \sqrt{-1}\,, \qquad \beta = \sqrt{2+\sqrt{5}}\,, \qquad \gamma = \frac{1}{\sqrt{-2-\sqrt{5}}}\,.$$

3. Groups of symmetries

3.1. **Symmetries of Markov equation.** Consider the following three groups of transformations of $\mathbb{Z}^3$:

Type I. The group G_1^c generated by transformations

$$\lambda_{i,j}^c\colon (a,b,c) \mapsto \left((-1)^i a, (-1)^{i+j} b, (-1)^j c\right), \qquad i,j \in \mathbb{Z}_2.$$

Type II. The group G_2^c generated by transformations

$$\sigma_1^c : (a,b,c) \mapsto (b,a,c), \quad \sigma_2^c : (a,b,c) \mapsto (a,c,b).$$

Type III. The group G_3^c generated by transformations

$$\begin{aligned} \tau_1^c &\colon (a,b,c) \mapsto (-a, \quad c, \quad b-ac),\\ \tau_2^c &\colon (a,b,c) \mapsto (b, \quad a-bc, \quad -c). \end{aligned}$$

We have $\tau_1^c\tau_2^c\tau_1^c = \tau_2^c\tau_1^c\tau_2^c$.

In these notations the superscript c stays for the word classical.

Remark 3.1. Let $\mathcal{B}_3$ be the braid group with three strands, and β_1, β_2 its standard generators (elementary braids) with $\beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2$. There is a group epimorphism

$$\phi\colon \mathcal{B}_3 \to G_3^c, \quad \beta_i \mapsto \tau_i^c, \quad i = 1,2.$$

The center $\mathcal{Z}(\mathcal{B}_3) = \langle (\beta_1\beta_2)^3 \rangle$ is contained in $\ker\phi$. Thus, the group

$$\mathcal{B}_3 \big/ \mathcal{Z}(\mathcal{B}_3) \cong \mathrm{PSL}(2,\mathbb{Z})$$

acts on the set of solutions of (1.1).

Proposition 3.2. *The set of nonzero Markov triples is invariant under the action of each of the groups G_1^c, G_2^c, G_3^c.* $\square$

3.2. **Markov and Viète groups.** Define the *Markov group* Γ_M^c as the group of transformations of $\mathbb{Z}^3$ generated by G_1^c, G_2^c, G_3^c,

$$\Gamma_M^c := \langle G_1^c, G_2^c, G_3^c \rangle. \tag{3.1}$$

Define the *Viète involutions* $v_1^c, v_2^c, v_3^c \in \Gamma_M^c$ by the formulas

$$\begin{aligned} v_1^c &\colon (a,b,c) \mapsto (bc-a, b, c),\\ v_2^c &\colon (a,b,c) \mapsto (a, ac-b, c),\\ v_3^c &\colon (a,b,c) \mapsto (a, b, ab-c). \end{aligned}$$

Define the *Viète group* Γ_V^c as the group generated by the Viète involutions v_1^c, v_2^c, v_3^c,

$$\Gamma_V^c := \langle v_1^c, v_2^c, v_3^c \rangle. \tag{3.2}$$

We have

$$v_1^c = \lambda_{1,1}^c\ \sigma_1^c\ \tau_2^c, \qquad v_2^c = \lambda_{1,0}^c\ \sigma_2^c\ \tau_1^c, \qquad v_3^c = \lambda_{1,1}^c\ \tau_1^c\ \sigma_2^c. \tag{3.3}$$

Theorem 3.3 ([EH74, Theorem 1]). *The group Γ_V^c is freely generated by v_1^c, v_2^c, v_3^c, that is, $\Gamma_V^c \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.* □

Proposition 3.4. *We have the following identities:*

$$\sigma_1^c \lambda_{k,l}^c \sigma_1^c = \lambda_{k+l,l}^c, \qquad \sigma_2^c \lambda_{k,l}^c \sigma_2^c = \lambda_{l,k+l}^c,$$

$$\begin{aligned} \sigma_1^c v_1^c \sigma_1^c &= v_2^c, & \sigma_2^c v_1^c \sigma_2^c &= v_1^c, \\ \sigma_1^c v_2^c \sigma_1^c &= v_1^c, & \sigma_2^c v_2^c \sigma_2^c &= v_3^c, \\ \sigma_1^c v_3^c \sigma_1^c &= v_3^c, & \sigma_2^c v_3^c \sigma_2^c &= v_2^c, \end{aligned}$$

$$\lambda_{k,l}^c v_i^c \lambda_{k,l}^c = v_i^c, \qquad i = 1, 2, 3.$$

□

Corollary 3.5. *We have $\Gamma_M^c = \langle \Gamma_V^c, G_1^c, G_2^c \rangle$. Moreover, Γ_V^c is a normal subgroup of Γ_M^c.*

Proof. The inclusion $\Gamma_M^c \supseteq \langle \Gamma_V^c, G_1^c, G_2^c \rangle$ is clear. We have $G_3^c \subseteq \langle \Gamma_V^c, G_1^c, G_2^c \rangle$, by equations (3.3). Hence $\Gamma_M^c = \langle \Gamma_V^c, G_1^c, G_2^c \rangle$. We have $g v_i^c g^{-1} \in \Gamma_V^c$ for any $g \in G_1^c, G_2^c$ by Proposition 3.4. □

Proposition 3.6. *We have $\Gamma_V^c \cap \langle G_1^c, G_2^c \rangle = \{\mathrm{id}\}$.*

Proof. Any element of Γ_V^c fixes the triple $(2, 2, 2)$. The only elements of $\langle G_1^c, G_2^c \rangle$ which fix $(2, 2, 2)$ are the elements of G_2^c.

Extend the action of both Γ_V^c and G_2^c to the space $\mathbb{C}^3$. The point $(0, 0, 0)$ is a fixed point for both actions. The Jacobian matrices at $(0, 0, 0)$ of the Viète transformations v_1^c, v_2^c, v_3^c are

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

respectively. Hence, any element of Γ_V^c has diagonal Jacobian matrix at $(0, 0, 0)$. The only transformation of G_2^c which can be represented by a diagonal matrix is the identity. □

Corollary 3.7. *For any element $g \in \Gamma_M^c$, there exist unique $v \in \Gamma_V^c$ and $h \in \langle G_1^c, G_2^c \rangle$ such that $g = vh$. This implies that $\Gamma_M^c = \Gamma_V^c \rtimes \langle G_1^c, G_2^c \rangle$.*

Proof. Since $\Gamma_M^c = \langle \Gamma_V^c, G_1^c, G_2^c \rangle$, any g can be expressed as a product

$$g = v_{i_1} a_{i_1} v_{i_2} a_{i_2} \ldots v_{i_k} a_{i_k}, \quad a_{i_j} \in \langle G_1^c, G_2^c \rangle.$$

We can factor g as

$$g = v_{i_1} v_{i_2} \ldots v_{i_k} a'_{i_1} a'_{i_2} \ldots a'_{i_k}, \quad a'_{i_j} \in \langle G_1^c, G_2^c \rangle, \tag{3.4}$$

by using the commutation rules described in Proposition 3.4. The decomposition in (3.4) is unique, by Proposition 3.6. □

3.3. **Symmetries of $*$-Markov equation.** Consider the following four groups of transformations of the space $\left(\mathbb{Z}[s_1, s_2, s_3^{\pm 1}]\right)^3$.

Type I. The group G_1 generated by transformations

$$\lambda_{i,j}\colon (a,b,c) \mapsto \left((-1)^i a, (-1)^{i+j} b, (-1)^j c\right), \qquad i,j \in \mathbb{Z}_2.$$

Type II. The group G_2 generated by transformations

$$\sigma_1 : (a,b,c) \mapsto (b,a,c), \qquad \sigma_2 : (a,b,c) \mapsto (a,c,b).$$

Type III. The group G_3 generated by transformations

$$\begin{aligned} \tau_1 &\colon (a,b,c) \mapsto (-a^*, \quad c^*, \quad b^* - ac), \\ \tau_2 &\colon (a,b,c) \mapsto (b^*, \quad a^* - bc, \quad -c^*). \end{aligned} \tag{3.5}$$

We have $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$.

Type IV. The group G_4 generated by transformations

$$\mu_{i,j}\colon (a,b,c) \mapsto (s_3^i a, s_3^{-i-j} b, s_3^j c), \qquad i,j \in \mathbb{Z}.$$

Proposition 3.8. *The set of all solutions of the $*$-Markov equation (2.1) is invariant under the action of each of the groups G_1, G_2, G_3, G_4.* □

As in the case of the Markov equation (1.1), we have the action of $\mathcal{B}_3/\mathcal{Z}(\mathcal{B}_3) \cong \mathrm{PSL}(2,\mathbb{Z})$ on the set of all solutions of the $*$-Markov equation (2.1). See Remark 3.1.

3.4. **$*$-Markov and $*$-Viète groups.** Define the $*$-*Markov group* Γ_M as the group of transformations of $\left(\mathbb{Z}[s_1, s_2, s_3^{\pm 1}]\right)^3$ generated by G_1, G_2, G_3, G_4,

$$\Gamma_M := \langle G_1, G_2, G_3, G_4 \rangle. \tag{3.6}$$

Define the $*$-*Viète involutions* $v_1, v_2, v_3 \in \Gamma_M$ by the formulas

$$\begin{aligned} v_1 &\colon (a,b,c) \mapsto (bc - a^*, b^*, c^*), \\ v_2 &\colon (a,b,c) \mapsto (a^*, ac - b^*, c^*), \\ v_3 &\colon (a,b,c) \mapsto (a^*, b^*, ab - c^*). \end{aligned}$$

Define the $*$-*Viète group* Γ_V as the group generated by Viète involutions v_1, v_2, v_3,

$$\Gamma_V := \langle v_1, v_2, v_3 \rangle.$$

We have

$$v_1 = \lambda_{1,1}\sigma_1\tau_2, \qquad v_2 = \lambda_{1,0}\sigma_2\tau_1, \qquad v_3 = \lambda_{1,1}\tau_1\sigma_2. \tag{3.7}$$

Proposition 3.9. *We have the following identities,*

$$\sigma_1\lambda_{k,l}\sigma_1 = \lambda_{k+l,l}, \qquad \sigma_2\lambda_{k,l}\sigma_2 = \lambda_{k,k+l},$$

$$\begin{aligned} \sigma_1 v_1 \sigma_1 &= v_2, & \sigma_2 v_1 \sigma_2 &= v_1, \\ \sigma_1 v_2 \sigma_1 &= v_1, & \sigma_2 v_2 \sigma_2 &= v_3, \\ \sigma_1 v_3 \sigma_1 &= v_3, & \sigma_2 v_3 \sigma_2 &= v_2, \end{aligned}$$

$$\lambda_{k,l} v_i \lambda_{k,l} = v_i, \qquad i = 1,2,3,$$

$$\lambda_{k,l}\,\mu_{i,j}\,\lambda_{k,l} = \mu_{i,j}\,, \qquad k,l \in \mathbb{Z}_2, \qquad i,j \in \mathbb{Z},$$
$$\sigma_1\mu_{i,j}\sigma_1 = \mu_{-i-j,j}\,, \qquad \sigma_2\mu_{i,j}\sigma_2 = \mu_{i,-i-j}\,,$$
$$\upsilon_k\,\mu_{i,j}\,\upsilon_k = \mu_{-i-j}\,, \qquad k = 1,2,3.$$

Proof. These identities are proved by straightforward computations. □

Corollary 3.10. *For any element $g \in \langle G_1, G_2, G_4\rangle$, there exist unique $g_1 \in G_1, g_2 \in G_2, g_4 \in G_4$ such that*

$$g = g_4 g_1 g_2. \tag{3.8}$$

Proof. Any $g \in \langle G_1, G_2, G_4\rangle$ can be put in the form (3.8) by Proposition 3.9. The uniqueness follows from the identities

$$G_4 \cap \langle G_1, G_2\rangle = \{\mathrm{id}\}\,, \quad G_1 \cap G_2 = \{\mathrm{id}\}\,.$$ □□

Corollary 3.11. *We have $\Gamma_M = \langle \Gamma_V, G_1, G_2, G_4\rangle$. Moreover Γ_V is a normal subgroup of Γ_M.*

Proof. The inclusion $\Gamma_M \supseteq \langle G_1, G_2, G_4, \Gamma_V\rangle$ is clear. We have $G_3 \subseteq \langle G_1, G_2, G_4, \Gamma_V\rangle$, by equations (3.7). Hence $\Gamma_M = \langle G_1, G_2, G_4, \Gamma_V\rangle$. It follows that $g\upsilon_i g^{-1} \in \Gamma_V$ for any $g \in G_1, G_2, G_4$, by Proposition 3.9. □

Proposition 3.12. *We have $\Gamma_V \cap \langle G_1, G_2, G_4\rangle = \{\mathrm{id}\}$.*

Proof. Let $g \in \Gamma_V \cap \langle G_1, G_2, G_4\rangle$. We have $g \in \ker\varphi_M$ by Proposition 3.6. The only elements of the form (3.8) which are in $\ker\varphi_M$ are the elements of G_4. Any element of Γ_V fixes the triple of constant polynomials $(2,2,2)$. The only element of G_4 which fixes $(2,2,2)$ is the identity. □

Corollary 3.13. *For any element $g \in \Gamma_M$, there exist unique $\upsilon \in \Gamma_V$ and $h \in \langle G_1, G_2, G_4\rangle$ such that $g = \upsilon h$. This implies that $\Gamma_M = \Gamma_V \rtimes \langle G_1, G_2, G_4\rangle$.*

Proof. Since $\Gamma_M = \langle \Gamma_V, G_1, G_2, G_4\rangle$, any g can be expressed as a product

$$g = \upsilon_{i_1} a_{i_1} \upsilon_{i_2} a_{i_2} \ldots \upsilon_{i_k} a_{i_k}, \quad a_{i_j} \in \langle G_1, G_2, G_4\rangle.$$

We can factor g as

$$g = \upsilon_{i_1}\upsilon_{i_2}\ldots\upsilon_{i_k} a'_{i_1} a'_{i_2}\ldots a'_{i_k}, \quad a'_{i_j} \in \langle G_1, G_2, G_4\rangle \tag{3.9}$$

by using the commutation rules described in Proposition 3.9. The decomposition in (3.9) is unique, by Proposition 3.12. □

Let $g \in \Gamma_M$. Consider its restriction $g|_{\mathbb{Z}^3}$ to the subset $\mathbb{Z}^3 \subseteq \left(\mathbb{Z}[s_1, s_2, s_3^{\pm1}]\right)^3$. Define the transformation $\varphi_M(g)\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ as the composition $\mathrm{Ev}_{\boldsymbol{s}_o} \circ g|_{\mathbb{Z}^3}$, i.e.

$$\mathbb{Z}^3 \xrightarrow{g|_{\mathbb{Z}^3}} \left(\mathbb{Z}[s_1, s_2, s_3^{\pm1}]\right)^3 \xrightarrow{\mathrm{Ev}_{\boldsymbol{s}_o}} \mathbb{Z}^3.$$

Proposition 3.14. *We have a group epimorphism*

$$\varphi_M\colon \Gamma_M \twoheadrightarrow \Gamma_M^c,$$

which acts on the generators as

$$\lambda_{\alpha,\beta} \mapsto \lambda^c_{\alpha,\beta}, \quad \sigma_i \mapsto \sigma^c_i, \quad \tau_i \mapsto \tau^c_i, \quad \mu_{\alpha,\beta} \mapsto \mathrm{id}. \tag{3.10}$$

Proof. Identities (3.10) are easily checked. Let $g, h \in \Gamma_M$. From the commutative diagram

$$\begin{tikzcd}
\left(\mathbb{Z}[s_1, s_2, s_3^{\pm 1}]\right)^3 \arrow[r, "g"] & \left(\mathbb{Z}[s_1, s_2, s_3^{\pm 1}]\right)^3 \arrow[r, "h"] \arrow[d, shift left, "\mathrm{Ev}_{\boldsymbol{s}_o}"] & \left(\mathbb{Z}[s_1, s_2, s_3^{\pm 1}]\right)^3 \arrow[d, "\mathrm{Ev}_{\boldsymbol{s}_o}"] \\
\mathbb{Z}^3 \arrow[u, "i"] \arrow[r, "\varphi_M(g)"'] & \mathbb{Z}^3 \arrow[u, shift left, "i"] \arrow[r, "\varphi_M(h)"'] & \mathbb{Z}^3
\end{tikzcd}$$

it readily follows that $\varphi_M(hg) = \varphi_M(h)\varphi_M(g)$. $\square$

Proposition 3.15. *We have* $\ker \varphi_M = G_4$, *so that* $\Gamma_M^c \cong \Gamma_M / G_4$.

Proof. Let $g \in \ker \varphi_M$. By Corollaries 3.13 and 3.10, there exist unique elements $v \in \Gamma_V, g_1 \in G_1, g_2 \in G_2, g_4 \in G_4$ such that

$$g = v g_4 g_1 g_2.$$

We have

$$\varphi_M(g) = \varphi_M(v)\varphi_M(g_1)\varphi_M(g_2) = \mathrm{id}.$$

By Corollary 3.7, together with $G_1^c \cap G_2^c = \{\mathrm{id}\}$, we have

$$\varphi_M(v) = \mathrm{id}, \quad \varphi_M(g_1) = \mathrm{id}, \quad \varphi_M(g_2) = \mathrm{id}.$$

This clearly implies that $g_1 = \mathrm{id}$ and $g_2 = \mathrm{id}$. The element v is of the form $v = \prod_{j=1}^n v_{i_j}$ with $i_j \in \{1, 2, 3\}$, so that

$$\mathrm{id} = \varphi_M(v) = \prod_{j=1}^n \varphi_M(v_{i_j}) = \prod_{j=1}^n v_{i_j}^c.$$

Since v_i^c freely generate Γ_V^c (by Theorem 3.3), we necessarily have $n = 0$, and $v = \mathrm{id}$. This shows that $\ker \varphi_M \subseteq G_4$. The opposite inclusion is obvious. $\square$

Lemma 3.16. *The morphism φ_M defines isomorphisms between the group G_i and G_i^c for $i = 1, 2, 3$, and between the group Γ_V and Γ_V^c.* $\square$

Lemma 3.17. *The evaluation morphism* $\mathrm{Ev}_{\boldsymbol{s}_o}$ *is* φ_M*-equivariant, i.e.*

$$\mathrm{Ev}_{\boldsymbol{s}_o}(g \cdot \boldsymbol{a}) = \varphi_M(g) \cdot \mathrm{Ev}_{\boldsymbol{s}_o}(\boldsymbol{a}), \quad g \in \Gamma_M, \quad \boldsymbol{a} \in \left(\mathbb{Z}[s_1, s_2, s_3^{\pm 1}]\right)^3. \qquad \square$$

4. Markov trees

4.1. **Decomposition of $\mathcal{M}^c$.** The set $\mathcal{M}^c$ of all nonzero solutions of the Markov equation (1.1) admits a partition in four subsets,

$$\mathcal{M}^c = \mathcal{M}_+^c \cup \mathcal{M}_{12}^c \cup \mathcal{M}_{13}^c \cup \mathcal{M}_{23}^c, \tag{4.1}$$

where $\mathcal{M}_+^c$ consists of all positive triples and $\mathcal{M}_{ij}^c$ consists of all triples with negative entries in the i-th and j-th position.

We have a projection $\pi : \mathcal{M}^c \to \mathcal{M}_+^c$, forgetting the minuses.

Lemma 4.1. *The action of the Viète group Γ_V^c on $\mathcal{M}^c$ preserves each of $\mathcal{M}_+^c, \mathcal{M}_{12}^c, \mathcal{M}_{13}^c, \mathcal{M}_{23}^c$. The action of the group $\langle G_2^c, \Gamma_V^c \rangle$ on $\mathcal{M}^c$ preserves the sets $\mathcal{M}_+^c$ and $\mathcal{M}_{12}^c \cup \mathcal{M}_{13}^c \cup \mathcal{M}_{23}^c$ and commutes with the projection $\pi : \mathcal{M}^c \to \mathcal{M}_+^c$.* $\square$

4.2. **Markov tree.** Solutions of the Markov equation (1.1) can be arranged in a graph, called the *Markov tree*.

Define $L := \sigma_2^c v_3^c,\ R := \sigma_1^c v_1^c \in \Gamma_M^c$. Given $(x, y, z) \in \mathcal{M}_+^c$, we have

$$\begin{tikzcd}
(x, xy - z, y) & & (y, yz - x, z) \\
& (x, y, z) \arrow[ul, "L"] \arrow[ur, "R"'] &
\end{tikzcd} \tag{4.2}$$

The Markov tree $\mathcal{T}$ is the infinite graph obtained by iterating the operations (4.2) starting from the initial solution $(3, 3, 3)$.

$$\begin{tikzcd}
& & & \cdots & & & \\
(3,102,39) & & (39,582,15) & & (15,1299,87) & & (87,507,6) \\
& (3,39,15) \arrow[ul] \arrow[ur] & & & & (15,87,6) \arrow[ul] \arrow[ur] & \\
& & & (3,15,6) \arrow[ull] \arrow[urr] & & & \\
& & & (3,6,3) \arrow[u] & & & \\
& & & (3,3,3) \arrow[u] & & &
\end{tikzcd}$$

Theorem 4.2 ([Mar79, Mar80][Ai13, Theorem 3.3])**.** *Up to permutations in G_2^c, all the elements of $\mathcal{M}_+^c$ appear exactly once in the Markov tree $\mathcal{T}$.* □

Corollary 4.3. *The group Γ_M^c acts transitively on the set $\mathcal{M}^c$.*

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}^c$. There exist $\gamma_1, \gamma_2 \in \langle G_1^c, G_2^c \rangle$ such that $\gamma_1 \boldsymbol{x}, \gamma_2 \boldsymbol{y}$ are vertices of $\mathcal{T}$, by Theorem 4.2. So, there exist $\delta_1, \delta_2 \in \Gamma_M^c$ such that $\delta_1(3, 3, 3) = \gamma_1 \boldsymbol{x}$ and $\delta_2(3, 3, 3) = \gamma_2 \boldsymbol{y}$. We have

$$\gamma_2^{-1} \delta_2 \delta_1^{-1} \gamma_1 \boldsymbol{x} = \boldsymbol{y}.$$ □

Theorem 4.4 ([Ai13, Lemma 3.1])**.** *The triples $(3, 3, 3)$ and $(3, 6, 3)$ are the only vertices of $\mathcal{T}$ with repeated numbers.* □

4.3. **Extended Markov tree.** Define the *extended Markov graph* as the infinite graph $\mathcal{T}^{\text{ext}}$, with vertex set $\mathcal{M}_+^c$. We connect two vertices (a, b, c), (a', b', c') of $\mathcal{T}^{\text{ext}}$ by an edge if $(a, b, c) = v_i^c(a', b', c')$ for some $i \in \{1, 2, 3\}$, where v_i^c are Viète involutions.

$$\begin{tikzcd}
 & & \cdots & & \\
 & (15,3,6) & & (3,15,6) & \\
\iddots & & (3,3,6) \arrow[ul] \arrow[ur] & & \ddots \\
(6,15,3) & & (3,3,3) \arrow[u] \arrow[dl] \arrow[dr] & & (15,6,3) \\
 & (6,3,3) \arrow[ul] \arrow[d] & & (3,6,3) \arrow[ur] \arrow[d] & \\
\ddots & (6,3,15) & & (3,6,15) & \iddots \\
 & & \cdots & &
\end{tikzcd}$$

Theorem 4.5. *The Viète group acts freely on the vertex set $\mathcal{M}^c_+$ of the extended Markov graph with one orbit, $\mathcal{M}^c_+ = \Gamma^c_V(3,3,3)$. Moreover $\mathcal{T}^{\rm ext}$ is a tree.*

The graph $\mathcal{T}^{\rm ext}$ is called the *extended Markov tree*.

The proof of Theorem 4.5 requires the following lemma. Define the function $m\colon \mathcal{M}^c_+ \to \mathbb{N}$, which assigns to a triple (x, y, z) its maximal entry. It is known that

$$\min m(\mathcal{M}^c_+) = 3,$$

and that such a minimum is achieved at $(3, 3, 3)$.

Lemma 4.6. *For any $(x, y, z) \in \mathcal{M}^c_+ \setminus \{(3, 3, 3)\}$ there exists a* unique *Viète transformation v^c_i, with $i \in \{1, 2, 3\}$, such that $m(v^c_i(x, y, z)) < m(x, y, z)$.*

Proof. We have

$$v^c_1(x, y, z) = (yz - x, y, z), \qquad v^c_2(x, y, z) = (x, xz - y, z), \qquad v^c_3(x, y, z) = (x, y, xy - z).$$

We claim that if $m(x, y, z) = x$, then the transformation is v^c_1; if $m(x, y, z) = y$, then the transformation is v^c_2; if $m(x, y, z) = z$, then the transformation is v^c_3.

To prove the first case we need to show that $yz - x \leqslant x$, $xz - y > x$, $xy - z > x$.

We may assume that $z \leqslant y < x$. Consider the function $\varphi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) := t^2 + y^2 + z^2 - tyz.$$

We have $\varphi(x) = \varphi(yz - x) = 0$, so that

$$\varphi(t) = (t - x)(t - (yz - x)).$$

If $yz - x \geqslant x$, so that $\varphi(t) > 0$ for all $t < x$. Then on the one hand we have $y < x$, but on the other hand we have

$$z^2 \leqslant y^2 \quad \Rightarrow \quad 2y^2 + z^2 \leqslant 3y^2 \leqslant zy^2 \quad \Rightarrow \quad \varphi(y) = 2y^2 + z^2 - y^2 z \leqslant 0.$$

This shows that the assumption $yz - x \geqslant x$ is contradictory. We also have

$$xz - y > 3x - y > 2x > x,$$
$$xy - z > 3x - z > 2x > x.$$

This completes the proof in the first case. The other two cases are proved similarly. $\square$

Corollary 4.7. *Any $(x,y,z)\in\mathcal{M}^c_+$ can be transformed to $(3,3,3)$ by an element of the Viète group Γ^c_V. Consequently, $\mathcal{M}^c_+=\Gamma^c_V(3,3,3)$.* □

Proof of Theorem 4.5. It is sufficient to prove that if $v(3,3,3)=(3,3,3)$ for some $v\in\Gamma^c_V$, then $v=\mathrm{id}$. Any element $v\in\Gamma^c_V$ is of the form

$$v=\prod_{k=1}^{n} v^c_{i_k}\quad i_k=1,2,3. \tag{4.3}$$

Define

$$m_j:=m\Big[\Big(\prod_{k=1}^{j} v^c_{i_k}\Big)(3,3,3)\Big],\quad j=1,\dots,n.$$

Define

$$M:=\max_{j=1,\dots,n} m_j,\quad J:=\min\{j\colon m_j=M\}\,.$$

We claim that $v^c_{i_{J+1}}=v^c_{i_J}$. Indeed, the assumption $v_{i_{J+1}}\neq v_{i_J}$ would imply that $m_{J+1}>m_J=M$, which is impossible. Hence, we can decrease the number of factors in (4.3) by two. By repeating the argument, we prove that all the factors in (4.3) cancel.

The same argument shows that the graph $\mathcal{T}^{\text{ext}}$ has no loops. □

Corollary 4.8. *For any i,j the Viète group Γ^c_V acts freely on the set $\mathcal{M}^c_{ij}$ with one orbit.* □

5. Distinguished representatives

5.1. **$*$-Markov group orbit of initial solution.** Let $\Gamma_M I$ be the orbit of the initial solution I of the $*$-Markov equation (2.1) under the action of the $*$-Markov group Γ_M. Any element of $\Gamma_M I$ is a solution of the $*$-Markov equation (2.1), see Proposition 3.8.

Proposition 5.1. *The evaluation morphism $\mathrm{Ev}_{\boldsymbol{s}_o}$ maps the set $\Gamma_M I$ onto the set $\mathcal{M}^c$ of all nonzero solutions of the Markov equation (1.1).*

Proof. If $\boldsymbol{a}\in\Gamma_M I$, then $\mathrm{Ev}_{\boldsymbol{s}_o}(\boldsymbol{a})\in\mathcal{M}^c$, by Proposition 2.1. We check surjectivity. Let $\boldsymbol{x}\in\mathcal{M}^c$. There exists $\gamma\in\Gamma^c_M$ such that $\boldsymbol{x}=\gamma\cdot(3,3,3)$, by Corollary 4.3. There exists $\tilde\gamma\in\Gamma_M$ such that $\varphi_M(\tilde\gamma)=\gamma$, by Proposition 3.14. We have that $\mathrm{Ev}_{\boldsymbol{s}_o}(\tilde\gamma\cdot(s_1,s_2^*,s_1))=\boldsymbol{x}$, by Lemma 3.17. □

5.2. **Initial solution and $*$-Viète group.** Let $p=(a,b,c)\in\mathcal{M}^c_+$. Let $v^p\in\Gamma^c_V$ be the unique element of the Viète group such that $v^p(3,3,3)=(a,b,c)$. Define the *distinguished element* $f^p\in\Gamma_M I$ by the formula

$$f^p=v^pI,$$

where v^p is considered as an element of the $*$-Viète group Γ_V. Notice that $\mathrm{ev}_{\boldsymbol{s}_o}(f^p)=(a,b,c)$.

Lemma 5.2. *For $p',p\in\mathcal{M}^c_+$, let $v^{p',p}\in\Gamma^c_V$ be the unique element such that $v^{p',p}p=p'$. Then $v^{p',p}f^p=f^{p'}$, where $v^{p',p}$ is considered as an element of Γ_V.*

Proof. We have $v^{p',p}f^p=v^{p',p}v^pI=v^{p'}I\,.$ □

Theorem 5.3. *Let $p=(a,b,c)$, $p'=(a',b',c')\in\mathcal{M}^c_+$ be such that the triple p' is a permutation of the triple p. Then $f^{p'}$ is obtained from f^p by the same permutation of coordinates of f^p composed with a transformation from the group G_4.*

Proof. Let $f^p = (f_1^p, f_2^p, f_3^p)$ and $f^{p'} = (f_1^{p'}, f_2^{p'}, f_3^{p'})$. Let ω be the permutation such that the evaluation of $\omega f^{p'}$ is (a,b,c), the same as the evaluation of f^p.

The triple $\omega f^{p'}$ lies in the orbit $\Gamma_M I$. So

$$\omega f^{p'} = v g_4 g_1 g_2 I = v g_4 g_2 I = v \tilde{g}_4 I, \tag{5.1}$$

where $v \in \Gamma_V$, $g_j \in G_j$, $\tilde{g}_4 \in G_4$. Here we may conclude that $g_1 = 1$, since (a,b,c) are positive. We may also conclude that $g_4 g_2 I = \tilde{g}_4 I$ for some $\tilde{g}_4 \in G_4$ since any permutation of coordinates of the initial solution $(s_1, s_1/s_3, s_1)$ can be performed by a transformation from G_4. On the other hand, we also have

$$f^p = vI, \tag{5.2}$$

where v in (5.2) is the same as in (5.1). We also know that $v\tilde{g}_4 = g_4' v$, for some $g_4' \in G_4$, by Proposition 3.9. Hence $\omega f^{p'} = v\tilde{g}_4 I = g_4' v I = g_4' f^p$. This proves the theorem. □

Theorem 5.4. *Let $p = (a,b,c) \in \mathcal{M}^c_+$. Let $p' = (a',b',c') \in \mathcal{M}^c$ be such that (a',b',c') is obtained from (a,b,c) by a permutation and possibly also by change of sign of two coordinates. Let $f' \in \Gamma_M I$ be an element, whose evaluation is p'. Then f' is obtained from f^p by an element of $\langle G_1, G_2, G_4\rangle$.*

Proof. We have $f' = gvI$, where $g \in \langle G_1, G_2, G_4\rangle$ and $v \in \Gamma_V$. The evaluation of vI has to be a permutation of (a,b,c),

$$\mathrm{Ev}_{\boldsymbol{s}_o}(vI) = \sigma(a,b,c), \qquad \sigma \in G_2\,.$$

Hence $vI = f^{\sigma(a,b,c)}$. By Theorem 5.3, $f^{\sigma(a,b,c)} = \mu\sigma f^{(a,b,c)}$, $\mu \in G_4$. Hence $f' = gvI = g\mu\sigma f^{(a,b,c)}$, that proves the theorem. □

6. Reduced polynomials solutions and $*$-Markov polynomials

6.1. **Degrees of a polynomial.** Let $f(s_1,s_2,s_3)$ be a polynomial. We consider two degrees of f: the *homogeneous degree* $d := \deg f$ with respect to weights $(1,1,1)$ and the *quasi-homogeneous degree* $q := \mathrm{Deg} f$ with respect to weights $(1,2,3)$.

Lemma 6.1. *Let $f(s_1,s_2,s_3)$ be a polynomial of homogeneous degree d not divisible by s_3, then*

$$g(s_1,s_2,s_3) := s_3^d\, f\Big(\frac{s_2}{s_3}, \frac{s_1}{s_3}, \frac{1}{s_3}\Big) \tag{6.1}$$

is a polynomial of homogeneous degree d not divisible by s_3. If additionally $f(s_1,s_2,s_3)$ is a quasi-homogeneous polynomial of quasi-homogeneous degree q, then $g(s_1,s_2,s_3)$ is a quasi-homogeneous polynomial of quasi-homogeneous degree $3d-q$.

Proof. If $s_1^{a_1}s_2^{a_2}s_3^{a_3}$ is a monomial entering the polynomial f with a nonzero coefficient, then $s_2^{a_1}s_1^{a_2}s_3^{d-(a_1+a_2+a_3)}$ is a monomial entering g with a nonzero coefficient. Hence g is a polynomial.

The homogeneous degree of $s_2^{a_1}s_1^{a_2}s_3^{d-(a_1+a_2+a_3)}$ equals $d-a_3$. Hence $\deg g \leqslant d$.

Since f is not divisible by s_3, there is a monomial $s_1^{a_1}s_2^{a_2}$ entering f. Hence the monomial $s_2^{a_1}s_1^{a_2}s_3^{d-(a_1+a_2)}$ enters g and has homogeneous degree d. Hence $\deg g = d$.

Since $\deg f = d$, there is a monomial $s_1^{a_1}s_2^{a_2}s_3^{a_3}$ entering f such that $a_1+a_2+a_3 = d$. Then the monomial $s_2^{a_1}s_1^{a_2}s_3^{d-(a_1+a_2+a_3)} = s_2^{a_1}s_1^{a_2}$ enters g and hence g is not divisible by s_3.

If additionally all monomials $s_1^{a_1}s_2^{a_2}s_3^{a_3}$ of f have the property $a_1+2a_2+3a_3 = q$, then the corresponding monomials $s_2^{a_1}s_1^{a_2}s_3^{d-(a_1+a_2+a_3)}$ of g have the property $2a_1 + a_2 + 3(d - (a_1 + a_2 + a_3)) = 3d - q$. $\square$

The polynomial g will be denoted by $\mu(f)$. Clearly

$$\mathrm{ev}_{\boldsymbol{s}_o}(f) = \mathrm{ev}_{\boldsymbol{s}_o}(g), \qquad \mu^2(f) = f. \tag{6.2}$$

The polynomials f, g are called *dual*. The *bi-degree vectors* of dual polynomials are

$$(d, q), \qquad (d, 3d-q). \tag{6.3}$$

The linear transformation

$$\mathbb{Z}^2 \to \mathbb{Z}^2, \qquad (d,q) \mapsto (d, 3d-q),$$

is an involution with invariant vector $(2, 3)$ and anti-invariant vector $(0, 1)$.

It is convenient to assign to the polynomial f the 2×2 *degree matrix*

$$M_f = \begin{pmatrix} d & d \\ q & 3d-q \end{pmatrix}, \tag{6.4}$$

whose columns are the bi-degrees of f and $\mu(f)$. Then

$$M_{\mu(f)} = \begin{pmatrix} d & d \\ 3d-q & q \end{pmatrix} = \begin{pmatrix} d & d \\ q & 3d-q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M_f P, \tag{6.5}$$

where P is the permutation matrix.

6.2. **Transformations of triples of polynomials.** Consider a triple $f = (f_1, f_2, f_3)$ of polynomials f_1, f_2, f_3 in s_1, s_2, s_3 such that

(1) each f_j is not divisible by s_3,
(2) each f_j is a quasi-homogeneous polynomial with respect to weights (1,2,3),
(3) denote by (d_j, q_j) the bi-degree vector of f_j, then

$$(d_1, q_1) + (d_3, q_3) = (d_2, q_2). \tag{6.6}$$

Such a triple (f_1, f_2, f_3) is called an *admissible* triple.

Equation (6.6) is equivalent to the equation

$$M_{f_1} + M_{f_3} = M_{f_2}\,. \tag{6.7}$$

Define new triples

$$Lf = (\mu(f_1), \mu(f_1)f_2 - s_3^{d_1} f_3, f_2), \tag{6.8}$$

$$Rf = (f_2, f_2\mu(f_3) - s_3^{d_3} f_1, \mu(f_3)). \tag{6.9}$$

The transformation $f \mapsto Lf$ is called the *left* transformation of an admissible triple f, because of the new first and second terms of Lf are on the left from the surviving term f_2. Similarly the transformation $f \mapsto Rf$ is called the *right* transformation, because of the new second and third terms of Rf are on the right from the surviving term f_2.

Theorem 6.2. *Let $f = (f_1, f_2, f_3)$ be an admissible triple of polynomials with bi-degree vectors $((d_1, q_1), (d_2, q_2), (d_3, q_3))$. Then the triples Lf and Rf are admissible. The bi-degree vectors of Lf are*

$$((d_1, 3d_1 - q_1), (d_1 + d_2, 3d_1 - q_1 + q_2), (d_2, q_2)) \tag{6.10}$$

and the bi-degree vectors of Rf are

$$((d_2, q_2), (d_2 + d_3, q_2 + 3d_3 - q_3), (d_3, 3d_3 - q_3)). \tag{6.11}$$

Proof. Clearly the polynomials $\mu(f_1)f_2 - s_3^{d_1} f_3$, $f_2\mu(f_3) - s_3^{d_3} f_1$ are nonconstant and are not divisible by s_3. The homogeneous degrees of Lf are $(d_1, d_1 + d_2, d_2)$. This follows from Lemma 6.1 and admissibility of the triple f. For the quasi-homogeneous degrees we have

$$\mathrm{Deg}(\mu(f_1)f_2) = 3d_1 - q_1 + q_2 = 3d_1 + q_3 = \mathrm{Deg}(s_3^{d_1} f_2),$$

by Lemma 6.1. Hence $\mu(f_1)f_2 - s_3^{d_1} f_3$ is a quasi-homogeneous polynomial of quasi-homogeneous degree $3d_1 - q_1 + q_2$. The quasi-homogeneous degree of $\mu(f_1)$ is $3d_1 - q_1$. This proves the statement for Lf. The argument for Rf is similar. $\square$

Corollary 6.3. *Let $f = (f_1, f_2, f_3)$ be an admissible triple of polynomials with degree matrices (M_1, M_2, M_3). Then the degree matrices of Lf and Rf are*

$$(M_1P,\ M_1P + M_2,\ M_2), \qquad (M_2,\ M_2 + M_3P,\ M_3P), \tag{6.12}$$

where P is the permutation matrix. $\square$

6.3. **Reduced polynomial solutions.** Any solution of the $*$-Markov equation (2.1) can be written in the form (f_1^*, f_2, f_3^*), where f_1, f_2, f_3 are Laurent polynomials. For any $m_1, m_3 \in \mathbb{Z}$, the triple

$$((s_3^{m_1} f_1)^*, s_3^{m_1+m_3} f_2, (s_3^{m_3} f_3)^*)$$

is also a solution. Given a solution (f_1^*, f_2, f_3^*) there exist unique $m_1, m_3 \in \mathbb{Z}$ such that $s_3^{m_1} f_1$, $s_3^{m_3} f_3$ are polynomials and each of $s_3^{m_1} f_1$, $s_3^{m_3} f_3$ is not divisible by s_3.

A solution (f_1^*, f_2, f_3^*) of (2.1) is called a *reduced polynomial solution* if each of f_1, f_2, f_3 is a nonconstant polynomial in s_1, s_2, s_3 not divisible by s_3. In this case we say that (f_1^*, f_2, f_3^*) is a *reduced polynomial presentation of the Markov triple* $(a, b, c) := \mathrm{Ev}_{\boldsymbol{s}_o}(f_1^*, f_2, f_3^*)$.

For example,

$$(s_1^*, s_1^2 - s_2, s_1^*), \qquad (s_2^*, s_2(s_1^2 - s_2) - s_3 s_1, (s_1^2 - s_2)^*) \tag{6.13}$$

are reduced polynomial presentations of the Markov triples $(3, 6, 3)$ and $(3, 15, 6)$.

Theorem 6.4.

(i) *Let (a, b, c) be a Markov triple, $0 < a < b$, $0 < c < b$, $6 \leqslant b$. Then there exists a* unique *reduced polynomial solution $(f_1^*, f_2, f_3^*) \in \Gamma_M I$, such that $\mathrm{Ev}_{\boldsymbol{s}_o}(f_1^*, f_2, f_3^*) = (a, b, c)$. Moreover, for that reduced polynomial solution (f_1^*, f_2, f_3^*) the triple $f = (f_1, f_2, f_3)$ is admissible.*

(ii) *Let (f_1^*, f_2, f_3^*) be the reduced polynomial presentation of a Markov triple (a, b, c) with $0 < a < b$, $0 < c < b$, $6 \leqslant b$. Denote $f = (f_1, f_2, f_3)$. Let $Lf = (\mu(f_1), \mu(f_1)f_2 -$*

$s_3^{d_1}f_3, f_2)$ and $Rf = (f_2, f_2\mu(f_3) - s_3^{d_3}f_1, \mu(f_3))$ be the left and right transformations of f. Then

$$(\mu(f_1)^*,\ \mu(f_1)f_2 - s_3^{d_1}f_3,\ f_2^*) \tag{6.14}$$

is the reduced polynomial presentation of the Markov triple $(a, ab-c, b)$ and

$$(f_2^*,\ f_2\mu(f_3) - s_3^{d_3}f_1,\ \mu(f_3)^*) \tag{6.15}$$

is the reduced polynomial presentation of the Markov triple $(b, bc-a, c)$.

Proof. First we prove the existence. The proof is by induction on the distance in the Markov tree from (a, b, c) to $(3, 3, 3)$.

Let us find the reduced polynomial presentations in $\Gamma_M I$ for the Markov triples $(3, 6, 3)$, $(3, 15, 6)$. We transform the initial solution as follows,

$$\begin{aligned} I = (s_1, s_2^*, s_1) &\mapsto (s_1^*, s_1^2 - s_2, s_1^*) = (s_2/s_3, s_1^2 - s_2, s_1^*) \\ &\mapsto (s_2, s_1^2 - s_2, (s_3s_1)^*) \mapsto (s_2^*, s_2(s_1^2 - s_2) - s_3s_1, (s_1^2 - s_2)^*). \end{aligned}$$

The triples

$$(s_1^*, s_1^2 - s_2, s_1^*), \qquad (s_2^*, s_2(s_1^2 - s_2) - s_3s_1, (s_1^2 - s_2)^*) \tag{6.16}$$

are desired reduced polynomial presentations of $(3, 6, 3)$ and $(3, 15, 6)$. For example, the polynomials $s_2, s_2(s_1^2-s_2)-s_3s_1, s_1^2-s_2$ are quasi-homogeneous of quasi-homogeneous degrees $(2, 4, 2)$ with $2 + 2 = 4$ as predicted and of homogeneous degrees $(1, 3, 2)$ with $1 + 2 = 3$. These three polynomials form an admissible triple.

Now assume that a Markov triple (a, b, c), $0 < a < b$, $0 < c < b$, $6 \leqslant b$, has a reduced polynomial presentation (f_1^*, f_2, f_3^*), where (f_1, f_2, f_3) is an admissible triple. Then

$$(f_1^*, f_2, f_3^*) = (\mu(f_1)/s_3^{d_1}, f_2, f_3^*) \mapsto (\mu(f_1), f_2, (s_3^{d_1}f_3)^*) \mapsto (\mu(f_1)^*, \mu(f_1)f_2 - s_3^{d_1}f_3, f_2^*) \tag{6.17}$$

and

$$(f_1^*, f_2, f_3^*) = (f_1^*, f_2, \mu(f_3)/s_3^{d_3}) \mapsto ((s_3^{d_3}f_1)^*, f_2, \mu(f_3)) \mapsto (f_2^*, \mu(f_3)f_2 - s_3^{d_3}f_1, \mu(f_3)^*) \tag{6.18}$$

are transformations by elements of the Markov group Γ_M. The triple $(\mu(f_1)^*, \mu(f_1)f_2 - s_3^{d_1}f_3, f_2^*)$ presents the Markov triple $(a, ab-c, b)$, and the triple $(f_2^*, \mu(f_3)f_2 - s_3^{d_3}f_1, \mu(f_3)^*)$ presents the Markov triple $(b, bc-a, c)$. These two triples satisfy the requirements of part (ii) of the theorem.

Let us prove the uniqueness. Let (f_1^*, f_2, f_3^*) and (h_1^*, h_2, h_3^*) be two reduced polynomial presentations of a Markov triple (a, b, c) with $0 < a < b$, $0 < c < b$, $6 \leqslant b$. By Theorem 5.4 (h_1^*, h_2, h_3^*) is obtained from (f_1^*, f_2, f_3^*) by a transformation of the form $g_2g_1g_4$, where $g_i \in G_i$. It is clear that a transformation g_4 cannot be used because it will destroy the property of $f_1f_2f_3$ to be not divisible by s_3. We also cannot use g_1 because it will destroy the fact that (f_1^*, f_2, f_3^*) represents a positive triple (a, b, c). If the numbers a, b, c are all distinct, we cannot use g_2. If $(a, b, c) = (3, 6, 3)$, the presentation $(s_1^*, s_1^2 - s_2, s_1^*)$ is symmetric with respect to the permutation of the first and third coordinates. The theorem is proved. ∎

6.4. **$*$-Markov polynomials.** We say that a polynomial $P(s_1, s_2, s_3)$ is a *$*$-Markov polynomial* if there exists a Markov triple (a, b, c), $0 < a < b$, $0 < c < b$, $6 \leqslant b$, with reduced polynomial presentation $(f_1^*, f_2, f_3^*) \in \Gamma_M I$, such that $P = f_2$.

In particular, this means that P is quasi-homogeneous and is not divisible by s_3.

The polynomial s_2 will also be called a *$*$-Markov polynomial*.

We say that a polynomial $Q(s_1, s_2, s_3)$ is a *dual $*$-Markov polynomial* if Q is not divisible by s_3 and $\mu(Q)$ is a $*$-Markov polynomial.

In particular this means that Q is quasi-homogeneous.

For example, $s_1^2 - s_2$, $s_2(s_1^2 - s_2) - s_3 s_1$ are $*$-Markov polynomials, since they appear as the middle terms in the reduced polynomial presentations in (6.13) and $s_2^2 - s_1 s_3$, $s_1(s_2^2 - s_1 s_3) - s_3 s_2$ are the corresponding dual $*$-Markov polynomials.

Corollary 6.5. *Let (a, b, c) be a Markov triple with $0 < a < b$, $0 < c < b$, $6 \leqslant b$. Let $(f_1^*, f_2, f_3^*) \in \Gamma_M I$ be the reduced polynomial presentation of (a, b, c). Then each of f_1, f_3 is either a $*$-Markov polynomial or a dual $*$-Markov polynomial. Moreover, if $(g_1, g_2, g_3) \in \Gamma_M I$ is any presentation of (a, b, c), then*

$$g_1 = s_3^{k_1} \mu(f_1), \qquad g_2 = s_3^{k_2} f_2, \qquad g_3 = s_3^{k_3} \mu(f_3), \tag{6.19}$$

for some $k_1, k_2, k_3 \in \mathbb{Z}$, and hence each of g_1, g_3 is either a $*$-Markov polynomial or a dual $*$-Markov polynomial multiplied by a power of s_3, and g_2 is a $*$-Markov polynomial multiplied by a power of s_3.

Proof. The first statement follows from Theorem 6.4 and the second statement follows from Theorem 5.4. □

Remark 6.6. Consider the three versions of the $*$-Markov equation,

$$aa^* + bb^* + cc^* - abc = \frac{3s_1 s_2 - s_1^3}{s_3}, \tag{6.20}$$

$$aa^* + bb^* + cc^* - ab^*c = \frac{3s_1 s_2 - s_1^3}{s_3}, \tag{6.21}$$

$$aa^* + bb^* + cc^* - a^*bc^* = \frac{3s_1 s_2 - s_1^3}{s_3}. \tag{6.22}$$

The first of them is the $*$-Markov equation (2.1), the second was considered in the introduction, see (1.4) and [CV20]. The third is a new one. All of the equations are obtained one from another by an obvious change of variables. For example, the third equation is obtained from the first by the change $(a, b, c) \to (a^*, b, c^*)$. That is, if (f_1^*, f_2, f_3^*) is a solution of the $*$-Markov equation (6.20), then (f_1, f_2, f_3) is a solution of equation (6.22).

Hence, by Theorem 6.4, for any Markov triple (a, b, c), $0 < a < b$, $0 < c < b$, $6 \leqslant b$, there exists a polynomial solution (f_1, f_2, f_3) of equation (6.22), such that $\mathrm{Ev}_{\boldsymbol{s}_o}(f_1, f_2, f_3) = (a, b, c)$.

7. Decorated planar binary trees

7.1. **Sets with involution and transformations.** A *set with involution and transformations* is a set S with an involution $\tau : S \to S$, $\tau^2 = \mathrm{id}_S$, a subset $T \subset S \times S \times S$ with a marked point $t^0 = (t_1^0, t_2^0, t_3^0) \in T$ and two maps

$$\begin{aligned} L : T \to T, &\qquad (t_1, t_2, t_3) \mapsto (\tau(t_1), L_2(t_1, t_2, t_3), t_2), \\ R : T \to T, &\qquad (t_1, t_2, t_3) \mapsto (t_2, R_2(t_1, t_2, t_3), \tau(t_3)), \end{aligned} \tag{7.1}$$

where $L_2, R_2 : T \to S$ are some functions.

A *morphism* $\varphi : (S, T, t^0, \tau, L, R) \to (S', T', t^{0'}, \tau', L', R')$ is a map $S \to S'$, which commutes with involutions and induces a map $(T, t^0) \to (T', t^{0'})$ commuting with transformations.

Here are examples.

7.1.1. Let S be the set of all polynomials in $\mathbb{Z}[s_1, s_2, s_3]$ not divisible by s_3 and $T \subset S^3$ the subset of all admissible triples. Let

$$t^0 = (s_2, s_2(s_1^2 - s_2) - s_3 s_1, s_1^2 - s_2), \tag{7.2}$$

$$\tau : S \to S, \qquad f \mapsto \mu(f), \tag{7.3}$$

$$L : T \to T, \qquad (f_1, f_2, f_3) \mapsto (\mu(f_1), \mu(f_1) f_2 - s_3^{d_1} f_3, f_2), \tag{7.4}$$

$$R : T \to T, \qquad (f_1, f_2, f_3) \mapsto (f_2, f_2 \mu(f_3) - s_3^{d_3} f_1, \mu(f_3)), \tag{7.5}$$

where μ is defined in Section 6.1.

7.1.2. Let $S = \mathbb{C}^2$ and $T \subset \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$ the subset of all triples of vectors w_1, w_2, w_3 such that $w_1 + w_3 = w_2$. Let

$$t^0 = ((1,2), (3,4), (2,2)), \tag{7.6}$$

$$\tau : \mathbb{C}^2 \to \mathbb{C}^2, \qquad (d, q) \mapsto (d, 3d - q), \tag{7.7}$$

$$L : T \to T, \qquad (w_1, w_2, w_3) \mapsto (\tau(w_1), \tau(w_1) + w_2, w_2), \tag{7.8}$$

$$R : T \to T, \qquad (w_1, w_2, w_3) \mapsto (w_2, w_2 + \tau(w_3), \tau(w_3)). \tag{7.9}$$

7.1.3. Let S be the set $\mathrm{Mat}(2, \mathbb{C})$ of all 2×2-matrices with complex entries and $T \subset \mathrm{Mat}(2, \mathbb{C})^3$ the subset of all triples of matrices M_1, M_2, M_3 such that $M_1 + M_3 = M_2$. Let

$$t^0 = \left(\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} \right), \tag{7.10}$$

$$\tau : \mathrm{Mat}(2, \mathbb{C}) \to \mathrm{Mat}(2, \mathbb{C}), \qquad M \mapsto MP, \tag{7.11}$$

$$L : T \to T, \qquad (M_1, M_2, M_3) \mapsto (M_1 P, M_1 P + M_2, M_2), \tag{7.12}$$

$$R : T \to T, \qquad (M_1, M_2, M_3) \mapsto (M_2, M_2 + M_3 P, M_3 P). \tag{7.13}$$

7.1.4. Let $S = \mathbb{C}$ and $T \subset \mathbb{C}^3$ the subset of all triples (w_1, w_2, w_3) such that $w_1 + w_3 = w_2$. Let

$$t^0 = (1, -1, -2), \tag{7.14}$$

$$\tau : \mathbb{C} \to \mathbb{C}, \qquad w \mapsto -w, \tag{7.15}$$

$$L : T \to T, \qquad (w_1, w_2, w_3) \mapsto (-w_1, -w_1 + w_2, w_2), \tag{7.16}$$

$$R : T \to T, \qquad (w_1, w_2, w_3) \mapsto (w_2, w_2 - w_3, -w_3). \tag{7.17}$$

7.1.5. Let $S=\mathbb{C}$ and $T=\mathbb{C}^3$. Let

$$\begin{aligned} t^0 &= (3,15,6), && (7.18)\\ \tau &= \mathrm{id}_{\mathbb{C}} && (7.19)\\ L &: T\to T, \qquad (a,b,c)\mapsto (a,ab-c,b), && (7.20)\\ R &: T\to T, \qquad (a,b,c)\mapsto (b,bc-a,c). && (7.21)\end{aligned}$$

7.1.6. Let $S=\mathbb{C}$ and $T=\mathbb{C}^3$. Let

$$\begin{aligned} t^0 &= (1,3,2), && (7.22)\\ \tau &= \mathrm{id}_{\mathbb{C}} && (7.23)\\ L &: T\to T, \qquad (a,b,c)\mapsto (a,a+b,b), && (7.24)\\ R &: T\to T, \qquad (a,b,c)\mapsto (b,b+c,c). && (7.25)\end{aligned}$$

7.1.7. *De-quantization.* Let S be the set with involution and transformations in Example 7.1.1 and S' the set with involution and transformations in Example 7.1.2. The map

$$\varphi : S\to S', \qquad f\mapsto (\deg(f),\mathrm{Deg}(f)),$$

defines a morphism of the sets with involution and transformations.

We may think of that $\varphi : S\to S'$ is a *de-quantization* of the set S with involution and transformations as explained in Section 1.4.3. Namely, Let $s_1=c_1e^{\alpha+\beta}$, $s_2=c_2e^{\alpha+2\beta}$, $s_3=c_3e^{\alpha+3\beta}$, where α,β are real parameters which tend to $+\infty$ and c_1,c_2,c_3 are fixed generic real numbers. If $f(s_1,s_2,s_3)$ is a quasi-homogeneous polynomial of bi-degree (d,q), then $\ln f(c_1e^{\alpha+\beta},c_1e^{\alpha+\beta},c_1e^{\alpha+\beta})$ has *leading term* $d\alpha+q\beta$ independent of the choice of c_1,c_2,c_3, which may be considered as a vector (d,q).

Taking the leading terms of all quasi-homogeneous polynomials in formulas of Example 7.1.1 we obtain the 2-vectors in formulas of Example 7.1.2. For instance, the triple of leading terms of the triple $(s_2, s_2(s_1^2-s_2)-s_3s_1, s_1^2-s_2)$ is the triple $(\alpha+2\beta,\ 3\alpha+4\beta,\ 2\alpha+2\beta)$, cf. (7.2) and (7.6).

7.1.8. Let S be the set with involution and transformations in Example 7.1.1 and S' the set with involution and transformations in Example 7.1.3. The map

$$\varphi : S\to S', \qquad f\mapsto M_f,$$

where M_f see in (6.5), defines a morphism of the sets with involution and transformations.

7.1.9. Let S be the set with involution and transformations in Example 7.1.3 and S' the set with involution and transformations in Example 7.1.4. The map

$$\varphi : S\to S', \qquad \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\mapsto a_{21}-a_{22},$$

defines a morphism of the sets with involution and transformations.

7.1.10. Let S be the set with involution and transformations in Example 7.1.1 and S' the set with involution and transformations in Example 7.1.5. The map

$$\varphi : S\to S', \qquad f\mapsto \mathrm{ev}_{\boldsymbol{s}_o}(f),$$

defines a morphism of the sets with involution and transformations.

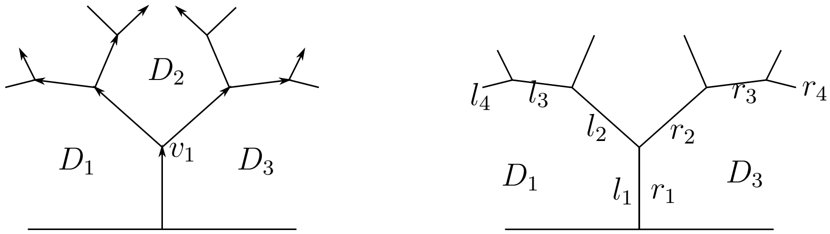

FIGURE 2.

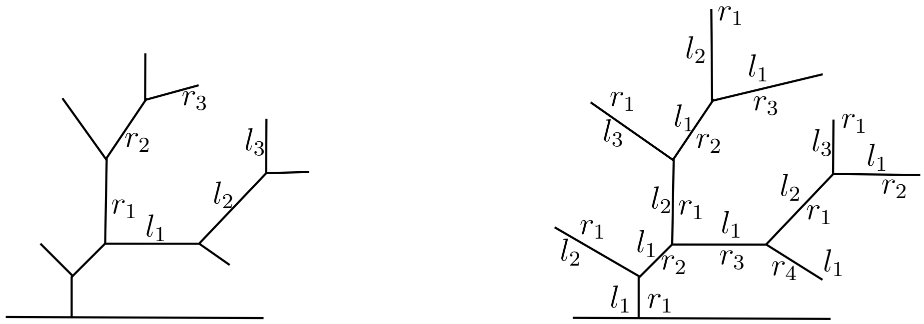

FIGURE 3.

7.1.11. Let S be the set with involution and transformations in Example 7.1.1 and S' the set with involution and transformations in Example 7.1.6. The map

$$\varphi : S \to S', \qquad f \mapsto \deg(f),$$

defines a morphism of the sets with involution and transformations.

7.2. **Planar binary tree.** Consider the *oriented binary planar tree*, growing from floor, and the domains of its complement, see Figure 2. The boundary of any domain of the complement has a distinguished vertex with shortest number of steps to the root along the tree.

There are two initial domains, which touch the floor. In Figure 2 they are D_1 and D_3. The root of the tree is the distinguished vertex of the two initial domains.

The boundary of the left initial domain D_1 consists of the left half-floor and the infinite sequence of edges $l_1, l_2, \dots$, see Figure 2. In the notation l_k, the letter l means that the domain D_1 is on the left from the edge, when we move from the root to this edge along the tree, and k means that it is the k-th edge counted from the root of the tree.

The boundary of the right initial domain D_3 consists of the right half-floor and the infinite sequence of edges $r_1, r_2, \dots$, see Figure 2.

The boundary of any other domain consists of two infinite sequences of edges $r_1, r_2, \dots$ and $l_1, l_2, \dots$, see Figure 3.

Every edge of the tree gets two labels, a label l_a from the left and a label r_b from the right. We denote such an edge with labels by $l_a|r_b$. The first edge of the tree has labels $l_1|r_1$. All other edges of the tree have labels

$$l_1|r_k \qquad \text{or} \qquad l_k|r_1 \qquad \text{with} \qquad k > 1,$$

FIGURE 4.

see Figure 3.

7.3. **Decorations.** Let (S, T, t^0, τ, L, R) be a set with involution and transformations. First we assign an element of the set T to every vertex of the planar binary tree different from the root vertex, and then assign an element of the set S to every domain of the complement. Thus the decoration procedure consists of two step.

Denote by v_1 the vertex of the tree surrounded by the domains D_1, D_2, D_3 in Figure 2. We assign to the vertex v_1 the marked triple $t^0 = (t_1^0, t_2^0, t_3^0)$.

Let v_2 be any other vertex of the tree different from the root. Let p be the path connecting v_1 and v_2 in the tree. The path is a sequence of turns $p_n p_{n-1} \dots p_2 p_1$, where p_j is the turn to the left or right on the way from v_1 to v_2. We assign to v_2 the element $t \in T$ obtained from t^0 by the application of the sequence of transformations L and R, where we apply L if p_j is the turn to the left and apply R if p_j is the turn to the right. For example, the element LRt^0 is assigned to the vertex v_2 in Figure 4.

This is the end of the first step of the decoration.

At the second step we assign to the initial domains D_1, D_3 in Figure 2 the elements t_1^0, t_3^0, respectively, where t_1^0, t_3^0 are the first and third coordinates of the initial triple (t_1^0, t_2^0, t_3^0).

Let C be any domain of the complement different from D_1, D_3. Let v be the distinguished vertex of the domain C, and $t = (t_1, t_2, t_3)$ the element of T assigned to v. We assign to C the element t_2.

For example we assign the element t_2^0 to the domain D_2 in Figure 2.

This is the end of the decoration procedure.

The decoration associated with (S, T, t^0, τ, L, R) is functorial with respect to morphisms of sets with involution and transformations.

Let us describe how to recover the element of T assigned to a vertex from the elements of S assigned to the domains of the complement.

Theorem 7.1. *Let v be a vertex surrounded by domains C_1, C_2, C_3 as in Figure 4. Let t_1, t_2, t_3 be elements of S assigned to C_1, C_2, C_3, respectively, at the second step of the decoration. Let the edge entering the vertex v has labels $l_a|r_b$. Then the element $(\tau^{a-1}(t_1), t_2, \tau^{b-1}(t_3))$ is an element of the set T and that element was assigned to v at the first step of the decoration.*

Proof. The proof is by induction on the distance from v to the root. $\square$

7.4. **Examples.**

7.4.1. Let (S, T, t^0, τ, L, R) be the set of Example 7.1.1. Then the domains of the complement to the binary tree are labeled by $*$-Markov polynomials. The resulting decorated tree is called the *$*$-Markov polynomial tree*, see Figure 5.

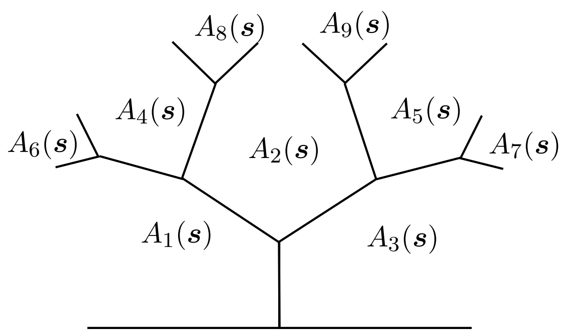

FIGURE 5.

The polynomials $A_i(\boldsymbol{s})$ are given by the formulas

$$\begin{aligned}
&A_1(\boldsymbol{s}) = s_2,\\
&A_2(\boldsymbol{s}) = s_2(s_1^2 - s_2) - s_1 s_3,\\
&A_3(\boldsymbol{s}) = s_1^2 - s_2,\\
&A_4(\boldsymbol{s}) = s_1^3 s_2 - s_1 s_2^2 - 2s_1^2 s_3 + s_2 s_3,\\
&A_5(\boldsymbol{s}) = s_1^2 s_2^3 - s_2^4 - s_1^3 s_2 s_3 + s_1^2 s_3^2 - s_2 s_3^2,\\
&A_6(\boldsymbol{s}) = s_1^3 s_2^2 - s_1 s_2^3 - 3s_1^2 s_2 s_3 + 2s_2^2 s_3 + s_1 s_3^2,\\
&A_7(\boldsymbol{s}) = s_1^4 s_2^3 - 2s_1^2 s_2^4 + s_2^5 - s_1^5 s_2 s_3 + s_1^3 s_2^2 s_3 + s_1^4 s_3^2 - 3s_1^2 s_2 s_3^2 + 2s_2^2 s_3^2 + s_1 s_3^3,\\
&A_8(\boldsymbol{s}) = s_1^4 s_2^3 - s_1^2 s_2^4 - s_1^5 s_2 s_3 - 2s_1^3 s_2^2 s_3 + 2s_1 s_2^3 s_3 + 2s_1^4 s_3^2 + s_1^2 s_2 s_3^2 - s_2^2 s_3^2 - s_1 s_3^3,\\
&A_9(\boldsymbol{s}) = s_1^3 s_2^5 - s_1 s_2^6 - 2s_1^4 s_2^3 s_3 + s_2^5 s_3 + s_1^5 s_2 s_3^2 + 2s_1^3 s_2^2 s_3^2 - s_1 s_2^3 s_3^2 - s_1^4 s_3^3 + s_1 s_3^4.
\end{aligned}$$

Let v be any vertex. It enters the boundary of three domains, which we denote by C_1, C_2, C_3 as in Figure 4. Let f_1, f_2, f_3 be the $*$-Markov polynomials, assigned to the domains C_1, C_2, C_3, respectively at the second step of the decoration. Let the edge entering the vertex v have labels $l_a|r_b$. Then the triple of polynomials $(\tau^{a-1}(f_1), f_2, \tau^{b-1}(f_3))$ is assigned to v at the first step of decoration, and the triple of polynomials

$$((\tau^{a-1}(f_1))^*, f_2, (\tau^{b-1}(f_3))^*)$$

is a reduced polynomial solution of the $*$-Markov equation (2.1).

7.4.2. Let (S, T, t^0, τ, L, R) be the set of Example 7.1.2. Then the domains of the complement to the binary tree are labeled by 2-vectors with positive integer coordinates. The resulting decorated tree is called the *2-vector tree*, see Figure 6.

7.4.3. Let (S, T, t^0, τ, L, R) be the set of Example 7.1.3. Then the domains of the complement to the binary tree are labeled by 2×2-matrices with positive integer coordinates. The resulting decorated tree is called the *matrix tree*, see Figure 7.

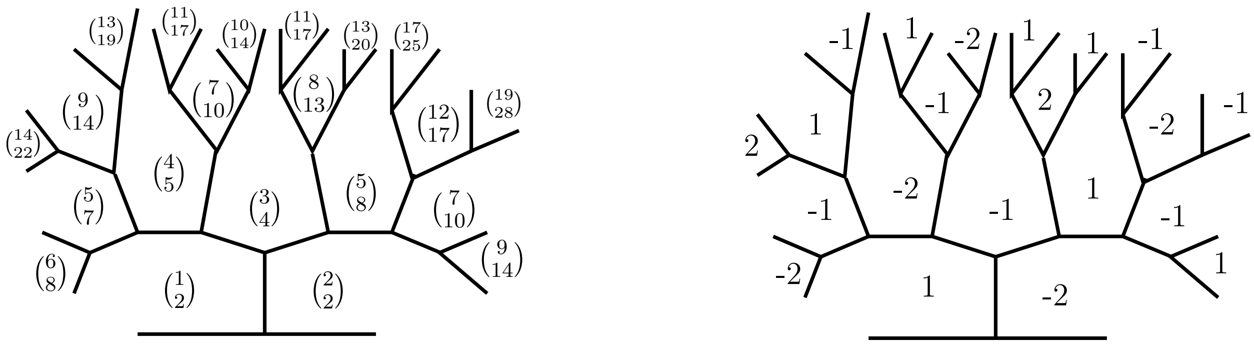

FIGURE 6.

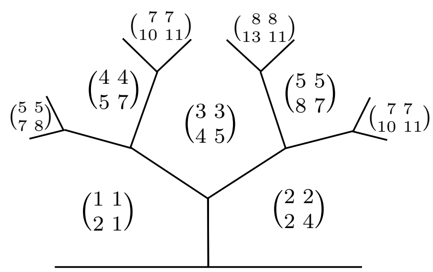

FIGURE 7.

7.4.4. Let (S, T, t^0, τ, L, R) be the set of Example 7.1.4. Then the domains of the complement to the binary tree are labeled by integers. The resulting decorated tree is called the *deviation tree*, see Figure 6.

7.4.5. Let (S, T, t^0, τ, L, R) be the set of Example 7.1.5. Then the domains of the complement to the binary tree are labeled by Markov numbers. The resulting decorated tree is called the *Markov tree*, see the left picture in Figure 8.

7.4.6. Let (S, T, t^0, τ, L, R) be the set of Example 7.1.6. Then the domains of the complement to the binary tree are labeled by positive integers. The resulting decorated tree is called the *Euclid tree*, see the right picture in Figure 8.

The decorated trees in Figures 6-8 can be obtained from the $*$-Markov polynomial tree in Figure 5. Namely the 2-vector tree is obtained by taking the bi-degree vectors of $*$-Markov polynomials; the matrix tree is obtained by taking the degree matrices of $*$-Markov polynomials; the deviation tree is obtained by assigning to a $*$-Markov polynomial with bi-degree (d, q) the number

$$q - (3d - q) = 2q - 3d\,;$$

the Markov tree is obtained by applying the evaluation map $\mathrm{ev}_{\boldsymbol{s}_o}$; the Euclid tree is obtained by taking the homogeneous degrees of $*$-Markov polynomials.

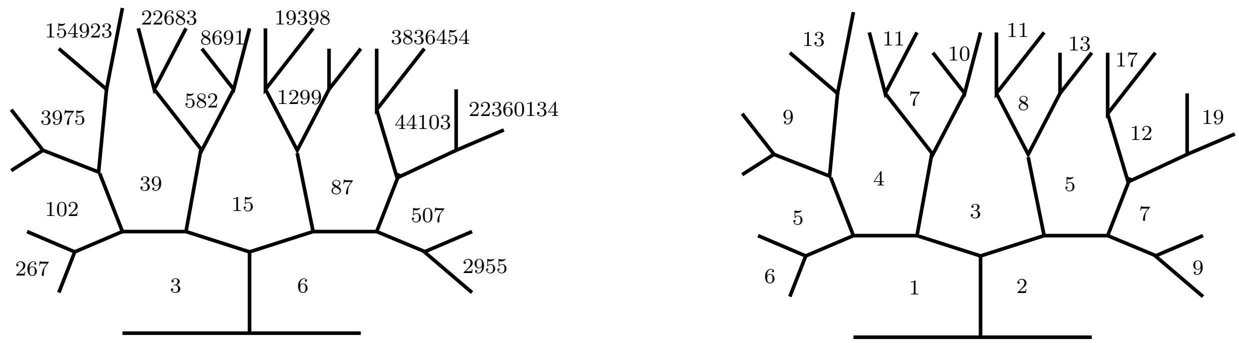

FIGURE 8.

7.5. **Do asymptotics exist?** Having a decorated tree it would be interesting to study asymptotics of the triples assigned to vertices along the infinite paths in the tree going from root to infinity. In [SoV19, SpV17, SpV18] the Markov and Euclid trees were considered. For any such a path the Lyapunov exponent was defined. The Lyapunov function on the space of paths was studied. Relations with hyperbolic dynamics were established.

The interrelations of the triples assigned to vertices of the Markov and Euclid trees were analyzed in [Za82] to study the growth of Markov numbers ordered in the increasing order. More precisely, if (u, v, w) is a Euclid triple with $u + w = v$, then the triple

$$a = 2\cosh\, u, \qquad b = 2\cosh\, v, \qquad c = 2\cosh\, w \tag{7.26}$$

is a solution of the modification of the Markov equation

$$a^2 + b^2 + c^2 - abc = 4, \tag{7.27}$$

considered by Mordell [Mo53]. This observation was used in [Za82] to evaluate asymptotics of Markov numbers in terms of asymptotics of Euclid numbers, see [SpV17].

Combining these remarks we observe a full circle of relations. We started with Markov triples and upgraded them to triples of $*$-Markov polynomials; taking the homogeneous degrees of $*$-Markov polynomials we obtained the Euclid triples; formulas (7.26) send us to triples solving the modified Markov equation (7.27); and the triples solving equation (7.27) approximate the true Markov triples. This circle of relations is a combination of "quantizations" and "de-quatizations".

7.6. **Values of $2q - 3d$.**

Theorem 7.2. *Let P be a $*$-Markov polynomial of bi-degree (d, q). Then $|2q - 3d| = 1$ if d is odd and $|2q - 3d| = 2$ if d is even. Moreover, the only triples of integers attached to vertices of the deviation tree are the elements of the set*

$$T^0 = \big\{(1,-1,-2),\ (-1,1,2),\ (-2,-1,1),\ (2,1,-1),\ (1,2,1),\ (-1,-2,-1)\big\}\,.$$

Proof. The statement is true for the triple $t^0 = (1, -1, -2)$ assigned to the first vertex of the deviation tree, see (7.14). It is easy to check that the set T^0 is preserved by the L and R transformations in formulas (7.16) and (7.17). This proves the theorem. $\square$

Corollary 7.3. *We have*

$$q = \frac{3d}{2} + \mathcal{O}(1) \quad \text{as} \quad d \to \infty. \tag{7.28}$$

□

7.7. **Newton polygons.** Let P be a $*$-Markov polynomial of bi-degree (d, q). Let N_P be the *Newton* polytope of P. Recall that for each monomial $s_1^{a_1} s_2^{a_2} s_3^{a_3}$, entering P with nonzero coefficient, we mark the point $(a_1, a_2, a_3) \in \mathbb{R}^3$, and the Newton polytope is the convex hull of marked points.

Since P is a quasi-homogeneous polynomial of degree q, the Newton polytope is a two-dimensional convex polygon, lying inside the *bounding polygon* $N_{d,q}$,

$$N_{d,q} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 + 2a_2 + 3a_3 = q;\ 0 \leqslant a_1, a_2, a_3 \leqslant d\}. \tag{7.29}$$

We divide all coordinates by d and obtain the *normalized Newton* polygon $\bar{N}_P$ inside the *normalized bounding* polygon $\bar{N}_{d,q}$,

$$\bar{N}_{d,q} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 + 2a_2 + 3a_3 = q/d;\ 0 \leqslant a_1, a_2, a_3 \leqslant 1\}. \tag{7.30}$$

It is convenient to project the polygons $\bar{N}_P$ and $\bar{N}_{d,q}$ along the a_3-axis to $\mathbb{R}^2$ with coordinates a_1, a_2 and obtain the *projected normalized Newton polygon* $\tilde{N}_P$ inside the *projected normalized bounding polygon* $\tilde{N}_{d,q}$.

7.8. **Limit $d \to \infty$.** The Euclid tree shows the distribution of the homogeneous degrees of $*$-Markov polynomials. The homogeneous degree d tends to infinity along the paths of the planar binary tree from root to infinity. Along these paths we have $q \to 3d/2$. In this limit the normalized bounding polygon $\bar{N}_{d,q}$ turns into the *quadrilateral* $\bar{N}_\infty$,

$$\bar{N}_\infty = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 + 2a_2 + 3a_3 = 3/2;\ 0 \leqslant a_1, a_2, a_3 \leqslant 1\}, \tag{7.31}$$

and the projected normalized bounding polygon $\tilde{N}_{d,q}$ turns into the *projected quadrilateral* $\tilde{N}_\infty$, the convex quadrilateral with vertices $(0, 0), (3/4, 0), (1/2, 1/2), (0, 3/4)$. See the pictures of $\bar{N}_\infty$ and $\tilde{N}_\infty$ in Figure 9.

Question. Could it be that for any infinite path from root to infinity, the projected normalized Newton polygon $\tilde{N}_P$ tends in an appropriate sense to a limiting shape inside the projected quadrilateral $\tilde{N}_\infty$?

We show that this is indeed so in the two examples of the left and right paths of the planar binary tree, which are related to the $*$-Fibonacci and $*$-Pell polynomials discussed in Sections 8 and 9. In the first case the limiting shape is the interval with vertices $(0, 0)$, $(1/2, 1/2)$, see Section 8.3. In the second case the limiting shape is the whole projected quadrilateral $\tilde{N}_\infty$, see Section 9.3.

7.8.1. Any $*$-Markov polynomial P has a monomial of the form $s_1 s_3^{a_3}$ or $s_2 s_3^{a_3}$ entering P with a nonzero coefficient and has no monomials of the form $s_3^{a_3}$. This easily follows by induction. Hence for any infinite path from root to infinity the point $(0, 0)$ is a limiting point of the projected normalized Newton polygon $\tilde{N}_P$.

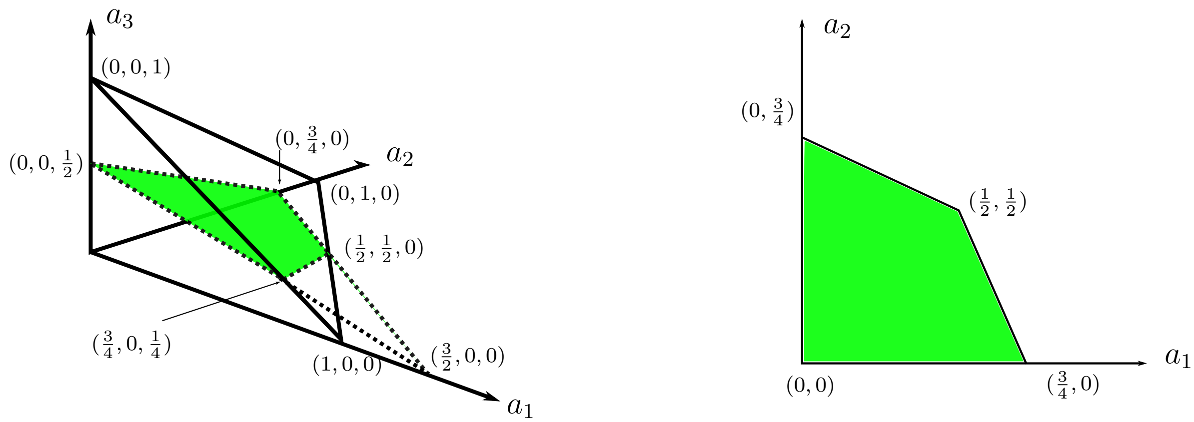

FIGURE 9.

7.8.2. Elementary computer experiments show that the expected limiting shape of the polygon $\tilde{N}_P$ along an infinite path is a 6-gon like in Figure 10, with width monotonically increasing from 0, for the $*$-Fibonacci polynomials, to the maximal value, for the $*$-Pell polynomials, when the path changes from the leftmost to the rightmost. This 6-gon is symmetric with respect to the diagonal $a_1 = a_2$, and hence its width completely determines the 6-gon. It looks like the speed of convergence to the limiting shape increases if the path has many changes of direction from left to right and back.

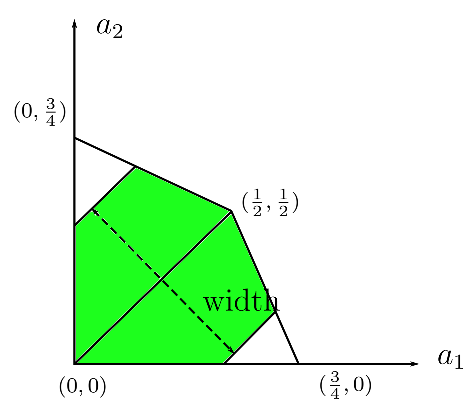

FIGURE 10.

7.9. **Planar binary tree decorated by convex sets.** The study of the limiting shapes of Newton polygons is closely related to the following decorated planar binary tree.

7.9.1. Consider $\mathbb{R}^2$ with coordinates a_1, a_2. For a subset $A \subset \mathbb{R}^2$ we denote by $\operatorname{conv}[A]$ the convex hull of A. We denote by $\mu(A)$ the subset A reflected with respect to the diagonal $a_1 = a_2$. For $d \in \mathbb{R}_{>0}$, we denote

$$dA = \{dv \mid v \in A\}.$$

For subsets $A, B \subset \mathbb{R}^2$ we denote by $A + B$ the Minkowski sum,

$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$

7.9.2. Define a set (S, T, t^0, τ, L, R) with involution and transformations. Let S be the set of all pairs (A, d), where A is a convex subset of $\mathbb{R}^2$ and d a positive number. Define the involution τ by the formula

$$\tau : S \to S, \qquad (A, d) \mapsto (\mu(A), d).$$

Let $T \subset S^3$ be the subset of all triples $((A_1, d_1), (A_2, d_2), (A_3, d_3))$ such that $d_2 = d_1 + d_3$. We fix the initial triple

$$t^0 = ((A_1^0, 1), (A_2^0, 3), (A_3^0, 2)) \in T, \tag{7.32}$$

where A_1^0 is the point $(0, 1)$, A_3^0 is the interval with vertices $(1, 0), (0, 1/2)$, and A_2^0 is the triangle with vertices $(1/3, 0), (2/3, 1/3), (0, 2/3)$.

Define the left and right transformations by the formulas

$$L \ :\ T \to T,\ ((A_1, d_1), (A_2, d_2), (A_3, d_3)) \mapsto ((\mu(A_1), d_1), (L_2, d_1 + d_2), (A_2, d_2)), \tag{7.33}$$
$$R \ :\ T \to T,\ ((A_1, d_1), (A_2, d_2), (A_3, d_3)) \mapsto ((A_2, d_2), (R_2, d_2 + d_3), (\mu(A_3), d_3)), \tag{7.34}$$

where

$$\begin{aligned} L_2 &= \operatorname{conv}\left[\left(\frac{d_1}{d_1 + d_2}\mu(A_1) + \frac{d_2}{d_1 + d_2}A_2\right) \cup \frac{d_3}{d_1 + d_2}A_3\right], \\ R_2 &= \operatorname{conv}\left[\left(\frac{d_2}{d_2 + d_3}A_2 + \frac{d_3}{d_2 + d_3}\mu(A_3)\right) \cup \frac{d_1}{d_2 + d_3}A_1\right]. \end{aligned} \tag{7.35}$$

Cf. (7.4) and (7.5).

Having this set with involution and transformations we may consider the associated decorated planar binary tree. The problem is to study the asymptotics of triples of convex subsets of $\mathbb{R}^2$ along the paths of the tree. Such asymptotics reflect the asymptotics of the Newton polygons of the $*$-Markov polynomials.

Remark 7.4. The triple of convex sets in (7.32) are the projected normalized Newton polygons of the triple $(s_2, s_2(s_1^2 - s_2) - s_3 s_1, s_1^2 - s_2)$.

The L and R transformations in (7.33) and (7.34) are just the reformulations of the L and R transformations of polynomials in (7.4) and (7.5) in the language of the their projected normalized Newton polygons.

8. Odd $*$-Fibonacci Polynomials

8.1. **Definition of odd $*$-Fibonacci polynomials.** The left boundary path of the Markov tree corresponds to the sequence of Markov triples $(3, 15, 6), (3, 39, 15), (3, 102, 19), \dots$, with general term $(3, 3\varphi_{2n+1}, 3\varphi_{2n-1})$, where $\varphi_{2n+1}, \varphi_{2n-1}$ are odd Fibonacci numbers,

$$\varphi_1 = 1, \quad \varphi_3 = 2, \quad \varphi_5 = 5, \quad \varphi_7 = 13, \quad \varphi_9 = 34, \quad \dots \quad ,$$

with recurrence relation

$$\varphi_{2n+3} = 3\varphi_{2n+1} - \varphi_{2n-1}. \tag{8.1}$$

We define the *odd $*$-Fibonacci polynomials* recursively by the formula

$$F_1(\boldsymbol{s}) = s_1, \qquad F_3(\boldsymbol{s}) = s_1^2 - s_2, \tag{8.2}$$
$$F_{2n+3}(\boldsymbol{s}) = g_n F_{2n+1}(\boldsymbol{s}) - s_3 F_{2n-1}(\boldsymbol{s}), \tag{8.3}$$

where $g_n = s_2$ if n is odd, and $g_n = s_1$ if n is even. In other words we have

$$\begin{aligned} F_{4n+3} &= s_1 F_{4n+1} - s_3 F_{4n-1}\,, \\ F_{4n+5} &= s_2 F_{4n+3} - s_3 F_{4n+1}\,. \end{aligned} \tag{8.4}$$

Lemma 8.1. *We have* $\mathrm{ev}_{\boldsymbol{s}_o}(F_{2n+1}) = 3\varphi_{2n+1}$. □

The first odd $*$-Fibonacci polynomials are

$$\begin{aligned}
F_1(\boldsymbol{s}) &= s_1, \\
F_3(\boldsymbol{s}) &= s_1^2 - s_2, \\
F_5(\boldsymbol{s}) &= s_1^2 s_2 - s_1 s_3 - s_2^2, \\
F_7(\boldsymbol{s}) &= s_1^3 s_2 - 2 s_1^2 s_3 - s_1 s_2^2 + s_2 s_3, \\
F_9(\boldsymbol{s}) &= s_1^3 s_2^2 - 3 s_1^2 s_2 s_3 - s_1 s_2^3 + s_1 s_3^2 + 2 s_2^2 s_3, \\
F_{11}(\boldsymbol{s}) &= s_1^4 s_2^2 - 4 s_1^3 s_2 s_3 - s_1^2 s_2^3 + 3 s_1^2 s_3^2 + 3 s_1 s_2^2 s_3 - s_2 s_3^2, \\
F_{13}(\boldsymbol{s}) &= s_1^4 s_2^3 - 5 s_1^3 s_2^2 s_3 - s_1^2 s_2^4 + 6 s_1^2 s_2 s_3^2 - s_1 s_3^3 + 4 s_1 s_2^3 s_3 - 3 s_2^2 s_3^2\,.
\end{aligned}$$

Theorem 8.2. *For $n > 1$ the triple*

$$(g_{n-1}^*, F_{2n+1}, F_{2n-1}^*) \tag{8.5}$$

is the reduced polynomial presentation of the Markov triple $(3, 3\varphi_{2n+1}, 3\varphi_{2n-1})$.

Proof. The proof is by induction on n. The statement is true for $n = 2$, since

$$(g_1^*, F_5, F_3^*) = (s_2^*, s_2(s_1^2 - s_2) - s_3 s_1, (s_1^2 - s_2)^*)$$

is the reduced polynomial presentation of the Markov triple $(3, 15, 6)$, see (6.13).

Assume that $(g_{n-1}^*, F_{2n+1}, F_{2n-1}^*)$ is the reduced polynomial presentation of the Markov triple $(3, 3\varphi_{2n+1}, 3\varphi_{2n-1})$. Denote $f = (f_1, f_2, f_3) := (g_{n-1}, F_{2n+1}, F_{2n-1})$. Let Lf be the triple defined in Theorem 6.4. By Theorem 6.4 the triple

$$\begin{aligned}
Lf &= (\mu(g_{n-1}), \mu(g_{n-1}) F_{2n+1} - s_3 F_{2n-1}, F_{2n+1}) \\
&= (g_n, g_n F_{2n+1} - s_3 F_{2n-1}, F_{2n+1}) \\
&= (g_n, F_{2n+3}, F_{2n+1})
\end{aligned}$$

is such that the triple $(g_n^*, F_{2n+3}, F_{2n+1}^*)$ is the reduced polynomial presentation of the Markov triple

$$(3, 9\varphi_{2n+1} - 3\varphi_{2n-1}, 3\varphi_{2n+1}) = (3, 3(3\varphi_{2n+1} - \varphi_{2n-1}), 3\varphi_{2n+1}) = (3, 3\varphi_{2n+3}, 3\varphi_{2n+1}).$$

This proves the theorem. □

Corollary 8.3. *The odd $*$-Fibonacci polynomials are $*$-Markov polynomials.* □

Remark 8.4. There are many q-deformations of (odd) Fibonacci numbers. For example, S. Morier-Genoud and V. Ovsienko [MO20] consider the odd Fibonacci polynomials $f_{2k+1}(q)$, defined by the relations

$$f_1(q) = q^{-1}, \qquad f_3(q) = 1+q, \tag{8.6}$$

$$f_{2n+3} = (1+q+q^2)f_{2n+1} - q^2 f_{2n-1}. \tag{8.7}$$

As V. Ovsienko informed us, our recurrence relation (8.3) turns into relation (8.7) under the specification $s_1 = s_2 = 1+q+q^2, s_3 = q^2$. Our initial conditions (8.2) turn into $1+q+q^2$, $(1+q+q^2)(q+q^2)$. Hence for any k the odd $*$-Fibonacci polynomials $F_{2k+1}(\boldsymbol{s})$ evaluated at $s_1 = s_2 = 1+q+q^2, s_3 = q^2$ equals $(q+q^2)f_{2k+1}(q)$.

8.2. **Formula for odd $*$-Fibonacci polynomials.**

Theorem 8.5. *For $n \geqslant 0$, we have*

$$F_{4n+1}(\boldsymbol{s}) = s_1 \sum_{i=0}^{n} \binom{2n-i}{i}(-s_3)^i (s_1 s_2)^{n-i} - s_2 \sum_{i=0}^{n-1} \binom{2n-1-i}{i}(-s_3)^i s_1^{n-1-i} s_2^{n-i}\,, \tag{8.8}$$

$$F_{4n+3}(\boldsymbol{s}) = s_1 \sum_{i=0}^{n} \binom{2n+1-i}{i}(-s_3)^i s_1^{n+1-i} s_2^{n-i} - s_2 \sum_{i=0}^{n} \binom{2n-i}{i}(-s_3)^i (s_1 s_2)^{n-i}\,. \tag{8.9}$$

Proof. The proof is by induction. The formulas correctly reproduce F_1, F_3. Then $s_1 F_{4n+1} - s_3 F_{4n-1}$ equals

$$s_1\Big(s_1 \sum_{i=0}^{n} \binom{2n-i}{i}(-s_3)^i (s_1 s_2)^{n-i} \;-\; s_2 \sum_{i=0}^{n-1} \binom{2n-1-i}{i}(-s_3)^i s_1^{n-1-i} s_2^{n-i}\Big)$$
$$-s_3\Big(s_1 \sum_{i=0}^{n-1} \binom{2n-1-i}{i}(-s_3)^i s_1^{n-i} s_2^{n-1-i} \;-\; s_2 \sum_{i=0}^{n-1} \binom{2n-2-i}{i}(-s_3)^i (s_1 s_2)^{n-1-i}\Big).$$

We have

$$s_1^2 \sum_{i=0}^{n} \binom{2n-i}{i}(-s_3)^i (s_1 s_2)^{n-i} - s_3 s_1 \sum_{i=0}^{n-1} \binom{2n-1-i}{i}(-s_3)^i s_1^{n-i} s_2^{n-1-i}$$
$$= s_1 \sum_{i=0}^{n} \binom{2n+1-i}{i}(-s_3)^i s_1^{n+1-i} s_2^{n-i}$$

and

$$s_1 s_2 \sum_{i=0}^{n-1} \binom{2n-1-i}{i}(-s_3)^i s_1^{n-1-i} s_2^{n-i} - s_3 s_2 \sum_{i=0}^{n-1} \binom{2n-2-i}{i}(-s_3)^i (s_1 s_2)^{n-1-i}$$
$$= s_2 \sum_{i=0}^{n} \binom{2n-i}{i}(-s_3)^i (s_1 s_2)^{n-i}\,.$$

Hence, $s_1 F_{4n+1} - s_3 F_{4n-1} = F_{4n+3}$. The other identity is proved similarly. □

Corollary 8.6. *For the ordinary Fibonacci integers we have formulae*

$$\varphi_{4n+1} = \sum_{i=0}^{n} \binom{2n-i}{i} (-1)^i 3^{2n-2i} - \sum_{i=0}^{n-1} \binom{2n-1-i}{i} (-1)^i 3^{2n-1-2i}, \tag{8.10}$$

$$\varphi_{4n+2} = \sum_{i=0}^{n} \binom{2n-i}{i} (-1)^i 3^{2n+1-2i} - 2\sum_{i=0}^{n} \binom{2n-i}{i} (-1)^i 3^{2n-2i}, \tag{8.11}$$

$$\varphi_{4n+3} = \sum_{i=0}^{n} \binom{2n+1-i}{i} (-1)^i 3^{2n+1-2i} - \sum_{i=0}^{n} \binom{2n-i}{i} (-1)^i 3^{2n-2i}, \tag{8.12}$$

$$\varphi_{4n+4} = \sum_{i=0}^{n} \binom{2n+1-i}{i} (-1)^i 3^{2n+1-2i}. \tag{8.13}$$

Proof. Formulae (8.10) and (8.12) follow from (8.8) and (8.9). Formulae (8.11) and (8.13) easily follow from the identities

$$\varphi_{4n+2} = \varphi_{4n+3} - \varphi_{4n+1}, \quad \varphi_{4n+4} = \varphi_{4n+3} + \varphi_{4n+2}. \qquad \square$$

8.3. **Newton polygons of odd $*$-Fibonacci polynomials.**

Lemma 8.7. *The odd $*$-Fibonacci polynomials F_{4n+1} and F_{4n+3} are of bi-degree $(2n+1, 3n+1)$ and $(2n+2, 3n+2)$, respectively.* $\square$

The Newton polygon $N_{F_{4n+1}}$ of F_{4n+1} is the convex hull of four points $(n+1, n, 0), (1, 0, n), (n-1, n+1, 0), (0, 2, n-1)$. The projected normalized Newton polygon $\tilde{N}_{F_{4n+1}}$ is the convex hull of four points

$$\Big(\frac{n+1}{2n+1}, \frac{n}{2n+1}\Big), \Big(\frac{1}{2n+1}, 0\Big), \Big(\frac{n-1}{2n+1}, \frac{n+1}{2n+1}\Big), \Big(0, \frac{2}{2n+1}\Big).$$

The limit of $\tilde{N}_{F_{4n+1}}$ as $n \to \infty$ is the interval with vertices $(0,0)$ and $(1/2, 1/2)$.

The Newton polygon $N_{F_{4n+3}}$ is the convex hull of four points $(n+2, n, 0), (2, 0, n), (n, n+1, 0), (0, 1, n)$. The projected normalized Newton polygon $\tilde{N}_{F_{4n+3}}$ is the convex hull of four points

$$\Big(\frac{n+2}{2n+2}, \frac{n}{2n+2}\Big), \Big(\frac{2}{2n+2}, 0\Big), \Big(\frac{n}{2n+2}, \frac{n+1}{2n+2}\Big), \Big(0, \frac{1}{2n+2}\Big).$$

The limit of $\tilde{N}_{F_{4n+3}}$ as $n \to \infty$ is the interval with vertices $(0,0)$ and $(1/2, 1/2)$, see Section 7.8.

8.4. **Generating function.** Introduce the generating power series of odd $*$-Fibonacci polynomials,

$$\mathcal{F}(\boldsymbol{s}, t) := \sum_{n=0}^{\infty} F_{2n+1}(\boldsymbol{s}) t^{2n+1}. \tag{8.14}$$

Theorem 8.8. *We have*

$$\mathcal{F}(\boldsymbol{s}, t) = \frac{s_2 s_3 t^7 + t^5 (s_2^2 - s_1 s_3) + t^3 (s_2 - s_1^2) - s_1 t}{-s_3^2 t^8 - (2s_3 - s_1 s_2) t^4 - 1}. \tag{8.15}$$

Proof. Split the series $\mathcal{F}(\boldsymbol{s},t)$ as follows

$$\mathcal{F}(\boldsymbol{s},t)=\underbrace{\sum_{k=0}^{\infty}F_{4k+1}(\boldsymbol{s})t^{4k+1}}_{\mathcal{F}_1(\boldsymbol{s},t)}+\underbrace{\sum_{n=0}^{\infty}F_{4k+3}(\boldsymbol{s})t^{4k+3}}_{\mathcal{F}_2(\boldsymbol{s},t)}. \tag{8.16}$$

From the recursive relation (8.2), we deduce

$$\underbrace{(1+s_3t^4)\mathcal{F}_1(\boldsymbol{s},t)}_{*}-t^2s_1\mathcal{F}_1(\boldsymbol{s},t)+(1+s_3t^4)\mathcal{F}_2(\boldsymbol{s},t)\underbrace{-t^2s_2\mathcal{F}_2(\boldsymbol{s},t)}_{*}=\underbrace{F_1t}_{*}-F_{-1}s_3t^3,$$

where $F_{-1}=s_2/s_3,\ F_1=s_1$. The terms marked by $*$ have only powers t^{4k+1}, with $k\geqslant 0$. The remaining terms have only powers t^{4k+3}, with $k\geqslant 0$. We have a linear system

$$\begin{pmatrix}1+s_3t^4 & -s_2t^2\\ -s_1t^2 & 1+s_3t^4\end{pmatrix}\begin{pmatrix}\mathcal{F}_1(\boldsymbol{s},t)\\ \mathcal{F}_2(\boldsymbol{s},t)\end{pmatrix}=\begin{pmatrix}F_1t\\ -F_{-1}s_3t^3\end{pmatrix}. \tag{8.17}$$

Hence

$$\mathcal{F}_1(\boldsymbol{s},t)=\frac{(s_1s_3-s_2^2)t^5+s_1t}{(s_3t^4+1)^2-s_1s_2t^4},\quad \mathcal{F}_2(\boldsymbol{s},t)=\frac{s_2s_3t^7+(s_2-s_1^2)t^3}{s_1s_2t^4-(s_3t^4+1)^2}. \tag{8.18}$$

Equation (8.15) follows from (8.16) and (8.18). □

Corollary 8.9. *For any $n\geqslant 0$, we have*

$$F_{2n+1}(\boldsymbol{s})=\frac{1}{(2n+1)!}\left.\frac{\partial^{2n+1}}{\partial t^{2n+1}}\right|_{t=0}\mathcal{F}(\boldsymbol{s},t). \tag{8.19}$$

□

Remark 8.10. At $\boldsymbol{s}=\boldsymbol{s}_o$, the generating function $\mathcal{F}(\boldsymbol{s},t)$ reduces to

$$\mathcal{F}(\boldsymbol{s}_o,t)=\frac{3(t-t^3)}{t^4-3t^2+1}=3\left(t+2t^3+5t^5+13t^7+34t^9+\dots\right),$$

the generating function of odd Fibonacci numbers multiplied by 3.

8.5. **Binet formula for odd $*$-Fibonacci polynomials.** In this section we consider the generating function $\mathcal{F}(\boldsymbol{s},t)$ as a rational function of t with coefficients depending on the parameters $\boldsymbol{s}$ varying in a neighborhood of $\boldsymbol{s}_o$.

The poles of $\mathcal{F}(\boldsymbol{s},t)$ are the roots of the polynomial $s_3^2t^8+(2s_3-s_1s_2)t^4+1$.

We will use the roots of $s_3^2t^2+(2s_3-s_1s_2)t+1$,

$$a_{\pm}(\boldsymbol{s}):=\frac{s_1s_2-2s_3\pm(s_1^2s_2^2-4s_1s_2s_3)^{\frac12}}{2s_3^2} \tag{8.20}$$

with $a_{\pm}(\boldsymbol{s}_0)=\frac{7\pm3\sqrt5}{2}$.

Introduce $\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma_1,\gamma_2$ by the formulas

$$\begin{aligned}&t((s_1s_3-s_2^2)t^4+s_1)=\alpha_1t^5+\alpha_0t,\\ &t^3(-s_2+s_1^2-s_2s_3t^4)=\beta_1t^7+\beta_0t^3,\\ &s_3^2t^8+(2s_3-s_1s_2)t^4+1=\gamma_2t^8+\gamma_1t^4+1.\end{aligned}$$

Then

$$\mathcal{F}_1(\boldsymbol{s},t) = \frac{\alpha_1 t^5 + \alpha_0 t}{\gamma_2 t^8 + \gamma_1 t^4 + 1}, \qquad \mathcal{F}_2(\boldsymbol{s},t) = \frac{\beta_1 t^7 + \beta_0 t^3}{\gamma_2 t^8 + \gamma_1 t^4 + 1}.$$

Theorem 8.11. *We have*

$$\mathcal{F}_1(\boldsymbol{s},t) = -\sum_{k=0}^{\infty}\Big(\frac{\alpha_1 a_+ + \alpha_0}{a_+^{k+1}(2\gamma_2 a_+ + \gamma_1)} + \frac{\alpha_1 a_- + \alpha_0}{a_-^{k+1}(2\gamma_2 a_- + \gamma_1)}\Big)t^{4k+1}, \tag{8.21}$$

$$\mathcal{F}_2(\boldsymbol{s},t) = -\sum_{k=0}^{\infty}\Big(\frac{\beta_1 a_+ + \beta_0}{a_+^{k+1}(2\gamma_2 a_+ + \gamma_1)} + \frac{\beta_1 a_- + \beta_0}{a_-^{k+1}(2\gamma_2 a_- + \gamma_1)}\Big)t^{4k+3}.$$

Proof. We prove the first formula. The proof of the second is similar.

The roots of $s_3^2 t^8 + (2s_3 - s_1 s_2)t^4 + 1$ are

$$b_n(\boldsymbol{s}) := \omega^n a_+(\boldsymbol{s})^{\frac{1}{4}}, \quad b_{4+n}(\boldsymbol{s}) := \omega^n a_-(\boldsymbol{s})^{\frac{1}{4}}, \quad n = 1,2,3,4, \tag{8.22}$$

where $\omega = e^{\pi\sqrt{-1}/2}$. We have $\mathcal{F}_1(\boldsymbol{s},t) = \sum_{n=1}^{8} \frac{A_n(\boldsymbol{s})}{t - b_n(\boldsymbol{s})}$, where

$$A_n(\boldsymbol{s}) = \mathrm{res}_{t=b_n(\boldsymbol{s})}\mathcal{F}_1(\boldsymbol{s},t) = \frac{\alpha_1 b_n(\boldsymbol{s})^4 + \alpha_0}{b_n(\boldsymbol{s})^2(8\gamma_2 b_n(\boldsymbol{s})^4 + 4\gamma_1)}\,.$$

Hence

$$\begin{aligned}\mathcal{F}_1(\boldsymbol{s},t) &= \sum_{n=1}^{8}\frac{A_n(\boldsymbol{s})}{t - b_n(\boldsymbol{s})} = -\sum_{n=1}^{8}\frac{A_n(\boldsymbol{s})}{b_n(\boldsymbol{s})}\frac{1}{1 - \frac{t}{b_n(\boldsymbol{s})}} \\ &= -\sum_{m=0}^{\infty}\sum_{n=1}^{8}\frac{A_n(\boldsymbol{s})}{b_n(\boldsymbol{s})^{m+1}}t^m = -\sum_{m=0}^{\infty}\sum_{n=1}^{8}\frac{\alpha_1 b_n(\boldsymbol{s})^4 + \alpha_0}{b_n(\boldsymbol{s})^{m+3}(8\gamma_2 b_n(\boldsymbol{s})^4 + 4\gamma_1)}t^m \\ &= -\sum_{k=0}^{\infty}\Big(\frac{\alpha_1 a_+(\boldsymbol{s}) + \alpha_0}{a_+(\boldsymbol{s})^{k+1}(2\gamma_2 a_+(\boldsymbol{s}) + \gamma_1)} + \frac{\alpha_1 a_-(\boldsymbol{s}) + \alpha_0}{a_-(\boldsymbol{s})^{k+1}(2\gamma_2 a_-(\boldsymbol{s}) + \gamma_1)}\Big)t^{4k+1}.\end{aligned}$$

□

Corollary 8.12. *We have*

$$F_{4k+1}(\boldsymbol{s}) = -\frac{\alpha_1 a_+ + \alpha_0}{a_+^{k+1}(2\gamma_2 a_+ + \gamma_1)} - \frac{\alpha_1 a_- + \alpha_0}{a_-^{k+1}(2\gamma_2 a_- + \gamma_1)}\,, \tag{8.23}$$

$$F_{4k+3}(\boldsymbol{s}) = -\frac{\beta_1 a_+ + \beta_0}{a_+^{k+1}(2\gamma_2 a_+ + \gamma_1)} - \frac{\beta_1 a_- + \beta_0}{a_-^{k+1}(2\gamma_2 a_- + \gamma_1)}.$$

If $\boldsymbol{s}$ is in a small neighborhood of $\boldsymbol{s}_o$, then $|a_+(\boldsymbol{s})| > |a_-(\boldsymbol{s})|$ and

$$F_{4k+1}(\boldsymbol{s}) \sim -\frac{\alpha_1 a_- + \alpha_0}{a_-^{k+1}(2\gamma_2 a_- + \gamma_1)}, \quad F_{4k+3}(\boldsymbol{s}) \sim -\frac{\beta_1 a_- + \beta_0}{a_-^{k+1}(2\gamma_2 a_- + \gamma_1)}, \tag{8.24}$$

$$s_2\,\frac{F_{4k+3}(\boldsymbol{s})}{F_{4k+1}(\boldsymbol{s})} \sim s_2\,\frac{\beta_1 a_- + \beta_0}{\alpha_1 a_- + \alpha_0}, \qquad s_1\,\frac{F_{4k+5}(\boldsymbol{s})}{F_{4k+3}(\boldsymbol{s})} \sim s_1\,\frac{\alpha_1 a_- + \alpha_0}{a_-(\beta_1 a_- + \beta_0)}, \tag{8.25}$$

as $n \to \infty$.

Lemma 8.13. *We have*

$$s_2\,\frac{\beta_1 a_\pm+\beta_0}{\alpha_1 a_\pm+\alpha_0}=s_1\,\frac{\alpha_1 a_\pm+\alpha_0}{a_\pm(\beta_1 a_\pm+\beta_0)}. \tag{8.26}$$

Proof. The proof is by direct verification. □

8.6. **Odd $*$-Fibonacci polynomials with negative indices.** The relations

$$F_{4n+3}=s_1F_{4n+1}-s_3F_{4n-1}\,,$$
$$F_{4n+5}=s_2F_{4n+3}-s_3F_{4n+1}\,.$$

can be reversed and written as

$$F_{-(4n+3)}\;=\;\frac{s_2}{s_3}\,F_{-(4n+1)}\;-\;\frac{1}{s_3}\,F_{-(4n-1)}\,,$$
$$F_{-(4n+5)}\;=\;\frac{s_1}{s_3}\,F_{-(4n+3)}\;-\;\frac{1}{s_3}\,F_{-(4n+1)}\,.$$

This allows us to define the $*$-Fibonacci Laurent polynomials with negative indices.

Theorem 8.14. *For any n we have $F_{-2n-1}=(F_{2n+1})^*$.* □

For example, $F_1=s_1$, $F_{-1}=(F_1)^*=\frac{s_2}{s_3}$.

8.7. **Cassini identity for odd $*$-Fibonacci polynomials.** The odd $*$-Fibonacci numbers satisfy the following identities:

$$\varphi_{2n+3}^2-\varphi_{2n+5}\varphi_{2n+1}=\varphi_{2n+1}^2-\varphi_{2n+3}\varphi_{2n-1}=\dots=-1\,.$$

Indeed, we have

$$\varphi_{2n+3}^2-\varphi_{2n+5}\varphi_{2n+1}=\varphi_{2n+3}(3\varphi_{2n+1}-\varphi_{2n-1})-\varphi_{2n+5}\varphi_{2n+1}$$
$$=\varphi_{2n+1}(3\varphi_{2n+3}-\varphi_{2n+5})-\varphi_{2n+3}\varphi_{2n-1}=\varphi_{2n+1}^2-\varphi_{2n+3}\varphi_{2n-1}\,.$$

Theorem 8.15. *The odd $*$-Fibonacci polynomials satisfy the following identities:*

$$g_{n+1}F_{2n+3}^2-g_nF_{2n+5}F_{2n+1}=s_3(g_nF_{2n+1}^2-g_{n-1}F_{2n+3}F_{2n-1})=s_3^{n-1}(s_1^3s_3+s_2^3-s_1^2s_2^2)\,.$$

Proof. We have

$$g_{n+1}F_{2n+3}^2-g_nF_{2n+5}F_{2n+1}=g_{n+1}F_{2n+3}(g_nF_{2n+1}-s_3F_{2n-1})-g_nF_{2n+5}F_{2n+1}$$
$$=g_nF_{2n+1}(g_{n+1}F_{2n+3}-F_{2n+5})-s_3g_{n+1}F_{2n+3}F_{2n-1}$$
$$=s_3(g_nF_{2n+1}^2-g_{n-1}F_{2n+3}F_{2n-1})$$

and

$$g_0F_1^2-g_{-1}F_{-1}F_3=s_3^{-1}(s_1^3s_3+s_2^3-s_1^2s_2^2)\,.$$ □

Corollary 8.16. *We have*

$$g_n\,\frac{F_{2n+5}}{F_{2n+3}}\;-\;g_{n-1}\,\frac{F_{2n+3}}{F_{2n+1}}\;=\;s_3^{n-1}\,\frac{s_1^2s_2^2-s_1^3s_3-s_2^3}{F_{2n+3}F_{2n+1}}\,. \tag{8.27}$$

□

Identity (8.27) evaluated at $\boldsymbol{s}=\boldsymbol{s}_o$ takes the form

$$\frac{\varphi_{2n+5}}{\varphi_{2n+3}}-\frac{\varphi_{2n+3}}{\varphi_{2n+1}}=\frac{1}{\varphi_{2n+5}\varphi_{2n+3}}\,.$$

If $\boldsymbol{s}$ lies in a small neighborhood of $\boldsymbol{s}_o$, then the right-hand and left-hand sides of formula (8.27) tend to zero as $n\to\infty$, see (8.24), (8.25), (8.26).

8.8. **Continued fractions for odd $*$-Fibonacci polynomials.** Consider the field $\mathbb{Q}(\boldsymbol{s})$ of rational functions in variables $\boldsymbol{s}$ with rational coefficients. Consider a continued fraction of the following form

$$\Big[a_0;\frac{b_1}{a_1},\dots,\frac{b_5}{a_5}\Big]:=a_0+\cfrac{b_1}{a_1+\frac{b_2}{a_2+\frac{b_3}{a_3+\frac{b_4}{a_4+\frac{b_5}{a_5}}}}}\ \in\ \mathbb{Q}(\boldsymbol{s})\,,$$

where $a_0,\dots,a_5\in\mathbb{Z}[s_1,s_2,s_3]$ and each of $b_1,\dots,b_5$ is of the form

$$s_1^{a_1}s_2^{a_2}s_3^{a_3},\qquad a_1,a_2\in\mathbb{Z}_{\geqslant 0},\ a_3\in\mathbb{Z}\,.$$

Similarly we define continued fractions $\Big[a_0;\frac{b_1}{a_1},\dots,\frac{b_n}{a_n}\Big]$ for any positive integer n.

For example,

$$\big[s_2;\frac{s_1}{s_2},\frac{s_1}{s_3}\big]=s_2+\frac{s_1}{s_2+\frac{s_1}{s_3}}=\frac{s_2^2s_3+s_1s_2+s_1s_3}{s_2s_3+s_1}\,.$$

Theorem 8.17. *For $n\geqslant 1$ we have*

$$\frac{F_{2n+3}}{F_{2n+1}}=\big[g_n;\frac{-s_3}{g_{n-1}},\frac{-s_3}{g_{n-2}},\dots,\frac{-s_3}{g_1},\frac{-s_3}{g_0},\frac{-s_2}{s_1}\big]\,.$$

Proof. The formula follows from the recurrence relations for the $*$-Fibonacci polynomials. $\square$

For example,

$$\frac{F_5}{F_3}=\big[g_1;\frac{-s_3}{g_0},\frac{-s_2}{s_1}\big]\ =\ g_1-\frac{s_3}{g_0-\frac{s_2}{s_1}}\ =\ s_2-\frac{s_3}{s_1-\frac{s_2}{s_1}}\,,$$

$$\frac{F_7}{F_5}=\big[g_2;\frac{-s_3}{g_1},\frac{-s_3}{g_0},\frac{-s_2}{s_1}\big]\ =\ g_2-\frac{s_3}{g_1-\frac{s_3}{g_0-\frac{s_2}{s_1}}}\ =\ s_1-\frac{s_3}{s_2-\frac{s_3}{s_1-\frac{s_2}{s_1}}}\,.$$

Remark 8.18. Formulas (8.25) show that the continued fraction of Theorem 8.17 converges to a an element of a quadratic extension of the field $\mathbb{Q}(\boldsymbol{s})$.

9. Odd $*$-Pell polynomials

9.1. **Definition of odd $*$-Pell polynomials.** The right boundary path of the Markov tree corresponds to the sequence of Markov triples $(3,15,6),(15,87,6),(87,507,6),\dots$, with general term $(3\psi_{2n-1},3\psi_{2n+1},6)$, where ψ_{2n+1}, ψ_{2n-1} are odd Pell numbers,

$$\psi_1=1,\qquad\psi_3=5,\quad\psi_5=29,\quad\psi_7=169,\quad\dots\quad,$$

with the recurrence relation

$$\psi_{2n+3}=6\psi_{2n+1}-\psi_{2n-1}\,. \tag{9.1}$$

We define the *odd $*$-Pell polynomials* recursively by the formula

$$\begin{aligned} &P_1(\boldsymbol{s}) = s_2, \qquad P_3(\boldsymbol{s}) = s_1^2 s_2 - s_1 s_3 - s_2^2, \\ &P_{2n+3}(\boldsymbol{s}) = h_n P_{2n+1}(\boldsymbol{s}) - s_3^2 P_{2n-1}(\boldsymbol{s})\,, \end{aligned} \tag{9.2}$$

where $h_n = s_2^2 - s_1 s_3$ if n is odd and $h_n = s_1^2 - s_2$ if n is even. In other words we have

$$\begin{aligned} P_{4n+3} &= (s_1^2 - s_2) P_{4n+1} - s_3^2 P_{4n-1}\,, \\ P_{4n+5} &= (s_2^2 - s_1 s_3) P_{4n+3} - s_3^2 P_{4n+1}\,. \end{aligned}$$

Lemma 9.1. *We have* $\mathrm{ev}_{\boldsymbol{s}_o}(P_{2n+1}) = 3\psi_{2n+1}$. □

The first odd $*$-Pell polynomials are

$$\begin{aligned} P_1(\boldsymbol{s}) &= s_2, \\ P_3(\boldsymbol{s}) &= s_1^2 s_2 - s_1 s_3 - s_2^2, \\ P_5(\boldsymbol{s}) &= s_1^2 s_2^3 - s_1^3 s_2 s_3 - s_2 s_3^2 + s_1^2 s_3^2 - s_2^4, \\ P_7(\boldsymbol{s}) &= s_1^4 s_2^3 + s_1^4 s_3^2 - s_1^5 s_2 s_3 + s_1^3 s_2^2 s_3 - 2 s_1^2 s_2^4 - 3 s_1^2 s_2 s_3^2 + s_1 s_3^3 + s_2^5 + 2 s_2^2 s_3^2, \\ P_9(\boldsymbol{s}) &= s_1^4 s_2^5 - 2 s_1^2 s_2^6 + s_2^7 - s_1 s_2^5 s_3 + 3 s_2^4 s_3^2 + 3 s_1^3 s_2^4 s_3 - 4 s_1^2 s_2^3 s_3^2 - 2 s_1^5 s_2^3 s_3 \\ &\quad - s_1 s_2^2 s_3^3 + s_2 s_3^4 + 4 s_1^3 s_2 s_3^3 + s_1^6 s_2 s_3^2 - 2 s_1^2 s_3^4 - s_1^5 s_3^3\,. \end{aligned}$$

Theorem 9.2. *For $n > 0$ the triple*

$$(P_{2n-1}^*, P_{2n+1}, h_{n-1}^*) \tag{9.3}$$

is the reduced polynomial presentation of the Markov triple $(3\psi_{2n-1}, 3\psi_{2n+1}, 6)$.

Proof. The proof is by induction on n. The statement is true for $n = 1$, since

$$(P_1^*, P_3, h_0^*) = (s_2^*, s_2(s_1^2 - s_2) - s_3 s_1, (s_1^2 - s_2)^*)$$

is the reduced polynomial presentation of the Markov triple $(3, 15, 6)$, see (6.13).

Assume that $(P_{2n-1}^*, P_{2n+1}, h_{n-1}^*)$ is the reduced polynomial presentation of the Markov triple $(3\psi_{2n-1}, 3\psi_{2n+1}, 6)$. Denote $f = (f_1, f_2, f_3) := (P_{2n-1}, P_{2n+1}, h_{n-1})$. Let Rf be the triple defined in Theorem 6.4. By Theorem 6.4 the triple

$$\begin{aligned} Rf &= (P_{2n+1}, P_{2n+1}\mu(h_{n-1}) - s_3^2 P_{2n-1}, \mu(h_{n-1})) \\ &= (P_{2n+1}, P_{2n+1} h_n - s_3^2 P_{2n-1}, h_n) \\ &= (P_{2n+1}, P_{2n+3}, h_n) \end{aligned}$$

is such that the triple $(P_{2n+1}^*, P_{2n+3}, h_n^*)$ is the reduced polynomial presentation of the Markov triple

$$(3\psi_{2n+1}, 18\psi_{2n+1} - 3\psi_{2n-1}, 6) = (3\psi_{2n+1}, 3(6\psi_{2n+1} - \psi_{2n-1}), 6) = (3\psi_{2n+1}, 3\psi_{2n+3}, 6).$$

This proves the theorem. □

Corollary 9.3. *The odd $*$-Pell polynomials are $*$-Markov polynomials.* □

9.2. **Formula for odd $*$-Pell polynomials.**

Theorem 9.4. *For $n \geqslant 0$, we have*

$$P_{4n+1}(\boldsymbol{s}) = s_2 \sum_{i=0}^{n} (-1)^i \binom{2n-i}{i} (h_0h_1)^{n-i} s_3^{2i} - s_1 \sum_{i=0}^{n-1} (-1)^i \binom{2n-1-i}{i} h_1 (h_0h_1)^{n-i-1} s_3^{2i+1}\,,$$

$$P_{4n+3}(\boldsymbol{s}) = s_2 \sum_{i=0}^{n} (-1)^i \binom{2n+1-i}{i} h_0 (h_0h_1)^{n-i} s_3^{2i} - s_1 \sum_{i=0}^{n} (-1)^i \binom{2n-i}{i} (h_0h_1)^{n-i} s_3^{2i+1}\,.$$

Proof. The proof is by induction. First one checks that the formulas correctly reproduce P_1, P_3. Then $h_0 P_{4n+1} - s_3^2 P_{4n-1}$ equals

$$\begin{aligned} &s_2 \sum_{i=0}^{n} (-1)^i \binom{2n-i}{i} h_0 (h_0h_1)^{n-i} s_3^{2i} - s_1 \sum_{i=0}^{n-1} (-1)^i \binom{2n-1-i}{i} (h_0h_1)^{n-i} s_3^{2i+1} \\ &-s_2 \sum_{i=0}^{n-1} (-1)^i \binom{2n-1-i}{i} h_0 (h_0h_1)^{n-i-1} s_3^{2i+2} + s_1 \sum_{i=0}^{n-1} (-1)^i \binom{2n-i-2}{i} (h_0h_1)^{n-i-1} s_3^{2i+3}. \end{aligned}$$

We have

$$\begin{aligned} &s_2 \left(\sum_{i=0}^{n} (-1)^i \binom{2n-i}{i} h_0 (h_0h_1)^{n-i} s_3^{2i} + \sum_{i=0}^{n-1} (-1)^{i+1} \binom{2n-1-i}{i} h_0 (h_0h_1)^{n-i-1} s_3^{2i+2} \right) \\ &= s_2 \left(\sum_{i=0}^{n} (-1)^i \binom{2n+1-i}{i} h_0 (h_0h_1)^{n-i} s_3^{2i} \right), \end{aligned}$$

$$\begin{aligned} &s_1 \left(\sum_{i=0}^{n-1} (-1)^{i+1} \binom{2n-1-i}{i} (h_0h_1)^{n-i} s_3^{2i+1} + \sum_{i=0}^{n-1} (-1)^i \binom{2n-i-2}{i} (h_0h_1)^{n-i-1} s_3^{2i+3} \right) \\ &= -s_1 \left(\sum_{i=0}^{n} (-1)^i \binom{2n-i}{i} (h_0h_1)^{n-i} s_3^{2i+1} \right) . \end{aligned}$$

Hence, $h_0 P_{4n+1} - s_3^2 P_{4n-1} = P_{4n+3}$. The other identity is proved similarly. $\square$

Corollary 9.5. *For the ordinary Pell numbers, we have*

$$\psi_{4n+1} = \sum_{i=0}^{n}(-1)^i\binom{2n-i}{i}6^{2n-2i} - \sum_{i=0}^{n-1}(-1)^i\binom{2n-i-1}{i}6^{2n-2i-1}, \tag{9.4}$$

$$\psi_{4n+2} = \frac{1}{2}\sum_{i=0}^{n}(-1)^i\binom{2n-i}{i}6^{2n-2i+1} - \sum_{i=0}^{n}(-1)^i\binom{2n-i}{i}6^{2n-2i}, \tag{9.5}$$

$$\psi_{4n+3} = \sum_{i=0}^{n}(-1)^i\binom{2n-i+1}{i}6^{2n-2i+1} - \sum_{i=0}^{n}(-1)^i\binom{2n-i}{i}6^{2n-2i}, \tag{9.6}$$

$$\begin{aligned}\psi_{4n+4} = {} & 2\sum_{i=0}^{n}(-1)^i\binom{2n-i+1}{i}6^{2n-2i+1} + \frac{1}{2}\sum_{i=0}^{n}(-1)^i\binom{2n-i}{i}6^{2n-2i+1} \\ & - 3\sum_{i=0}^{n}(-1)^i\binom{2n-i}{i}6^{2n-2i}.\end{aligned} \tag{9.7}$$

Proof. Formulas (9.5), (9.7) follow from the recurrence relation for Pell numbers

$$\psi_{n+1} = 2\psi_n + \psi_{n-1}, \quad n \geqslant 1.$$

□

9.3. **Limiting Newton polygons of odd $*$-Pell polynomials.**

Lemma 9.6. *The odd $*$-Pell polynomials P_{4n+1} and P_{4n+3} are of bi-degree $(4n+1, 6n+2)$ and $(4n+3, 6n+4)$, respectively.* □

The Newton polygon $N_{P_{4n+1}}$ of P_{4n+1} contains the points $(0,1,2n), (2n,2n+1,0), (0,3n+1,0), (3n,1,n)$. Hence the limit of $\tilde{N}_{P_{4n+1}}$ as $n \to \infty$ contains the points $(0,0)$, $(1/2,1/2)$, $(0,3/4)$, $(3/4,0)$. Therefore the limit of $\tilde{N}_{P_{4n+1}}$ is the projected quadrilateral $\tilde{N}_\infty$.

Similarly one checks that the limit of $\tilde{N}_{P_{4n+3}}$ as $n \to \infty$ is the projected quadrilateral $\tilde{N}_\infty$, see Section 7.8.

9.4. **Generating function.** Introduce the generating power series of odd $*$-Pell polynomials

$$\mathcal{P}(\boldsymbol{s}, t) := \sum_{n=0}^{\infty} P_{2n+1}(\boldsymbol{s})t^{2n+1}. \tag{9.8}$$

Theorem 9.7. *We have*

$$\mathcal{P}(\boldsymbol{s}, t) = \frac{-s_1s_3^3t^7 + \left(\left(s_1^2+s_2\right)s_3^2 - s_1s_2^2s_3\right)t^5 + \left(\left(s_1^2-s_2\right)s_2 - s_1s_3\right)t^3 + s_2t}{s_3^4t^8 + \left(s_1^3s_3 - s_1s_2s_3 + s_2^3 - s_1^2s_2^2 + 2s_3^2\right)t^4 + 1}. \tag{9.9}$$

Proof. The proof is similar to the proof of the corresponding theorem on the odd Fibonacci polynomials. □

Corollary 9.8. *For any $n \geqslant 0$, we have*

$$P_{2n+1}(\boldsymbol{s}) = \frac{1}{(2n+1)!}\left.\frac{\partial^{2n+1}}{\partial t^{2n+1}}\right|_{t=0}\mathcal{P}(\boldsymbol{s}, t), \tag{9.10}$$

where $\mathcal{P}(\boldsymbol{s}, t)$ is given by (9.9).

Remark 9.9. At $\boldsymbol{s} = \boldsymbol{s}_o$, the generating function $\mathcal{P}(\boldsymbol{s}, t)$ reduces to

$$\mathcal{P}(\boldsymbol{s}_o, t) = \frac{3\,(t - t^3)}{t^4 - 6t^2 + 1} = 3(t + 5t^3 + 29t^5 + 169t^7 + 985t^9 + \dots), \tag{9.11}$$

namely the generating series of odd Pell numbers multiplied by 3.

9.5. **Other properties of ∗-Pell polynomials.** The ∗-Pell polynomials have properties similar to the properties of ∗-Fibonacci polynomials discussed in Section 8. In particular one easily obtains a Binet-type formula like in Corollary 8.12.

As examples of properties of ∗-Pell polynomials we formulate the continued fraction property and an analog of the Cassini identity.

Theorem 9.10. *For $n \geqslant 1$ we have*

$$\frac{P_{2n+3}}{P_{2n+1}} = \left[h_n; \frac{-s_3^2}{h_{n-1}}, \frac{-s_3^2}{h_{n-2}}, \dots, \frac{-s_3^2}{h_1}, \frac{-s_3^2}{h_0}, \frac{-s_1 s_3}{s_2}\right].$$

□

Theorem 9.11. *The odd ∗-Pell polynomials satisfy the following identities:*

$$\begin{aligned} &h_{n+1}P_{2n+3}^2 - h_n P_{2n+5}P_{2n+1} = s_3^2(h_n P_{2n+1}^2 - h_{n-1}P_{2n+3}P_{2n-1}) \\ &= s_3^{2n-1}\Big((s_1^2 - s_2)s_2^2 s_3 \;-\; s_1\,(s_2^2 - s_1 s_3)\big((s_1^2 - s_2)s_2 - s_1 s_3\big)\Big). \end{aligned}$$

□

10. ∗-Markov group actions

In this section we study the action of the ∗-Markov group on $\mathbb{C}^6$ with coordinates $(a, a^*,\, b, b^*,\, c, c^*)$. It is convenient to denote these coordinates by $(x_1, \dots, x_6)$.

10.1. **Space $\mathbb{C}^6$ with involution and polynomials.** Consider $\mathbb{C}^6$ with coordinates $\boldsymbol{x} = (x_1, \dots, x_6)$, involution

$$\nu : \mathbb{C}^6 \to \mathbb{C}^6, \qquad (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_2, x_1, x_4, x_3, x_6, x_5),$$

polynomials

$$H_1 = x_1x_2 + x_3x_4 + x_5x_6 - x_1x_3x_5, \qquad H_2 = x_1x_2 + x_3x_4 + x_5x_6 - x_2x_4x_6.$$

The ∗-Markov group Γ_M acts on $\mathbb{C}^6$ by the formulas

$$\begin{aligned} \lambda_{i,j}\colon \quad & \boldsymbol{x} \mapsto \big((-1)^i x_1, (-1)^i x_2, (-1)^{i+j} x_3, (-1)^{i+j} x_4, (-1)^j x_5, (-1)^j x_6\big), \\ \sigma_1\colon \quad & \boldsymbol{x} \mapsto (x_3, x_4, x_1, x_2, x_5, x_6), \\ \sigma_2\colon \quad & \boldsymbol{x} \mapsto (x_1, x_2, x_5, x_6, x_3, x_4), \\ \tau_1\colon \quad & \boldsymbol{x} \mapsto (-x_2,\, -x_1,\, x_6,\, x_5,\, x_4 - x_1x_5,\, x_3 - x_2x_6), && (10.1) \\ \tau_2\colon \quad & \boldsymbol{x} \mapsto (x_4,\, x_3,\, x_2 - x_3x_5,\, x_1 - x_4x_6,\, -x_6,\, -x_5), && (10.2) \end{aligned}$$

and the elements $\mu_{i,j}$ act on $\mathbb{C}^6$ by the identity maps.

The action of the ∗-Markov group on $\mathbb{C}^6$ commutes with the involution ν.

Lemma 10.1. *The Γ_M-action preserves each of the polynomials H_1, H_2, and*

$$\nu^\star H_1 = H_2, \qquad \nu^\star H_2 = H_1.$$

Hence the differential forms dH_1, dH_2, $dH_1\wedge dH_2$ are Γ_M-invariant, and $dH_1\wedge dH_2$ is ν anti-invariant. □

Lemma 10.2. *The holomorphic volume form*

$$dV := dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dx_5\wedge dx_6$$

is $\lambda_{i,j},\sigma_1,\sigma_2$ invariant and τ_1,τ_2,ν anti-invariant. □

Lemma 10.3. *The differential 4-form*

$$\begin{aligned}\Omega \;=\;\; & x_1x_3\, dx_2\wedge dx_4\wedge dx_5\wedge dx_6 \;+\; x_1x_4\, dx_2\wedge dx_3\wedge dx_5\wedge dx_6 \\ & -x_1x_5\, dx_2\wedge dx_3\wedge dx_4\wedge dx_6 \;-\; x_1x_6\, dx_2\wedge dx_3\wedge dx_4\wedge dx_5 \\ & +x_2x_3\, dx_1\wedge dx_4\wedge dx_5\wedge dx_6 \;+\; x_2x_4\, dx_1\wedge dx_3\wedge dx_5\wedge dx_6 \\ & -x_2x_5\, dx_1\wedge dx_3\wedge dx_4\wedge dx_6 \;-\; x_2x_6\, dx_1\wedge dx_3\wedge dx_4\wedge dx_5 \\ & +x_3x_5\, dx_1\wedge dx_2\wedge dx_4\wedge dx_6 \;+\; x_3x_6\, dx_1\wedge dx_2\wedge dx_4\wedge dx_5 \\ & +x_4x_5\, dx_1\wedge dx_2\wedge dx_3\wedge dx_6 \;+\; x_4x_6\, dx_1\wedge dx_2\wedge dx_3\wedge dx_5\,. \end{aligned} \tag{10.3}$$

is $\lambda_{i,j}$, τ_1,τ_2 invariant and σ_1,σ_2,ν anti-invariant.

Proof. The proof is by direct verification. For example, we have

$$\begin{aligned}&\tau_1\Omega = x_5(x_4-x_1x_5)dx_1\wedge dx_2\wedge dx_3\wedge dx_6 + x_6(x_3-x_2x_6)dx_1\wedge dx_2\wedge dx_4\wedge dx_5\\ &+x_5(x_3-x_2x_6)(dx_1\wedge dx_2\wedge dx_4\wedge dx_6 - x_1dx_1\wedge dx_2\wedge dx_5\wedge dx_6)\\ &-x_6(x_4-x_1x_5)(-dx_1\wedge dx_2\wedge dx_3\wedge dx_5 - x_2dx_1\wedge dx_2\wedge dx_5\wedge dx_6)\\ &+x_2x_5(-x_1(x_6dx_1\wedge dx_2\wedge dx_5\wedge dx_6 - dx_1\wedge dx_3\wedge dx_5\wedge dx_6) + x_6dx_1\wedge dx_2\wedge dx_4\wedge dx_6\\ &-dx_1\wedge dx_3\wedge dx_4\wedge dx_6) - x_2(x_4-x_1x_5)(x_6dx_1\wedge dx_2\wedge dx_5\wedge dx_6 - dx_1\wedge dx_3\wedge dx_5\wedge dx_6)\\ &+x_2(x_3-x_2x_6)dx_1\wedge dx_4\wedge dx_5\wedge dx_6 - x_2x_6(-x_6dx_1\wedge dx_2\wedge dx_4\wedge dx_5\\ &-x_2dx_1\wedge dx_4\wedge dx_5\wedge dx_6 + dx_1\wedge dx_3\wedge dx_4\wedge dx_5) + x_1(x_4-x_1x_5)dx_2\wedge dx_3\wedge dx_5\wedge dx_6\\ &+x_1x_5(x_5dx_1\wedge dx_2\wedge dx_3\wedge dx_6 + x_1dx_2\wedge dx_3\wedge dx_5\wedge dx_6 - dx_2\wedge dx_3\wedge dx_4\wedge dx_6)\\ &+x_1(x_3-x_2x_6)(x_5dx_1\wedge dx_2\wedge dx_5\wedge dx_6 + dx_2\wedge dx_4\wedge dx_5\wedge dx_6)\\ &-x_1x_6(x_5(-dx_1\wedge dx_2\wedge dx_3\wedge dx_5 - x_2dx_1\wedge dx_2\wedge dx_5\wedge dx_6) + dx_2\wedge dx_3\wedge dx_4\wedge dx_5\\ &-x_2dx_2\wedge dx_4\wedge dx_5\wedge dx_6) = \Omega.\end{aligned}$$

See another proof of the lemma in Corollary 10.10. That other proof also provides reasons for the existence of such a 4-form Ω. □

10.2. **Casimir subalgebra.** Denote

$$h_1 = x_1x_2, \quad h_2 = x_3x_4, \quad h_3 = x_5x_6, \quad h_4 = x_1x_3x_5, \quad h_5 = x_2x_4x_6, \tag{10.4}$$

and $\boldsymbol{h} = (h_1,\dots,h_5)$. Then

$$h_1h_2h_3 - h_4h_5 = 0$$

and

$$H_1 = h_1+h_2+h_3-h_4, \qquad H_2 = h_1+h_2+h_3-h_5.$$

Define the *Casimir subalgebra* $\mathcal{C} \subset \mathbb{C}[\boldsymbol{x}]$ to be the subalgebra generated by $h_1, \dots, h_5$.

Theorem 10.4. *The Casimir subalgebra is ν and Γ_M invariant. More precisely,*

$$
\begin{aligned}
&\nu : \boldsymbol{h} \mapsto (h_1,\, h_2,\, h_3,\, h_5,\, h_4),\\
&\tau_1 : \boldsymbol{h} \mapsto (h_1,\, h_3,\, h_2 + h_1h_3 - h_4 - h_5,\, -h_5 + h_1h_3,\, -h_4 + h_1h_3),\\
&\tau_2 : \boldsymbol{h} \mapsto (h_2,\, h_1 + h_2h_3 - h_4 - h_5,\, h_3,\, -h_5 + h_2h_3,\, -h_4 + h_2h_3),\\
&\sigma_1 : \boldsymbol{h} \mapsto (h_2,\, h_1,\, h_3,\, h_4,\, h_5),\\
&\sigma_2 : \boldsymbol{h} \mapsto (h_1,\, h_3,\, h_2,\, h_4,\, h_5),
\end{aligned}
$$

and the elements $\lambda_{i,j}, \mu_{i,j} \in \Gamma_M$ fix elements of $\mathcal{C}$ point-wise. □

10.3. **Space $\mathbb{C}^5$ with polynomials.** Consider $\mathbb{C}^5$ with coordinates $\boldsymbol{y} = (y_1, \dots, y_5)$, and involution

$$\nu \colon \mathbb{C}^5 \to \mathbb{C}^5, \quad (y_1, y_2, y_3, y_4, y_5) \mapsto (y_1, y_2, y_3, y_5, y_4).$$

The $*$-Markov group acts on $\mathbb{C}^5$ by the formulas of Theorem 10.4. The $*$-Markov group action on $\mathbb{C}^5$ commutes with the involution ν.

Denote

$$J = y_1y_2y_3 - y_4y_5, \quad J_1 = y_1 + y_2 + y_3 - y_4, \quad J_2 = y_1 + y_2 + y_3 - y_5,$$

$$dW = dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 \wedge dy_5.$$

Lemma 10.5. *The polynomials J, J_1, J_2 are Γ_M-invariant. We have*

$$\nu^\star J = J, \qquad \nu^\star J_1 = J_2, \qquad \nu^\star J_2 = J_1.$$

The differential form dW is τ_1, τ_2 invariant and σ_1, σ_2, ν anti-invariant. The differential form $dJ \wedge dJ_1 \wedge dJ_2$ is τ_1, τ_2, σ_1, σ_2 invariant and ν anti-invariant. □

Let $Y = \{\boldsymbol{y} \in \mathbb{C}^5 \mid J(\boldsymbol{y}) = 0\}$ be the zero level hypersurface of the polynomial J. The hypersurface Y has a well-defined holomorphic nonzero differential 4-form at its nondegenerate points, $\omega = dW/dJ$, called the Gelfand-Leray residue form. It is uniquely determined by the property

$$dW = dJ \wedge \omega. \tag{10.5}$$

For example, if at some point $q \in Y$ we have $\frac{\partial J}{\partial y_1}(q) \neq 0$, then at a neighborhood of that point

$$\omega = \frac{1}{\frac{\partial J}{\partial y_1}} dy_2 \wedge dy_3 \wedge dy_4 \wedge dy_5 = \frac{1}{y_2y_3} dy_2 \wedge dy_3 \wedge dy_4 \wedge dy_5$$

with property (10.5).

Corollary 10.6. *The form*

$$\omega = \frac{1}{y_2y_3} dy_2 \wedge dy_3 \wedge dy_4 \wedge dy_5,$$

restricted to Y, extends to a nonzero differential 4-form on the regular part of Y. That form is τ_1, τ_2 invariant and σ_1, σ_2, ν anti-invariant. □

Consider the map $F : \mathbb{C}^6 \to \mathbb{C}^5$ defined by the formulas

$$y_1 = x_1x_2, \quad y_2 = x_3x_4, \quad y_3 = x_5x_6, \quad y_4 = x_1x_3x_5, \quad y_5 = x_2x_4x_6. \tag{10.6}$$

Lemma 10.7. *We have the following statements:*

(i) *The map F commutes with the actions of the $*$-Markov group on $\mathbb{C}^6$ and $\mathbb{C}^5$.*
(ii) *The Casimir subalgebra $\mathcal{C} \subset \mathbb{C}[\boldsymbol{x}]$ is the preimage of the algebra $\mathbb{C}[\boldsymbol{y}]$ under the map F.*
(iii) *The image of F lies in the hypersurface Y.* □

Corollary 10.8. *The preimage $F^\star\omega$ of the differential form ω under the map F is τ_1, τ_2 invariant and σ_1, σ_2, ν anti-invariant.* □

Lemma 10.9. *We have $F^\star\omega = \Omega$, where Ω is defined in (10.3).*

Proof. The lemma easily follows by direct verification from the formula

$$\begin{aligned} F^\star\omega \quad = \quad & \Big(\frac{dx_3}{x_3} + \frac{dx_4}{x_4}\Big) \wedge \Big(\frac{dx_5}{x_5} + \frac{dx_6}{x_6}\Big) \\ & \wedge\Big(x_3x_5dx_1 + x_1x_5dx_3 + x_1x_3dx_5\Big) \wedge \Big(x_4x_6dx_2 + x_2x_6dx_4 + x_2x_4dx_6\Big). \end{aligned} \tag{10.7}$$

□

Corollary 10.10. *The differential form Ω is τ_1, τ_2 invariant and σ_1, σ_2, ν anti-invariant.* □

Cf. Lemma 10.3.

11. Poisson structures on $\mathbb{C}^6$ and $\mathbb{C}^5$

11.1. **Nambu-Poisson manifolds.**

Definition 11.1 ([Ta94])**.** Let A be the algebra of functions of a manifold Y. The manifold Y is a *Nambu-Poisson manifold* of order n if there exists a multi-linear map

$$\{\,,\ldots,\,\} : A^{\otimes n} \to A\,,$$

a *Nambu bracket* of order n, satisfying the following properties.

(i) Skew-symmetry,
$$\{f_1, \ldots, f_n\} = (-1)^{\epsilon(\sigma)}\{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\}\,,$$
for all $f_1, \ldots, f_n \in A$ and $\sigma \in S_n$.

(ii) Leibniz rule,
$$\{f_1f_2, f_3, \ldots, f_{n+1}\} = f_1\{f_2, f_3, \ldots, f_{n+1}\} + f_2\{f_1, f_3, \ldots, f_{n+1}\}\,,$$
for all $f_1, \ldots, f_{n+1} \in A$.

(iiii) Fundamental Identity (FI),
$$\begin{aligned} \{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = & \{\{f_1, \ldots, f_{n-1}, g_1\}, g_2, \ldots, g_n\} \\ & + \{g_1, \{f_1, \ldots, f_{n-1}, g_2\}, g_3, \ldots, g_n\} + \ldots \\ & + \{g_1, \ldots, g_{n-1}, \{f_1, \ldots, f_{n-1}, g_n\}\}\,, \end{aligned} \tag{11.1}$$
for all $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n \in A$.

In particular, for $n = 2$ this is the standard Poisson structure.

Remark 11.2. The brackets with properties (i-ii) were considered by Y. Nambu [Na73], who was motivated by problems of quark dynamics. The notion of a Nambu-Poisson manifold was introduced by L. Takhtajan [Ta94] in order to formalize mathematically the n-ary generalization of Hamiltonian mechanics proposed by Y. Nambu. The fundamental identity was discovered by V. Filippov [Fi85] as a generalization of the Jacobi identity for an n-ary Lie algebra and then later and independently by Takhtajan [Ta94] for the Nambu-Poisson setting.

The fundamental identity is also called the Filippov identity.

The dynamics associated with the Nambu bracket on a Nambu-Poisson manifold of order n is specified by $n-1$ Hamiltonians $H_1, \dots, H_{n-1} \in A$, and the time evolution of $f \in A$ is given by the equation

$$\frac{df}{dt} = \{H_1, \dots, H_{n-1}, f\}\,. \tag{11.2}$$

Let φ_t be the flow associated with equation (11.2) and U_t the one-parameter group acting on A by $f \mapsto U_t(f) = f \circ \varphi_t$.

Theorem 11.3 ([Ta94])**.** *The flow preserves the Nambu bracket,*

$$U_t(\{f_1, \dots, f_n\}) = \{U_t(f_1), \dots, U_t(f_n)\}\,, \tag{11.3}$$

for all $f_1, \dots, f_n \in A$. □

A function $f \in A$ is called an *integral of motion* for the system defined by equation (11.2) if it satisfies $\{H_1, \dots, H_{n-1}, f\} = 0$.

Theorem 11.4 ([Ta94])**.** *Given $H_1, \dots, H_{n-1}$, the Nambu bracket of n integrals of motion is also an integral of motion.* □

These two theorems follow from the fundamental identity.

11.2. **Examples.** An example of a Nambu-Poisson manifold of order n is $\mathbb{C}^n$ with standard coordinates $x_1, \dots, x_n$ and *canonical Nambu bracket* given by

$$\{f_1, \dots, f_n\} = \det{}_{i,j=1}^{n}\Big(\frac{\partial f_i}{\partial x_j}\Big). \tag{11.4}$$

This example was considered by Nambu [Na73]. Other examples of Nambu-Poisson manifolds see in [CT96, Ta94]. See also [BF75, Ch96, DT97, GM00, OR02].

It turns out that any Nambu-Poisson manifold of order $n > 2$ has presentation (11.4) locally.

Theorem 11.5. *Let Y be an m-dimensional manifold which is a Nambu-Poisson manifold of order n, $m \geqslant n > 2$, with bracket $\{\,, \dots, \,\}$. Let $x \in X$ be a point such that $\{\,, \dots, \,\}$ is nonzero at x. Then there exists local coordinates $x_1, \dots, x_m$ in a neighborhood of x such that*

$$\{f_1, \dots, f_n\} = \det{}_{i,j=1}^{n}\Big(\frac{\partial f_i}{\partial x_j}\Big)\,.$$

This statement was conjectured by L. Takhtajan [Ta94], proved in [AG96, Ga96]. It was discovered eventually that the theorem is a consequence of an old result in [We23], reproduced in the textbook by Schouten [Sc89], Chap. II, Sections 4 and 6, formula (6.7). See on that in [DFST97].

11.3. **Hierarchy of Nambu-Poisson structures.** A Nambu-Poisson manifold structure of order n on a manifold X induces an infinite family of subordinated Nambu-Poisson manifold structures on X of orders $n-1$ and lower, including a family of Poisson structures, [Ta94].

Indeed for $H_1,\dots,H_{n-k}\in A$ define the k-bracket $\{\,,\dots,\,\}_H$ by the formula

$$\{h_1,\dots,h_k\}_H=\{H_1,\dots,H_{n-k},h_1,\dots,h_k\}\,. \tag{11.5}$$

Clearly, the bracket $\{\,,\dots,\,\}_H$ is skew-symmetric and satisfies the Leibnitz rule. The fundamental identity for $\{\,,\dots,\,\}_H$ follows from the fundamental identity (11.1) for the original bracket.

For example, for $n=6$ and $k=4$, the fundamental identity for the bracket

$$\{h_1,h_2,h_3,h_4\}_{H_1,H_2}:=\{H_1,H_2,h_1,h_2,h_3,h_4\} \tag{11.6}$$

and 7 functions $u_1,u_2,u_3,v_1,v_2,v_3,v_4$ follows from the fundamental identity for $n=6$ and 11 functions $f_1,f_2,f_3,f_4,f_5,g_1,g_2,g_3,g_4,g_5,g_6$ if

$$(f_1,f_2,f_3,f_4,f_5,g_1,g_2,g_3,g_4,g_5,g_6)=(H_1,H_2,u_1,u_2,u_3,H_1,H_2,v_1,v_2,v_3,v_4)\,.$$

The family of subordinated k-brackets, obtained by this construction from a given n-bracket, satisfy the matching conditions described in [Ta94].

Example. Consider $\mathbb{C}^3$ with coordinates a,b,c and canonical Nambu bracket of order 3,

$$\{f_1,f_2,f_3\}=\frac{df_1\wedge df_2\wedge df_3}{da\wedge db\wedge dc}\,.$$

The braid group $\mathcal{B}_3$ acts on $\mathbb{C}^3$ by the formulas,

$$\tau_1:(a,b,c)\mapsto(-a,c,b-ac)\,,$$
$$\tau_2:(a,b,c)\mapsto(b,a-bc,-c)\,.$$

The polynomial $H=a^2+b^2+c^2-abc$ is braid group invariant. The subordinated 2-bracket

$$\{h_1,h_2\}_H:=\{H,h_1,h_2\}$$

is the braid group invariant Dubrovin Poisson structure on $\mathbb{C}^3$,

$$\{a,b\}_H=2c-ab,\qquad\{b,c\}_H=2a-bc,\qquad\{c,a\}_H=2b-ac.$$

11.4. **Poisson structure on $\mathbb{C}^6$.** Let us return to the space $\mathbb{C}^6$ with involution ν, two polynomials H_1, H_2, the holomorphic volume form dV, differential form Ω, considered in Section 10.

On $\mathbb{C}^6$ consider the canonical Nambu bracket of order 6,

$$\{f_1,\dots,f_6\}=\frac{df_1\wedge df_2\wedge df_3\wedge df_4\wedge df_5\wedge df_6}{dV}\,, \tag{11.7}$$

and associated brackets

$$\{f_1, f_2, f_3, f_4\}_{H_1,H_2} = \frac{dH_1 \wedge dH_2 \wedge df_1 \wedge df_2 \wedge df_3 \wedge df_4}{dV}, \tag{11.8}$$

$$\{f_1, f_2\}_\Omega = \frac{\Omega \wedge df_1 \wedge df_2}{dV}. \tag{11.9}$$

Theorem 11.6. *The Nambu bracket* $\{\,,\dots,\,\}_{H_1,H_2}$ *defines a Nambu-Poisson manifold structure on* $\mathbb{C}^6$ *of order 4. The structure* $\{\,,\dots,\,\}_{H_1,H_2}$ *is* $\lambda_{i,j}, \sigma_1, \sigma_2, \nu$ *invariant and* τ_1, τ_2 *anti-invariant.*

Proof. The theorem is a corollary of Lemmas 10.1 and 10.2. □

Theorem 11.7. *The bracket* $\{\,,\,\}_\Omega$ *defines a Poisson structure on* $\mathbb{C}^6$. *The Poisson structure* $\{\,,\,\}_\Omega$ *is* $\lambda_{i,j}, \nu$ *invariant and* $\tau_1, \tau_2, \sigma_1, \sigma_2$ *anti-invariant.*

Proof. By formula (10.7), the form Ω is the wedge-product of four differentials. Hence the bracket $\{\,,\,\}_\Omega$ defines a Nambu-Poisson manifold structure on $\mathbb{C}^6$ of order 2. The invariance properties of it follow from Lemmas 10.2 and 10.3. □

Lemma 11.8. *The Poisson structure* $\{\,,\,\}_\Omega$ *is log-canonical. The Poisson brackets* $\{x_i, x_j\}_\Omega$ *are given by the following matrix,*

$$\begin{pmatrix} 0 & 0 & -x_1x_3 & x_1x_4 & x_1x_5 & -x_1x_6 \\ 0 & 0 & x_2x_3 & -x_2x_4 & -x_2x_5 & x_2x_6 \\ x_1x_3 & -x_2x_3 & 0 & 0 & -x_3x_5 & x_3x_6 \\ -x_1x_4 & x_2x_4 & 0 & 0 & x_4x_5 & -x_4x_6 \\ -x_1x_5 & x_2x_5 & x_3x_5 & -x_4x_5 & 0 & 0 \\ x_1x_6 & -x_2x_6 & -x_3x_6 & x_4x_6 & 0 & 0 \end{pmatrix}.$$

Proof. Explicit computations give

$$\begin{aligned} \Omega \wedge dx_1 \wedge dx_2 &= 0, \\ \Omega \wedge dx_1 \wedge dx_3 &= -x_1x_3\, dV, \\ \Omega \wedge dx_1 \wedge dx_4 &= x_1x_4\, dV, \\ \Omega \wedge dx_1 \wedge dx_5 &= x_1x_5\, dV, \end{aligned}$$

and so on. □

Lemma 11.9. *For any* $f \in \mathbb{C}[\boldsymbol{x}]$ *and* $h \in \mathcal{C}$, *we have* $\{f, h\}_\Omega = 0$. *In particular,* $\{f, H_1\}_\Omega = \{f, H_2\}_\Omega = 0$ *for any* $f \in \mathbb{C}[\boldsymbol{x}]$.

This statement justifies the name of the Casimir subalgebra for the subalgebra $\mathcal{C} \subset \mathbb{C}[\boldsymbol{x}]$.

Proof. The equalities $\{x_i, h_j\}_\Omega = 0$, $i = 1, \dots, 6$, $j = 1, \dots, 5$, are easily checked directly. The statement also follows from the fact that the image of F lies in Y and ω is a top degree form on Y. □

Lemma 11.10. *The symplectic leaves of the Poisson structure* $\{\,,\,\}_\Omega$ *are at most two-dimensional and lie in fibers of the map* F. □

11.5. **Remarks.**

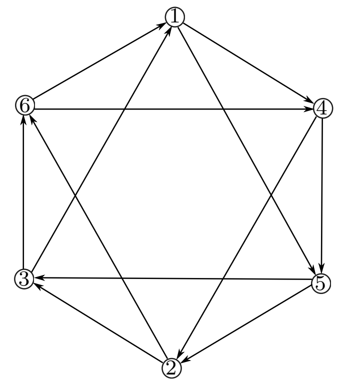

FIGURE 11.

11.5.1. The log-canonical Poisson structure $\{\,,\,\}_\Omega$ can be encoded by the quiver in Figure 11. It would be interesting to determine if some of the $*$-Markov group transformations can be obtained as a sequence of mutations in the cluster algebra of that quiver. We were able to represent in this way only the action of the permutations σ_1, σ_2. To obtain σ_1 one needs to mutate the cluster variables at vertex 1, then at vertex 3, then at vertex 1 and so on as in the sequence 1 3 1 3 1 2 4 2 4 2 (10 mutations). The permutation σ_2 is obtained by the sequence of mutations 3 5 3 5 3 4 6 4 6 4. Cf. [CS20], where the braid group action was presented by mutations for the A_n quivers.

11.5.2. We say that a Poisson structure on $\mathbb{C}^6$ is *quadratic*, if the $\{x_i, x_j\}$ are homogeneous quadratic polynomials in $\boldsymbol{x}$.

Lemma 11.11. *The Poisson structure $\{\,,\,\}_\Omega$ is the unique, up to rescaling by a nonzero constant, nonzero quadratic Poisson structure on $\mathbb{C}^6$ having both H_1 and H_2 as Casimir elements.*

Proof. Let $\{x_i, x_j\} = \sum_{k,l} Q_{ij}^{kl} x_k x_l$ with unknown coefficients $Q_{ij}^{kl} \in \mathbb{C}$. Assume the skew-symmetry of $\{\cdot,\cdot\}$, and that both H_1 and H_2 are Casimir elements. This gives a system of linear equations for Q_{ij}^{kl}. A computer assistant calculation shows that the matrix of coefficients of that system has rank 1, and the space of solutions is spanned by the Poisson tensor $\{\,,\,\}_\Omega$. $\square$

11.5.3. Let $\{x_i, x_j\} = Q_{i,j}(\boldsymbol{x})$ be a polynomial Poisson structure on $\mathbb{C}^6$ having H_1, H_2 as Casimir elements. Expand the coefficients $Q_{ij}(x)$ at the origin, $Q_{ij}(x) = \sum_{k=1}^{6} Q_{ij}^k x_k + \dots$, $Q_{ij}^k \in \mathbb{C}$. A computer assistant calculation shows that $Q_{i,j}^k = 0$ for all i, j, k.

11.5.4. A computer assistant calculation shows that the Poisson structure $\{\,,\,\}_\Omega$ is the unique, up to rescaling by a nonzero constant, nonzero log-canonical Poisson structure on $\mathbb{C}^6$, which remains to be log-canonical after the action on it by any element of the braid group $\mathcal{B}_3$.

11.6. **Poisson structure on $\mathbb{C}^5$.** Consider $\mathbb{C}^5$ with coordinates $\boldsymbol{y} = (y_1, \dots, y_5)$, and objects discussed in Section 10.3.

Consider on $\mathbb{C}^5$ the canonical Nambu bracket of order 5,

$$\{f_1, \dots, f_5\} = \frac{df_1 \wedge df_2 \wedge df_3 \wedge df_4 \wedge df_5}{dW}\,, \tag{11.10}$$

and the associated bracket

$$\{f_1, f_2\}_{J,J_1,J_2} = \frac{dJ \wedge dJ_1 \wedge dJ_2 \wedge df_1 \wedge df_2}{dW}\,. \tag{11.11}$$

Theorem 11.12. *The bracket $\{\ ,\ \}_{J,J_1,J_2}$ defines a Poisson structure on $\mathbb{C}^5$ with Casimir elements J, J_1, J_2. The Poisson structure $\{\ ,\ \}_{J,J_1,J_2}$ is $\lambda_{i,j}, \tau_1, \tau_2, \nu$ invariant and σ_1, σ_2 anti-invariant.*

Proof. The theorem follows from Lemma 10.5. □

Introduce new linear coordinates on $\mathbb{C}^5$, $(u_1, u_2, u_3, u_4, u_5) := (y_1, y_2, y_3, J_1, J_2)$.

Lemma 11.13. *The Poisson brackets $\{u_i, u_j\}_{J,J_1,J_2}$ are given by the formulas:*

$$\begin{aligned}
\{u_1, u_2\}_{J,J_1,J_2} &= u_1 u_2 - 2(u_1 + u_2 + u_3) + u_4 + u_5,\\
\{u_2, u_3\}_{J,J_1,J_2} &= u_2 u_3 - 2(u_1 + u_2 + u_3) + u_4 + u_5,\\
\{u_3, u_1\}_{J,J_1,J_2} &= u_3 u_1 - 2(u_1 + u_2 + u_3) + u_4 + u_5,
\end{aligned}$$

and $\{u_i, u_4\}_{J,J_1,J_2} = \{u_i, u_5\}_{J,J_1,J_2} = 0$ for all i. □

Notice the similarity of these formulas with Dubrovin's formulas (11.7). Similarly to Dubrovin's case the linear and quadratic parts of the Poisson structure $\{\ ,\ \}_{J,J_1,J_2}$ form a pencil of Poisson structures, that, is any linear combination of them is a Poisson structure too.

Remark 11.14. It would be interesting to compare the Poisson structures of this Section 11 with numerous examples in [OR02].

12. $*$-Analog of Horowitz Theorem

In this section we discuss analogs of the Horowitz Theorem 1.4.

12.1. **Algebra $\mathcal{R}$ and ν-endomorphisms.** Let $\boldsymbol{x} = (x_1, \dots, x_6)$. Define an involution ν on the polynomial algebra $\mathcal{R} := \mathbb{Z}[s_1, s_2, s_3^{\pm 1}][\boldsymbol{x}]$ as follows. For an element

$$f = \sum_{\boldsymbol{a} \in \mathbb{N}^6} A_{\boldsymbol{a}}\, x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} x_5^{a_5} x_6^{a_6}, \qquad A_{\boldsymbol{a}} \in \mathbb{Z}[s_1, s_2, s_3^{\pm 1}],$$

define

$$\nu f = \sum_{\boldsymbol{a} \in \mathbb{N}^6} A_{\boldsymbol{a}}^*\, x_1^{a_2} x_2^{a_1} x_3^{a_4} x_4^{a_3} x_5^{a_6} x_6^{a_5}.$$

A $\mathbb{Z}[s_1, s_2, s_3^{\pm 1}]$-algebra endomorphism $\varphi\colon \mathcal{R} \to \mathcal{R}$ is called a *ν-endomorphism* if

$$\varphi(\nu f) = \nu \varphi(f), \qquad f \in \mathcal{R}.$$

If φ is invertible, then φ is called a *ν-automorphism* of $\mathcal{R}$. The group of ν-automorphisms of $\mathcal{R}$ is denoted by $\mathrm{Aut}_\nu(\mathcal{R})$.

There is a one-to-one correspondence between ν-endomorphisms of $\mathcal{R}$ and triples of polynomials $(P,Q,R)\in\mathcal{R}^3$. Such a triple defines a ν-endomorphism

$$\begin{aligned} &x_1\mapsto P, \quad &&x_3\mapsto Q, \quad &&x_5\mapsto R,\\ &x_2\mapsto \nu P, \quad &&x_4\mapsto \nu Q, \quad &&x_6\mapsto \nu R. \end{aligned}$$

In what follows we often define a ν-endomorphism by giving a triple (P,Q,R).

12.2. **Markov group of ν-automorphism.** Consider the following four groups of ν-automorphisms of $\mathcal{R}$:

Type I. The group $G_1^{\rm aut}$ of ν-automorphisms generated by transformations

$$\Lambda_{i,j}\colon x_1\mapsto (-1)^i x_1, \quad x_3\mapsto (-1)^{i+j}x_3, \quad x_5\mapsto (-1)^j x_5, \qquad i,j\in\mathbb{Z}_2.$$

Type II. The group $G_2^{\rm aut}$ of ν-automorphisms generated by transformations,

$$\begin{aligned} \Sigma_1 &: \ x_1\mapsto x_3, \quad x_3\mapsto x_1, \quad x_5\mapsto x_5,\\ \Sigma_2 &: \ x_1\mapsto x_1, \quad x_3\mapsto x_5, \quad x_5\mapsto x_3. \end{aligned}$$

Type III. The group $G_3^{\rm aut}$ of ν-automorphisms generated by transformations

$$\begin{aligned} \mathrm{T}_1 &: \ x_1\mapsto -x_2, \qquad x_3\mapsto x_6-x_1x_3, \qquad x_5\mapsto x_4,\\ \mathrm{T}_2 &: \ x_1\mapsto x_4-x_1x_5, \qquad x_3\mapsto x_2, \qquad x_5\mapsto -x_6. \end{aligned}$$

We have $\mathrm{T}_1\mathrm{T}_2\mathrm{T}_1=\mathrm{T}_2\mathrm{T}_1\mathrm{T}_2$.

Type IV. The group $G_4^{\rm aut}$ of ν-automorphisms generated by transformations

$$\mathrm{M}_{i,j}\colon x_1\mapsto s_3^{-i}x_1, \quad x_3\mapsto s_3^{i+j}x_3, \quad x_5\mapsto s_3^{-j}x_5, \qquad i,j\in\mathbb{Z}.$$

Define the $*$-*Markov group* $\Gamma_M^{\rm aut}$ of ν-automorphisms of $\mathcal{R}$ as the group generated by $G_1^{\rm aut},G_2^{\rm aut},G_3^{\rm aut},G_4^{\rm aut}$.

Define the *Viète ν-involutions* $V_1,V_2,V_3\in\Gamma_M^{\rm aut}$ by the formulas

$$\begin{aligned} V_1&: \quad x_1\mapsto x_3x_5-x_2, \quad && x_3\mapsto x_4, \quad && x_5\mapsto x_6,\\ V_2&: \quad x_1\mapsto x_2, \quad && x_3\mapsto x_1x_5-x_4, \quad && x_5\mapsto x_6,\\ V_3&: \quad x_1\mapsto x_2, \quad && x_3\mapsto x_4, \quad && x_5\mapsto x_1x_3-x_6. \end{aligned}$$

We denote by $\Gamma_V^{\rm aut}$ the group generated by V_1,V_2,V_3. We have

$$V_1=\Lambda_{1,1}\ \Sigma_1\ \mathrm{T}_2, \qquad V_2=\Lambda_{1,0}\ \Sigma_2\ \mathrm{T}_1, \qquad V_3=\Lambda_{1,1}\ \mathrm{T}_1\ \Sigma_2.$$

Theorem 12.1. *We have the following identities:*

$$\begin{aligned} \Sigma_1\Lambda_{i,j}\Sigma_1&=\Lambda_{i+j,j}\,,\\ \Sigma_2\Lambda_{i,j}\Sigma_2&=\Lambda_{i,i+j}\,, \end{aligned}$$

$$\begin{aligned} \Sigma_1V_1\Sigma_1&=V_2, \qquad &\Sigma_2V_1\Sigma_2&=V_1,\\ \Sigma_1V_2\Sigma_1&=V_1, \qquad &\Sigma_2V_2\Sigma_2&=V_3,\\ \Sigma_1V_3\Sigma_1&=V_3, \qquad &\Sigma_2V_3\Sigma_2&=V_2,\\ \Lambda_{k,l}V_i\Lambda_{k,l}&=V_i\,, \qquad &i&=1,2,3, \end{aligned}$$

$$\begin{aligned}
&\Lambda_{k,l}\mathrm{M}_{i,j}\Lambda_{k,l} = \mathrm{M}_{i,j}\,, && k,l\in\mathbb{Z}_2, \quad i,j\in\mathbb{Z},\\
&\Sigma_1\mathrm{M}_{i,j}\Sigma_1 = \mathrm{M}_{-i-j,j}\,, && \Sigma_2\mathrm{M}_{i,j}\Sigma_2 = \mathrm{M}_{i,-i-j}\,,\\
&V_k\mathrm{M}_{i,j}V_k = \mathrm{M}_{-i,-j}\,, && k=1,2,3.
\end{aligned}$$

Proof. These identities are proved by straightforward computations. □

Theorem 12.2. *We have an epimorphism of groups $\iota\colon \Gamma_M \to \Gamma_M^{\mathrm{aut}}$ defined on generators by*

$$\iota(\lambda_{i,j}) := \Lambda_{i,j}, \quad \iota(\sigma_i) := \Sigma_i, \quad \iota(v_j) := V_j, \quad \iota(\mu_{i,j}) := \mathrm{M}_{i,j}. \tag{12.1}$$

Proof. Let us show that the morphism $\iota\colon \Gamma_M \to \Gamma_M^{\mathrm{aut}}$ is well defined. First, notice that (12.1) uniquely extends to a group morphism on each of the groups G_1, G_2, G_4, Γ_V. Any $g\in\Gamma_M$ admits a unique decomposition $g = vg_4g_1g_2$ with $v\in\Gamma_V$, $g_1\in G_1$, $g_2\in G_2$, $g_4\in G_4$, by Corollaries 3.10 and 3.13. We define

$$\iota(g) := \iota(v)\iota(g_4)\iota(g_1)\iota(g_2).$$

Given $\tilde g\in\Gamma_M$, we have to show that $\iota(g\tilde g) = \iota(g)\iota(\tilde g)$. We have

$$g\tilde g = vg_4g_1g_2\tilde v\tilde g_4\tilde g_1\tilde g_2 = v\tilde v' g_4\tilde g_4' g_1\tilde g_1' g_2\tilde g_2,$$

where in the second line we use the commutations relations of Proposition 3.9. The map ι preserves the commutations relations among the generators $\Lambda_{\alpha,\beta}$, Σ_i, V_j, $\mathrm{M}_{\alpha,\beta}$, by Theorem 12.1. So, we have

$$\begin{aligned}
\iota(g\tilde g) &= \iota(v\tilde v')\iota(g_4\tilde g_4')\iota(g_1\tilde g_1')\iota(g_2\tilde g_2)\\
&= \iota(v)\iota(\tilde v')\iota(g_4)\iota(\tilde g_4')\iota(g_1)\iota(\tilde g_1')\iota(g_2)\iota(\tilde g_2)\\
&= \iota(v)\iota(g_4)\iota(g_1)\iota(g_2)\iota(\tilde v)\iota(\tilde g_4)\iota(\tilde g_1)\iota(\tilde g_2)\\
&= \iota(g)\iota(\tilde g).
\end{aligned}$$

This completes the proof. □

Let $P(\boldsymbol{x})\in\mathcal{R}$. Define the map $\widehat{P}\colon \left(\mathbb{Z}[s_1,s_2,s_3^{\pm1}]\right)^3 \to \mathbb{Z}[s_1,s_2,s_3^{\pm1}]$ by the formula

$$\widehat{P}\colon (f_1,f_2,f_3)\mapsto P(f_1,f_1^*,f_2,f_2^*,f_3,f_3^*).$$

Proposition 12.3. *For any $P\in\mathcal{R}$, $g\in\Gamma_M$, $(f_1,f_2,f_3)\in\left(\mathbb{Z}[s_1,s_2,s_3^{\pm1}]\right)^3$, we have*

$$\widehat{\iota(g)P}(f_1,f_2,f_3) = \widehat{P}\left(g^{-1}(f_1,f_2,f_3)\right).$$

Proof. This is easily checked on the generators of Γ_M. □

Proposition 12.4. *The morphism ι is an isomorphism.*

Proof. Consider the polynomials $x_1,x_3,x_5\in\mathcal{R}$. They define the natural projections

$$\begin{aligned}
&\widehat{x}_1\colon \left(\mathbb{Z}[s_1,s_2,s_3^{\pm1}]\right)^3\to\mathbb{Z}[s_1,s_2,s_3^{\pm1}], \quad (f_1,f_2,f_3)\mapsto f_1,\\
&\widehat{x}_3\colon \left(\mathbb{Z}[s_1,s_2,s_3^{\pm1}]\right)^3\to\mathbb{Z}[s_1,s_2,s_3^{\pm1}], \quad (f_1,f_2,f_3)\mapsto f_2,\\
&\widehat{x}_5\colon \left(\mathbb{Z}[s_1,s_2,s_3^{\pm1}]\right)^3\to\mathbb{Z}[s_1,s_2,s_3^{\pm1}], \quad (f_1,f_2,f_3)\mapsto f_3.
\end{aligned}$$

Let $g\in\ker\iota$. By Proposition 12.3, we have

$$\widehat{x}_1(g^{-1}(f_1,f_2,f_3)) = f_1, \quad \widehat{x}_3(g^{-1}(f_1,f_2,f_3)) = f_2, \quad \widehat{x}_5(g^{-1}(f_1,f_2,f_3)) = f_3.$$

Hence $g^{-1} = \mathrm{id}$. □

12.3. **ν-Endomorphisms of maximal rank.** Let $\varphi\colon \mathcal{R}\to\mathcal{R}$ be a ν-endomorphism, defined by a triple $P,Q,R\in\mathcal{R}$,

$$x_1\mapsto P,\quad x_3\mapsto Q,\quad x_5\mapsto R,\quad x_2\mapsto \nu P,\quad x_4\mapsto \nu Q,\quad x_6\mapsto \nu R.$$

For any fixed $\boldsymbol{p}\in\mathbb{C}^3$, denote by $P_{\boldsymbol{p}},Q_{\boldsymbol{p}},R_{\boldsymbol{p}},\nu P_{\boldsymbol{p}},\nu Q_{\boldsymbol{p}},\nu R_{\boldsymbol{p}}\in\mathbb{C}[\boldsymbol{x}]$ the specialization of $P,Q,R,\nu P,\nu Q,\nu R$ at $\boldsymbol{z}=\boldsymbol{p}$.

For any fixed $\boldsymbol{p}\in\mathbb{C}^3$, we have a polynomial map $\varphi_{\boldsymbol{p}}\colon\mathbb{C}^6\to\mathbb{C}^6$ defined by

$$\boldsymbol{q}\mapsto\left(P_{\boldsymbol{p}}(\boldsymbol{q}),\quad \nu P_{\boldsymbol{p}}(\boldsymbol{q}),\quad Q_{\boldsymbol{p}}(\boldsymbol{q}),\quad \nu Q_{\boldsymbol{p}}(\boldsymbol{q}),\quad R_{\boldsymbol{p}}(\boldsymbol{q}),\quad \nu R_{\boldsymbol{p}}(\boldsymbol{q})\right).$$

A ν-endomorphism φ of $\mathcal{R}$ is said to be *of maximal rank* if there exist $\boldsymbol{p}\in\mathbb{C}^3$ and $\boldsymbol{q}\in\mathbb{C}^6$ such that the Jacobian matrix of $\varphi_{\boldsymbol{p}}$ at the point $\boldsymbol{q}$ is invertible.

12.4. **Horowitz type theorem for $*$-Markov group.** Define the ν-Horowitz group $G_{\rm Hor}$ as the group of ν-automorphisms of $\mathcal{R}$ which preserve the polynomial

$$H=x_1x_2+x_3x_4+x_5x_6-x_1x_3x_5.$$

Define $\Gamma^{\max}$ to be the set of ν-endomorphisms of $\mathcal{R}$ of maximal rank, which preserve the polynomial H.

We have $\Gamma_M^{\rm aut}\subseteq G_{\rm Hor}\subseteq\Gamma^{\max}$.

Theorem 12.5. *We have $\Gamma_M^{\rm aut}=G_{\rm Hor}=\Gamma^{\max}$. In particular, any element of $\Gamma^{\max}$ is a ν-automorphism.*

Proof. It is sufficient to prove that $\Gamma^{\max}\subseteq\Gamma_M^{\rm aut}$. The proof is an adaptation of the original argument of [Ho75, Theorem 2]. Let

$$x_1\mapsto P,\quad x_3\mapsto Q,\quad x_5\mapsto R,\quad x_2\mapsto \nu P,\quad x_4\mapsto \nu Q,\quad x_6\mapsto \nu R,\tag{12.2}$$

be an element of $\Gamma^{\max}$, where $P,Q,R\in\mathcal{R}$. Up to an action of $G_2^{\rm aut}$, we can assume that the total degrees of P,Q,R with respect to x_1,x_2,x_3,x_4,x_5,x_6 are in ascending order, i.e.

$$\operatorname{Deg}P\leqslant\operatorname{Deg}Q\leqslant\operatorname{Deg}R.$$

Set

$$\begin{aligned}P&=P_p+P_{p-1}+\dots+P_0,\\ Q&=Q_q+Q_{q-1}+\dots+Q_0,\\ R&=R_r+R_{r-1}+\dots+R_0,\end{aligned}\tag{12.3}$$

where P_k,Q_k,R_k are homogeneous polynomials in x_1,x_2,x_3,x_4,x_5,x_6 of degree k. Necessarily, we must have $p,q,r\geqslant 1$, otherwise (12.2) does not define an endomorphism of maximal rank. Since (12.2) is an element of $\Gamma^{\max}$, we have

$$P\cdot\nu P+Q\cdot\nu Q+R\cdot\nu R-PQR=x_1x_2+x_3x_4+x_5x_6-x_1x_3x_5.\tag{12.4}$$

Suppose that $p=q=r=1$. By comparison of the highest degree terms of the l.h.s. and r.h.s. of (12.4), we deduce that $P_1Q_1R_1=x_1x_3x_5$. Since x_1,x_3,x_5 are irreducible, unique factorization implies that up to reordering of P_1,Q_1,R_1 we have

$$P_1=\gamma_1x_1,\quad Q_1=\gamma_3x_3,\quad R_1=\gamma_5x_5,\tag{12.5}$$

where $\gamma_1,\gamma_3,\gamma_5\in\mathbb{Z}[s_1,s_2,s_3^{\pm1}]$ and $\gamma_1\gamma_3\gamma_5=1$. Hence each of γ_j is of the form $\pm s_3^{a_j}$.

If we substitute (12.5) in (12.3), and expand (12.4), we deduce that $P_0 = Q_0 = R_0 = 0$ (since the r.h.s. of (12.4) has no terms x_1x_3, x_1x_5, x_3x_5). Thus, the only possible form of (12.2) is

$$\begin{aligned} &x_1 \mapsto (-1)^i s_3^{a_3} x_1, \qquad x_3 \mapsto (-1)^{i+j} s_3^{a_3} x_3, \qquad x_5 \mapsto (-1)^j s_3^{a_5} x_5, \\ &x_2 \mapsto (-1)^i s_3^{-a_1} x_2, \quad\ x_4 \mapsto (-1)^{i+j} s_3^{-a_3} x_4, \quad\ x_6 \mapsto (-1)^j s_3^{-a_5} x_6, \end{aligned}$$

where $i, j \in \mathbb{Z}_2$ and $a_1, a_2, a_3 \in \mathbb{Z}$, $a_1 + a_2 + a_3 = 0$. All these transformations are in $\langle G_1^{\mathrm{aut}}, G_4^{\mathrm{aut}} \rangle \subseteq \Gamma_M^{\mathrm{aut}}$.

Now we proceed by induction on the maximum r of the degrees of P, Q, R. If we expand (12.4) using (12.3), we obtain

$$P_p \cdot \nu P_p + Q_q \cdot \nu Q_q + R_r \cdot \nu R_r - P_p Q_q R_r + \cdots = x_1x_2 + x_3x_4 + x_5x_6 - x_1x_3x_5, \tag{12.6}$$

where the dots denote lower degree terms. The term $P_pQ_qR_r$ is of degree at least 4. The degree of every term of the r.h.s. of (12.6) is less than 4. Hence $P_pQ_qR_r$ must cancel with another term of the l.h.s. This is possible if and only if $r = p + q$. If $r = p + q$, then the terms of highest degree are $R_r \cdot \nu R_r$ and $P_pQ_qR_r$, and we must have $R_r \cdot \nu R_r - P_pQ_qR_r = 0$. Thus,

$$\nu R_r = P_p Q_q. \tag{12.7}$$

The transformation

$$x_1 \mapsto \nu P, \quad x_3 \mapsto \nu Q, \quad x_5 \mapsto PQ - \nu R, \tag{12.8}$$

extends to a ν-endomorphism of $\mathcal{R}$. Such an endomorphism is the composition of (12.2) with $V_3 \in \Gamma_M^{\mathrm{aut}}$. The endomorphism (12.8) has the highest degree less than r, because of (12.7). Hence, by induction hypothesis, the endomorphism (12.8) is a ν-automorphism in Γ_M^{aut}. This completes the proof. □

12.5. **Horowitz type theorem for $\mathbb{C}^5$.** Recall the action of the $*$-Markov group on $\mathbb{C}^5$ with coordinates $(y_1, \ldots, y_5)$. In particular, the $*$-Viète involutions act on $\mathbb{C}^5$ by the formulas

$$\begin{aligned} v_1 &: (y_1, \ldots, y_5) \mapsto (y_1 + y_2y_3 - y_4 - y_5, y_2, y_3, -y_5 + y_2y_3, -y_4 + y_2y_3), \\ v_2 &: (y_1, \ldots, y_5) \mapsto (y_1, y_2 + y_1y_3 - y_4 - y_5, y_3, -y_5 + y_1y_3, -y_4 + y_1y_3), \\ v_2 &: (y_1, \ldots, y_5) \mapsto (y_1, y_2, y_3 + y_1y_2 - y_4 - y_5, -y_5 + y_1y_2, -y_4 + y_1y_2). \end{aligned}$$

Theorem 12.6. *Let $\psi : \mathbb{C}^5 \to \mathbb{C}^5$ be a maximal rank polynomial map, which preserves the polynomials*

$$J = y_1y_2y_3 - y_4y_5, \qquad J_1 = y_1 + y_2 + y_3 - y_4, \qquad J_2 = y_1 + y_2 + y_3 - y_5.$$

Then ψ is invertible and lies in the image of the $*$-Markov group.

Remark 12.7. One can easily add the parameters $\mathbb{Z}[s_1, s_2, s_3^{\pm 1}]$ and reformulate Theorem 12.6 similarly to Theorem 12.5.

Corollary 12.8. *If the map ψ satisfies the assumptions of Theorem 12.6, then it commutes with the involution*

$$\nu : (y_1, \ldots, y_5) \mapsto (y_1, y_2, y_3, y_5, y_4).$$

Proof of Theorem 12.6. Let ψ send $(y_1,\dots,y_5)$ to $(P_1,\dots,P_5)$. Then

$$P_1P_2P_3 - P_4P_5 = y_1y_2y_3 - y_4y_5.$$

$$P_4 - P_1 - P_2 - P_3 = y_4 - y_1 - y_2 - y_3, \qquad P_5 - P_1 - P_2 - P_3 = y_5 - y_1 - y_2 - y_3.$$

Hence

$$P_4 = y_4 - y_1 - y_2 - y_3 + P_1 + P_2 + P_3,$$
$$P_5 = y_5 - y_1 - y_2 - y_3 + P_1 + P_2 + P_3,$$

and the map ψ is completely determined by the three polynomials P_1, P_2, P_3.

First assume that ψ is a linear map, $P_i = P_{i,0} + a_i$, where $a_i \in \mathbb{C}$ and $P_{i,0}$ are homogeneous polynomials in $\boldsymbol{y}$ of degree 1. Then

$$y_1y_2y_3 = P_{1,0}P_{2,0}P_{3,0}.$$

Hence after a permutation of P_1, P_2, P_3 we will have

$$P_i = b_iy_i + a_i, \qquad i = 1,2,3, \quad b_i \in \mathbb{C}, \quad b_1b_2b_3 = 1.$$

We have

$$\begin{aligned} P_4 \;&=\; y_4 - y_1 - y_2 - y_3 + P_1 + P_2 + P_3 \\ &= y_4 + (b_1-1)y_1 + (b_2-1)y_2 + (b_3-1)y_3 + a_1 + a_2 + a_3, \\ P_5 \;&=\; y_5 - y_1 - y_2 - y_3 + P_1 + P_2 + P_3 \\ &= y_4 + (b_1-1)y_1 + (b_2-1)y_2 + (b_3-1)y_3 + a_1 + a_2 + a_3, \end{aligned}$$

Hence

$$\begin{aligned} &\big(y_4 + (b_1-1)y_1 + (b_2-1)y_2 + (b_3-1)y_3 + a_1 + a_2 + a_3\big) \\ &\times\big(y_5 + (b_1-1)y_1 + (b_2-1)y_2 + (b_3-1)y_3 + a_1 + a_2 + a_3\big) \\ &-(b_1y_1 + a_1)(b_2y_2 + a_2)(b_3y_3 + a_3) = y_4y_5 - y_1y_2y_3. \end{aligned}$$

This means that $b_i = 1$ for all i and hence

$$\begin{aligned} &\big(y_4 + a_1 + a_2 + a_3\big)\big(y_5 + a_1 + a_2 + a_3\big) \\ &\qquad - (y_1 + a_1)(y_2 + a_2)(y_3 + a_3) = y_4y_5 - y_1y_2y_3. \end{aligned}$$

This implies that $a_i = 0$ for all i.

Equation $P_4P_5 - P_1P_2P_3 = y_4y_5 - y_1y_2y_3$ can be rewritten as

$$\begin{aligned} (y_4 - y_1 - y_2 - y_3 + P_1 + P_2 + P_3)(y_5 - y_1 - y_2 - y_3 + P_1 + P_2 + P_3) - P_1P_2P_3 \\ = y_4y_5 - y_1y_2y_3. \end{aligned} \tag{12.9}$$

Let us write $P_i = P_{i,1} + \dots$, $i = 1,2,3$, where $P_{i,1}$ is the top degree homogeneous component of P_i. Denote d_i the degree of P_i. Since ψ is of maximal rank, we have $d_i > 0$.

Assume that the maximum of d_1, d_2, d_3 is greater than 1. After a permutation of the first three coordinates we may assume that $d_1 \geqslant d_2 \geqslant d_3$. Then equation (12.9) implies that $d_1 = d_2 + d_3$ and there are exactly two terms of degree $2d_1$ which have to cancel,

$$P_{1,1}P_{2,1}P_{3,1} - P_{1,1}^2 = P_{1,1}(P_{2,1}P_{3,1} - P_{1,1}) = 0.$$

Let us compose ψ with involution v_1. Then $v_1 \circ \psi$ sends $(y_1, \dots, y_5)$ to $(\tilde{P}_1, \dots, \tilde{P}_5)$, where $\tilde{P}_2 = P_2$, $\tilde{P}_3 = P_3$,

$$\begin{aligned}
\tilde{P}_1 &= P_1 + P_2P_3 - P_4 - P_5 \\
&= P_1 + P_2P_3 - (y_4 - y_1 - y_2 - y_3 + P_1 + P_2 + P_3) \\
&\quad - (y_5 - y_1 - y_2 - y_3 + P_1 + P_2 + P_3) \\
&= P_2P_3 - P_1 - 2P_2 - 2P_3 - y_4 - y_5 + 2y_1 + 2y_3 + 2y_3,
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_4 &= -(y_5 - y_1 - y_2 - y_3 + P_1 + P_2 + P_3) + P_2P_3, \\
\tilde{P}_5 &= -(y_4 - y_1 - y_2 - y_3 + P_1 + P_2 + P_3) + P_2P_3,
\end{aligned}$$

These formulas show that $\deg \tilde{P}_1 < \deg P_1$, while $\deg \tilde{P}_i = \deg P_i$ for $i = 2, 3$, and the theorem follows from the iteration of this procedure. $\square$

Appendix A. Horowitz type theorems

A.1. **Classical setting.** Let $a_0, \dots, a_n \in \mathbb{Z}$ be non-zero integers such that

$$a_j \quad \text{divides} \quad a_0, \quad \text{for} \quad j = 1, \dots, n.$$

Consider the polynomial in n variables $x_1, \dots, x_n$,

$$H := \sum_{j=1}^{n} a_j x_j^2 - a_0 \prod_{j=1}^{n} x_j \,.$$

The polynomial H is quadratic with respect to each variable x_j. This ensures that the polynomial has a nontrivial group of symmetries.

Theorem A.1. *Let $\sigma \in \mathfrak{S}_n$ be such that $(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = (a_1, \dots, a_n)$. Then the permutation $(x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ preserves the polynomial H.* $\square$

Theorem A.2. *For any $i = 1, \dots, n$, the transformation*

$$v_i : (x_1, \dots, x_n) \mapsto \left(x_1, \dots, x_{i-1}, -x_i + \frac{a_0}{a_i} \prod_{j \neq i} x_j, x_{i+1}, \dots, x_n \right)$$

is an involution preserving the polynomial H.

Proof. We check this for v_1. Denote $y = \frac{a_0}{a_1} \prod_{j>1} x_j$. Then

$$\begin{aligned}
&H(-x_1 + y, x_2, \dots, x_n) = a_1(-x_1 + y)^2 - a_0(-x_1 + y) \prod_{j>1} x_j + \sum_{l>1} x_l^2 \\
&= a_1 x_1^2 - 2x_1 a_0 \prod_{j>1} x_j + \frac{a_0^2}{a_1} \prod_{j>1} x_j^2 + a_0 x_1 \prod_{j>1} x_j - \frac{a_0^2}{a_1} \prod_{j>1} x_j^2 + \sum_{l>1} x_l^2 \\
&= H(x_1, x_2, \dots, x_n).
\end{aligned}$$

$\square$

The permutations of Theorem A.1 and the Viète maps of Theorem A.2 are automorphisms of the algebra $\mathbb{Z}[\boldsymbol{x}]$.

We say that an endomorphism of algebras $\varphi \colon \mathbb{Z}[\boldsymbol{x}] \to \mathbb{Z}[\boldsymbol{x}]$ defined by

$$x_j \mapsto P_j(\boldsymbol{x}), \quad P_j \in \mathbb{Z}[\boldsymbol{x}], \quad j = 1, \dots, n,$$

is *of maximal rank* if there exists a point $\boldsymbol{q} \in \mathbb{C}^n$ such that the Jacobian matrix of φ at $\boldsymbol{q}$ is invertible.

The following is a stronger version of the original Horowitz Theorem.

Theorem A.3. *Any endomorphism of maximal rank preserving the polynomial H is an automorphism. The group of all automorphisms of $\mathbb{Z}[\boldsymbol{x}]$ preserving H is generated by the Viète transformations, by the permutation of variables preserving the n-tuple $(a_1, \dots, a_n)$, and by multiplication by -1 of an even number of variables.*

Proof. The argument is the same of the proof of Theorem 12.5. □

A.2. **$*$-Setting.** Let m, n be two positive integers. Let $a_0, \dots, a_n$ be symmetric Laurent polynomials in $z_1, \dots, z_m$ with integer coefficients and such that

$$a_j^* = a_j, \quad \text{and} \quad a_j \text{ divides } a_0 \quad \text{for } j = 1, \dots, n,$$

in the algebra $\mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_m}$ of symmetric Laurent polynomials.

Consider the polynomial in $2n$ variables $x_1, x_2 \dots, x_{2n-1}, x_{2n}$,

$$H := \sum_{j=1}^n a_j x_{2j-1} x_{2j} - a_0 \prod_{j=1}^n x_{2j-1}\,.$$

The algebra $\mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_m}[\boldsymbol{x}]$ admits an involution

$$\nu\colon x_{2j-1} \mapsto x_{2j}, \quad x_{2j} \mapsto x_{2j-1}, \qquad j = 1, \dots, n.$$

The notions of a ν-endomorphism and a ν-automorphism given in Section 12.1 obviously extend to the algebra $\mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_m}[\boldsymbol{x}]$.

The polynomial H has a nontrivial group of symmetries.

Theorem A.4. *Let $\sigma \in \mathfrak{S}_n$ be such that $(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = (a_1, \dots, a_n)$. Then the permutation $\boldsymbol{x} \mapsto (x_{2\sigma(1)-1}, x_{2\sigma(1)} \dots, x_{2\sigma(n)-1}, x_{2\sigma(n)})$ preserves the polynomial H.* □

Theorem A.5. *For any $i = 1, \dots, n$, the transformation v_i defined by*

$$\begin{gathered}
x_1 \mapsto x_2, \qquad x_2 \mapsto x_1, \quad \dots\,, \quad x_{2i-3} \mapsto x_{2i-2}, \qquad x_{2i-2} \mapsto x_{2i-3}, \\
x_{2i-1} \mapsto -x_{2i} + \frac{a_0}{a_i} \prod_{j \neq i} x_{2j-1}, \qquad x_{2i} \mapsto -x_{2i-1} + \frac{a_0}{a_i} \prod_{j \neq i} x_{2j}, \\
x_{2i+1} \mapsto x_{2i+2}, \quad x_{2i+2} \mapsto x_{2i+1}, \quad \dots\,, \quad x_{2n-1} \mapsto x_{2n}, \quad x_{2n} \mapsto x_{2n-1},
\end{gathered}$$

is an involution preserving the polynomial H.

Proof. The proof is by straightforward calculation, the same as for $n = 3$. □

The permutations of Theorem A.4 and the Viète maps of Theorem A.5 are ν-automorphisms of the algebra $\mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_m}[\boldsymbol{x}]$.

Given $P \in \mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_m}[\boldsymbol{x}]$ and $\boldsymbol{p} \in \mathbb{C}^m$ denotes by $P_{\boldsymbol{p}} \in \mathbb{C}[\boldsymbol{x}]$ the specialization of P at $\boldsymbol{z} = \boldsymbol{p}$.

Let $\varphi\colon \mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_m}[\boldsymbol{x}] \to \mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_m}[\boldsymbol{x}]$ be a ν-endomorphism defined by

$$x_{2j-1} \mapsto P_j(\boldsymbol{x}), \quad x_{2j} \mapsto \nu P_j(\boldsymbol{x}), \quad j = 1, \dots, n.$$

For any $\boldsymbol{p} \in \mathbb{C}^m$ there is a map $\varphi_{\boldsymbol{p}} \colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ defined by

$$\boldsymbol{x} \mapsto (P_{1,\boldsymbol{p}}(\boldsymbol{x}), \nu P_{1,\boldsymbol{p}}(\boldsymbol{x}), \ldots, P_{n,\boldsymbol{p}}(\boldsymbol{x}), \nu P_{n,\boldsymbol{p}}(\boldsymbol{x})).$$

The ν-endomorphism $\varphi \colon \mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_m}[\boldsymbol{x}] \to \mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_m}[\boldsymbol{x}]$ is said to be *of maximal rank* if there exist a point $\boldsymbol{p} \in \mathbb{C}^m$ and a point $\boldsymbol{q} \in \mathbb{C}^{2n}$ such that the Jacobian matrix of $\varphi_{\boldsymbol{p}}$ at q is invertible.

Theorem A.6. *Any ν-endomorphism of maximal rank preserving H is a ν-automorphism. The group of all ν-automorphisms of $\mathbb{Z}[\boldsymbol{z}^{\pm 1}]^{\mathfrak{S}_m}[\boldsymbol{x}]$ preserving H is generated by the Viète transformations of Theorem A.5, by the permutation of variables preserving $(a_1, \ldots, a_n)$, by multiplication by -1 of an even number of variables, and by multiplication of variables by powers of $s_m := \prod_{j=1}^m z_j$.*

Proof. The proof is the same as for $n = 3$. □

Appendix B. $*$-Equations for $\mathbb{P}^3$ and Poisson structures

B.1. **$*$-Equations for $\mathbb{P}^3$.** As we wrote in the Introduction, a T-full exceptional collection (E_1, E_2, E_3) in $\mathcal{D}^b_T(\mathbb{P}^2)$ has the matrix $(\chi_T(E_i^* \otimes E_j))$ of equivariant Euler characteristics of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, where (a, b, c) are symmetric Laurent polynomials in the equivariant parameters z_1, z_2, z_3 satisfying the $*$-Markov equation

$$aa^* + bb^* + cc^* - ab^*c = 3 - \frac{z_1^3 + z_2^3 + z_3^3}{z_1 z_2 z_3}. \tag{B.1}$$

Similar objects and equations are available for any projective space $\mathbb{P}^n$. For example, for $\mathbb{P}^3$ the matrix of equivariant Euler characteristics has the form $\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where (a, b, c, d, e, f) are symmetric Laurent polynomials in the equivariant parameters z_1, z_2, z_3, z_4 satisfying the system of equations

$$\begin{aligned} &aa^* + bb^* + cc^* + dd^* + ee^* + ff^* \\ &\quad - a^*bd^* - a^*ce^* - b^*cf^* - d^*ef^* + a^*cd^*f^* \\ &\quad = 4 + \frac{z_2 z_3 z_4}{z_1^3} + \frac{z_1 z_3 z_4}{z_2^3} + \frac{z_1 z_2 z_4}{z_3^3} + \frac{z_1 z_2 z_3}{z_4^3}, \end{aligned} \tag{B.2}$$

$$\begin{aligned} &-2aa^* - 2bb^* - 2cc^* - 2dd^* - 2ee^* - 2ff^* \\ &\quad + ab^*d + a^*bd^* + ac^*e + a^*ce^* + b^*cf^* + bc^*f + de^*f + d^*ef^* \\ &\quad - ab^*ef^* - a^*be^*f - bc^*d^*e - b^*cde^* \\ &\quad + aa^*ff^* + bb^*ee^* + cc^*dd^* \\ &\quad = -6 + \frac{z_2^2 z_4^2}{z_1^2 z_3^2} + \frac{z_2^2 z_3^2}{z_1^2 z_4^2} + \frac{z_1^2 z_2^2}{z_3^2 z_4^2} + \frac{z_3^2 z_4^2}{z_1^2 z_2^2} + \frac{z_1^2 z_4^2}{z_2^2 z_3^2} + \frac{z_1^2 z_3^2}{z_2^2 z_4^2}, \end{aligned} \tag{B.3}$$

$$\begin{aligned} &aa^* + bb^* + cc^* + dd^* + ee^* + ff^* \\ &\quad - ab^*d - ac^*e - bc^*f - de^*f + ac^*df \\ &\quad = 4 + \frac{z_1^3}{z_2z_3z_4} + \frac{z_2^3}{z_1z_3z_4} + \frac{z_3^3}{z_1z_2z_4} + \frac{z_4^3}{z_1z_2z_3}, \end{aligned} \tag{B.4}$$

see [CV20, Formulas (3.24)-(3.26)]. One may study this system of equations similarly to our study of the $*$-Markov equation.

In this appendix we briefly discuss the analogs for the system of equations (B.2)-(B.4) of the Poisson structure on $\mathbb{C}^6$ constructed in Section 11. It will be a family of Poisson structures on $\mathbb{C}^{12}$.

B.2. **Poisson structures on $\mathbb{C}^{12}$.** Consider $\mathbb{C}^{12}$ with coordinates $\boldsymbol{x} = (x_1, \dots, x_{12})$, involution

$$\nu\colon \mathbb{C}^{12} \to \mathbb{C}^{12}, \qquad x_{2j-1} \mapsto x_{2j}, \qquad x_{2j} \mapsto x_{2j-1}, \qquad j = 1, \dots, 6,$$

and polynomials

$$\begin{aligned} H_1(\boldsymbol{x}) = \quad & x_1x_2 + x_3x_4 + x_5x_6 + x_7x_8 + x_9x_{10} + x_{11}x_{12} \\ & -x_2x_3x_8 - x_2x_5x_{10} - x_4x_5x_{12} - x_8x_9x_{12} + x_2x_5x_8x_{12}, \\ H_2(\boldsymbol{x}) = \quad & -2x_1x_2 - 2x_3x_4 - 2x_5x_6 - 2x_7x_8 - 2x_9x_{10} - 2x_{11}x_{12} \\ & +x_3x_8x_2 + x_5x_{10}x_2 + x_1x_4x_7 + x_1x_6x_9 + x_3x_6x_{11} \\ & +x_7x_{10}x_{11} + x_4x_5x_{12} + x_8x_9x_{12} - x_3x_{10}x_{11}x_2 + x_1x_{11}x_{12}x_2 \\ & +x_5x_6x_7x_8 - x_3x_6x_8x_9 - x_4x_5x_7x_{10} \\ & +x_3x_4x_9x_{10} - x_1x_4x_9x_{12}, \\ H_3(\boldsymbol{x}) = \quad & x_1x_2 + x_3x_4 + x_5x_6 + x_7x_8 + x_9x_{10} + x_{11}x_{12} \\ & -x_1x_4x_7 - x_1x_6x_9 - x_3x_6x_{11} - x_7x_{10}x_{11} + x_1x_6x_7x_{11}. \end{aligned}$$

We have $\nu^\star H_1 = H_3$, $\nu^\star H_2 = H_2$, $\nu^\star H_3 = H_1$.

Consider the braid group $\mathcal{B}_4$ with standard generators τ_1, τ_2, τ_3. The group $\mathcal{B}_4$ acts on $\mathbb{C}^{12}$,

$$\begin{aligned}
&\tau_1^\star x_1 = -x_2, && \tau_2^\star x_1 = x_3 - x_1x_7, && \tau_3^\star x_1 = x_1, \\
&\tau_1^\star x_2 = -x_1, && \tau_2^\star x_2 = x_4 - x_2x_8, && \tau_3^\star x_2 = x_2, \\
&\tau_1^\star x_3 = -x_2x_3 + x_7, && \tau_2^\star x_3 = x_1, && \tau_3^\star x_3 = x_5 - x_3x_{11}, \\
&\tau_1^\star x_4 = -x_1x_4 + x_8, && \tau_2^\star x_4 = x_2, && \tau_3^\star x_4 = x_6 - x_4x_{12}, \\
&\tau_1^\star x_5 = -x_2x_5 + x_9, && \tau_2^\star x_5 = x_5, && \tau_3^\star x_5 = x_3, \\
&\tau_1^\star x_6 = -x_1x_6 + x_{10}, && \tau_2^\star x_6 = x_6, && \tau_3^\star x_6 = x_4, \\
&\tau_1^\star x_7 = x_3, && \tau_2^\star x_7 = -x_8, && \tau_3^\star x_7 = x_9 - x_7x_{11}, \\
&\tau_1^\star x_8 = x_4, && \tau_2^\star x_8 = -x_7, && \tau_3^\star x_8 = x_{10} - x_8x_{12}, \\
&\tau_1^\star x_9 = x_5, && \tau_2^\star x_9 = -x_8x_9 + x_{11}, && \tau_3^\star x_9 = x_7,
\end{aligned}$$

$$\tau_1^\star x_{10} = x_6, \qquad \tau_2^\star x_{10} = -x_7x_{10} + x_{12}, \qquad \tau_3^\star x_{10} = x_8,$$
$$\tau_1^\star x_{11} = x_{11}, \qquad \tau_2^\star x_{11} = x_9, \qquad \tau_3^\star x_{11} = -x_{12},$$
$$\tau_1^\star x_{12} = x_{12}, \qquad \tau_2^\star x_{12} = x_{10}, \qquad \tau_3^\star x_{12} = -x_{11}.$$

Theorem B.1. *The space V of all quadratic Poisson structures on $\mathbb{C}^{12}$, which have H_1, H_2, H_3 as Casimir elements, is a 3-dimensional vector space consisting of log-canonical structures. For suitable coordinates $\boldsymbol{b} = (b_1, b_2, b_3)$ on V, the Poisson structures have the form:*

$$\{x_1, x_2\} = 0, \quad \{x_3, x_4\} = 0, \quad \{x_5, x_6\} = 0, \quad \{x_7, x_8\} = 0, \quad \{x_9, x_{10}\} = 0, \quad \{x_{11}, x_{12}\} = 0,$$

$$\begin{aligned}
\{x_1, x_{11}\} &= b_2x_1x_{11}, & \{x_1, x_{12}\} &= -b_2x_1x_{12},\\
\{x_2, x_{11}\} &= -b_2x_2x_{11}, & \{x_2, x_{12}\} &= b_2x_2x_{12},\\
\{x_3, x_9\} &= -(b_1 - b_2 + b_3)x_3x_9, & \{x_3, x_{10}\} &= (b_1 - b_2 + b_3)x_3x_{10},\\
\{x_4, x_9\} &= (b_1 - b_2 + b_3)x_4x_9, & \{x_4, x_{10}\} &= -(b_1 - b_2 + b_3)x_4x_{10},\\
\{x_5, x_7\} &= (b_1 - b_3)x_5x_7, & \{x_5, x_8\} &= -(b_1 - b_3)x_5x_8,\\
\{x_6, x_7\} &= -(b_1 - b_3)x_6x_7, & \{x_6, x_8\} &= (b_1 - b_3)x_6x_8,\\
\{x_1, x_3\} &= -b_3x_1x_3, & \{x_1, x_4\} &= b_3x_1x_4,\\
\{x_2, x_3\} &= b_3x_2x_3, & \{x_2, x_4\} &= -b_3x_2x_4,\\
\{x_1, x_5\} &= -(b_3 - b_2)x_1x_5, & \{x_1, x_5\} &= (b_3 - b_2)x_1x_6,\\
\{x_2, x_5\} &= (b_3 - b_2)x_2x_5, & \{x_2, x_6\} &= -(b_3 - b_2)x_2x_6,\\
\{x_1, x_7\} &= -b_3x_1x_7, & \{x_1, x_8\} &= b_3x_1x_8,\\
\{x_2, x_7\} &= b_3x_2x_7, & \{x_2, x_8\} &= -b_3x_2x_8,\\
\{x_1, x_9\} &= -(b_3 - b_2)x_1x_9, & \{x_1, x_{10}\} &= (b_3 - b_2)x_1x_{10},\\
\{x_2, x_9\} &= (b_3 - b_2)x_2x_9, & \{x_2, x_{10}\} &= -(b_3 - b_2)x_2x_{10},\\
\{x_3, x_5\} &= -(b_1 - b_2)x_3x_5, & \{x_3, x_6\} &= (b_1 - b_2)x_3x_6,\\
\{x_4, x_5\} &= (b_1 - b_2)x_4x_5, & \{x_4, x_6\} &= -(b_1 - b_2)x_4x_6,\\
\{x_3, x_7\} &= -b_3x_3x_7, & \{x_3, x_8\} &= b_3x_3x_8,\\
\{x_4, x_7\} &= b_3x_4x_7, & \{x_4, x_8\} &= -b_3x_4x_8,\\
\{x_3, x_{11}\} &= -(b_1 - b_2)x_3x_{11}, & \{x_3, x_{12}\} &= (b_1 - b_2)x_3x_{12},\\
\{x_4, x_{11}\} &= (b_1 - b_2)x_4x_{11}, & \{x_4, x_{12}\} &= -(b_1 - b_2)x_4x_{12},\\
\{x_5, x_7\} &= (b_1 - b_3)x_5x_7, & \{x_5, x_8\} &= -(b_1 - b_3)x_5x_8,\\
\{x_6, x_7\} &= -(b_1 - b_3)x_6x_7, & \{x_6, x_8\} &= (b_1 - b_3)x_6x_8,\\
\{x_5, x_9\} &= -(b_3 - b_2)x_5x_9, & \{x_5, x_{10}\} &= (b_3 - b_2)x_5x_{10},\\
\{x_6, x_9\} &= (b_3 - b_2)x_6x_9, & \{x_6, x_{10}\} &= -(b_3 - b_2)x_6x_{10},\\
\{x_5, x_{11}\} &= -(b_1 - b_2)x_5x_{11}, & \{x_5, x_{12}\} &= (b_1 - b_2)x_5x_{12},\\
\{x_6, x_{11}\} &= (b_1 - b_2)x_6x_{11}, & \{x_6, x_{12}\} &= -(b_1 - b_2)x_6x_{12},
\end{aligned}$$

$$\begin{aligned}
\{x_7, x_9\} &= -b_1 x_7 x_9, & \{x_7, x_{10}\} &= b_1 x_7 x_{10},\\
\{x_8, x_9\} &= b_1 x_8 x_9, & \{x_8, x_{10}\} &= -b_1 x_8 x_{10},\\
\{x_7, x_{11}\} &= -b_1 x_7 x_{11}, & \{x_7, x_{12}\} &= b_1 x_7 x_{12},\\
\{x_8, x_{11}\} &= b_1 x_8 x_{11}, & \{x_8, x_{12}\} &= -b_1 x_8 x_{12},\\
\{x_9, x_{11}\} &= -b_1 x_9 x_{11}, & \{x_9, x_{12}\} &= b_1 x_9 x_{12},\\
\{x_{10}, x_{11}\} &= b_1 x_{10} x_{11}, & \{x_{10}, x_{12}\} &= -b_1 x_{10} x_{12}.
\end{aligned}$$

Each of these Poisson structures is ν-invariant. If $(b_1, b_2, b_3) \neq (0,0,0)$, then the Poisson structure is of rank 2.

The Poisson structure with parameters $\boldsymbol{b}$ is denoted by $\{\, ,\, \}_{\boldsymbol{b}}$.

Proof. A computer assistant calculation shows that the only requirements on a quadratic bracket $\{\, ,\, \}$ to be skew-symmetric and have H_1, H_2, H_3 as Casimir elements uniquely determines the Poisson structures above. □

Another computer assistant calculation shows that if a polynomial Poisson structure $\{\, ,\, \}$ on $\mathbb{C}^{12}$ has H_1, H_2, H_3 as Casimir elements, then its Taylor expansion at the origin, has to start with at least quadratic terms.

B.3. **Braid group $\mathcal{B}_4$ action.** Given a Poisson bracket $\{\, ,\, \}$ on $\mathbb{C}^{12}$ define the Poisson bracket $\{\, ,\, \}^{\tau_i}$ by

$$\{f, g\}^{\tau_i} := \tau_i^\star \{f \circ \tau_i^{-1}, g \circ \tau_i^{-1}\}, \quad i = 1, 2, 3.$$

These formulas define a braid group $\mathcal{B}_4$ action on the space of Poisson structures on $\mathbb{C}^{12}$.

Theorem B.2. *The three-parameter family of Poisson structures $\{\, ,\, \}_{\boldsymbol{b}}$ is invariant with respect to the braid group $\mathcal{B}_4$-action on the space of all Poisson structures. The induced braid group $\mathcal{B}_4$ action ρ on the space of parameters V is a vector representation defined by the formulas,*

$$\begin{aligned}
\tau_1\colon &\quad (b_1, b_2, b_3) \mapsto (b_1 - b_2, \quad -b_2, \quad -b_3),\\
\tau_2\colon &\quad (b_1, b_2, b_3) \mapsto (-b_1, \quad -b_1 + b_2 - b_3, \quad -b_3),\\
\tau_3\colon &\quad (b_1, b_2, b_3) \mapsto (-b_1, \quad -b_2, \quad -b_2 + b_3).
\end{aligned}$$

The representation ρ factors through a representation of the symmetric group $\mathfrak{S}_4$, $\rho(\tau_i^2) = \mathrm{id}$, $i = 1, 2, 3$. The representation $\rho : \mathfrak{S}_4 \to \mathrm{GL}(V)$ is irreducible and is isomorphic to the standard three-dimensional representation tensored with the sgn *representation.* □

By Theorem B.2, there is no a $\mathcal{B}_4$-invariant or $\mathcal{B}_4$-anti-invariant quadratic Poisson structure on $\mathbb{C}^{12}$, which has H_1, H_2, H_3 as Casimir elements.

Remark B.3. A computer assistant calculation shows that if a log-canonical Poisson structure $\{\, ,\, \}$ on $\mathbb{C}^{12}$ remains to be log-canonical after the action on it by any element of the braid group $\mathcal{B}_4$, then $\{\, ,\, \}$ is one of the Poisson structures $\{\, ,\, \}_{\boldsymbol{b}}$ in Theorem B.1.

B.4. **Coefficients of $\{\ ,\ \}_{\boldsymbol{b}}$ and elements of weight lattice.** Consider $\mathbb{C}^4$ with standard Euclidean quadratic form $(\ ,\ )$. Denote $(1,-1,0,0)$, $(0,1,-1,0)$, $(0,0,1,-1) \in \mathbb{C}^4$ by v_1, v_2, v_3. We identify the space of parameters V with the subspace $\{\boldsymbol{t} \in \mathbb{C}^4 \mid \sum_{i=1}^4 t_i = 0\}$, by sending a point of V with coordinates (b_1, b_2, b_3) to the point $b_1v_1 + b_2v_2 + b_3v_3$. The vectors v_1, v_2, v_3 generate the *root lattice* in V.

For $i = 1, 2, 3$, the linear map $\rho(\tau_i) : V \to V$ permutes the i-th and $i+1$-st coordinates of vectors of V and multiplies the vectors by -1.

The *weight lattice* in V is the lattice of the elements $t = (t_1, t_2, t_3, t_4) \in \mathbb{C}^4$ such that $\sum_{i=1}^4 t_i = 0$ and $(t, v_i) \in \mathbb{Z}$, $i = 1, 2, 3$. The weight lattice has a basis $w_1 = (3, -1, -1, -1)/4$, $w_2 = (2, 2, -2, -2)/4$, $w_3 = (1, 1, 1, -3)/4$ with the property $(w_i, v_j) = \delta_{ij}$ for all i, j.

There are exactly 8 vectors of the weight lattice of square length 12/16,

$$\pm w_1, \quad \pm w_1 \mp w_2, \quad \pm w_2 \mp w_3, \quad \pm w_3, \tag{B.5}$$

and there are exactly 6 vectors of the weight lattice of square length 1,

$$\pm w_2, \quad , \pm w_1 \mp w_2 \pm w_3, \quad \pm w_1 \mp w_3. \tag{B.6}$$

All other vectors of the root lattice are longer. These two groups of vectors form two $\mathfrak{S}_4$-orbits.

The scalar products of these 14 vectors with the vector $b_1v_1 + b_2v_2 + b_3v_3$ give us the linear functions in b_1, b_2, b_3,

$$\pm b_1, \quad \pm b_1 \mp b_2, \quad \pm b_2 \mp b_3, \quad \pm b_3, \quad \pm b_2, \quad , \pm b_1 \mp b_2 \pm b_3, \quad \pm b_1 \mp b_3.$$

These are exactly the linear functions appearing as coefficients of the Poisson structure $\{\ ,\ \}_{\boldsymbol{b}}$ of Theorem B.1.

B.5. **Casimir subalgebra.** Denote by $\mathcal{C}$ the subalgebra of $\mathbb{C}[\boldsymbol{x}]$ generated by the following 20 monomials:

$$\begin{array}{llll}
m_1 = x_1x_2, & m_2 = x_3x_4, & m_3 = x_5x_6, & m_4 = x_7x_8, \\
m_5 = x_9x_{10}, & m_6 = x_{11}x_{12}, & m_7 = x_2x_3x_8, & m_8 = x_2x_5x_{10}, \\
m_9 = x_4x_5x_{12}, & m_{10} = x_8x_9x_{12}, & m_{11} = x_1x_4x_7, & m_{12} = x_1x_6x_9, \\
m_{13} = x_3x_6x_{11}, & m_{14} = x_7x_{10}x_{11}, & m_{15} = x_2x_5x_8x_{12}, & m_{16} = x_2x_3x_{10}x_{11}, \\
m_{17} = x_3x_6x_8x_9, & m_{18} = x_4x_5x_7x_{10}, & m_{19} = x_1x_4x_9x_{12}, & m_{20} = x_1x_6x_7x_{11}.
\end{array}$$

It is easy to see that the polynomials H_1, H_2, H_3 are elements of the subalgebra $\mathcal{C}$.

Theorem B.4. *For every $\boldsymbol{b}$ each element of the subalgebra $\mathcal{C}$ is a Casimir element of the Poisson structure $\{\ ,\ \}_{\boldsymbol{b}}$. The subalgebra $\mathcal{C}$ is ν-invariant and the braid group $\mathcal{B}_4$ action invariant.*

Proof. The theorem is proved by direct verification. For example, easy calculations lead to formulas like

$$\begin{aligned}
\tau_1^\star m_2 &= -m_{11} + m_1m_2 + m_4 - m_7, \\
\tau_3^\star m_{18} &= m_{17} - m_{10}m_2 - m_{13}m_4 + m_2m_4m_6,
\end{aligned}$$

which prove the braid group invariance of $\mathcal{C}$. $\square$

B.6. **Symplectic leaves.** Since $\{\ ,\ \}_{\boldsymbol{b}}$ is of rank 2, the symplectic leaves of $\{\ ,\ \}_{\boldsymbol{b}}$ are two-dimensional. In logarithmic coordinates $\log x_i$, $i = 1, \dots, 12$, they are two-dimensional affine subspaces. More precisely, we have the following statement.

Theorem B.5. *Given $(b_1, b_2, b_3) \neq (0, 0, 0)$, then the function*

$$C_{\boldsymbol{b}}(\boldsymbol{x}) = (b_1 - b_2) \log x_1 + (b_2 - b_3) \log x_3 + b_3 \log x_5$$

is a Casimir element of $\{\ ,\ \}_{\boldsymbol{b}}$; the $\mathbb{C}$-span of $C_{\boldsymbol{b}}$ and the functions $\log m_i$, $i = 1, \dots, 20$, is 10-dimensional, while the $\mathbb{C}$-span of the functions $\log m_i$, $i = 1, \dots, 20$, is 9-dimensional. □

Hence the symplectic leaves of $\{\ ,\ \}_{\boldsymbol{b}}$ are the surfaces, on which the functions of this 10-dimensional $\mathbb{C}$-span are constant. In particular, the leaves do depend on $\boldsymbol{b}$.

We may also conclude that $x_1^{b_1-b_2} x_3^{b_2-b_3} x_5^{b_3}$ is a Casimir element of $\{\ ,\ \}_{\boldsymbol{b}}$ functionally independent of the Casimir elements m_i, $i = 1, \dots, 20$.

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