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Poles of Intégrale Tritronquée and Anharmonic Oscillators. A WKB Approach

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Abstract

Poles of solutions to the Painlevé-I equations are intimately related to the theory of the cubic anharmonic oscillator. In particular, poles of intégrale tritronquée are in bijection with cubic oscillators that admit the simultaneous solutions of two quantization conditions. We analyze this pair of quantization conditions by means of a suitable version of the complex WKB method.

1 Introduction

The aim of the present paper is to study the distribution of poles of the solutions $y = y(z)$ to the Painlevé first equation (P-I)

$$y'' = 6y^2 - z\,,\ z \in \mathbb{C} \quad ,$$

with particular attention to the poles of the *intégrale tritronquée*.

As it is well-known, any local solution of P-I extends to a global meromorphic function $y(z), z \in \mathbb{C}$, with an essential singularity at infinity [GLS00]. Global solutions of P-I are called Painlevé-I transcendents, since they cannot be expressed via elementary functions or classical special functions [Inc56]. The intégrale tritronquée is a special P-I transcendent, which was discovered by Boutroux in his classical paper [Bou13] (see [JK88] and [Kit94] for a modern review). Boutroux characterized the intégrale tritronquée as the unique solution of P-I with the following asymptotic behaviour at infinity

$$y(z) \sim -\sqrt{\frac{z}{6}}, \quad \text{if} \quad |\arg z| < \frac{4\pi}{5}\ .$$

Nowadays Painlevé first equation is studied in many areas of mathematics and physics. Indeed, it is remarkable that special solutions of P-I describe semiclassical asymptotics of a wealth of different problems (see [Kap04] and

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references therein). In particular, in [DGK09] it is discovered that the intégrale tritronquée provides the universal correction to the dispersionless limit of solutions to the focusing nonlinear Schrödinger equation.

Theoretical and numerical evidences led the authors of [DGK09] to the following inspiring

Conjecture. *If $a \in \mathbb{C}$ is a pole of the intégrale tritronquée then* $|\arg a| \geq \frac{4\pi}{5}$.

Following the *isomonodromic* approach to P-I [Kap04], any solution $y(z)$ gives rise to an *isomonodromic deformation* of the following linear equation with an irregular singularity

$$\overrightarrow{\Phi}_\lambda(\lambda, z) \quad = \quad \begin{pmatrix} y'(z) & 2\lambda^2 + 2\lambda y(z) - z + 2y^2(z) \\ 2(\lambda - y(z)) & -y'(z) \end{pmatrix} \overrightarrow{\Phi}(\lambda, z) \, .$$

The deformation of the equation is manifestly singular at every pole $a \in \mathbb{C}$ of y, however in Theorem 2 we show that at the singularity this equation can be replaced with a simpler one, which has the same *monodromy data* (cf. [IN86] for Painlevé II). This is the following Schrödinger equation with cubic potential

$$\frac{d^2\psi(\lambda)}{d\lambda^2} = V(\lambda; a, b)\psi(\lambda) \, , \quad V(\lambda; a, b) = 4\lambda^3 - 2a\lambda - 28b \, .$$

Here a is the location of the pole of y and b is a complex number entering into the Laurent expansion of y around a (see formula (7) below).

The isomonodromy property implies that there exists a natural injective map $\mathcal{M}$ from the space of solutions of P-I to the space of *monodromy data* of the above equations (see Lemma 2), while the Schrödinger equation defines naturally a map $\mathcal{T}$ from the space of cubic potentials to the space of monodromy data.

Our first main result is Theorem 3, which states that $a \in \mathbb{C}$ is a pole of $y(z)$ *if and only if* there exists $b \in \mathbb{C}$ such that $\mathcal{M}(y) = \mathcal{T}(V(\lambda; a, b))$.

In particular, because of the special monodromy data related to the intégrale tritronquée (see Theorem 1, due to Kapaev), we will show that the poles of the intégrale tritronquée are in bijection with the simultaneous solutions of two different *quantization conditions*.

The above approach naturally embeds the study of poles of Painlevé-I transcendents into the Nevanlinna's theory of branched coverings of the sphere and the complex WKB method.

The beautiful theory of R. Nevanlinna (see [Nev70] and [Elf34]) relates bijectively the Schrödinger equations with a polynomial potential to the branched coverings of the sphere with logarithmic branch points, considered up to conformal equivalence. Using this theory we are able to prove the surjectivity of the map $\mathcal{M}$ (see Theorem 5).

Moreover Nevanlinna's theory provides the poles of any solution of P-I with an unexpected and remarkable rich structure. In particular, poles of the tritronquée solution can be labelled by the monodromy of coverings of the Riemann sphere with 3 logarithmic branch points. In a subsequent paper, we are going to use this topological description to complete the WKB analysis of the present paper.

The WKB analysis of P-I developed in [KT05] has never been applied to the direct study of the distributions of poles. To achieve such a goal we follow the Fedoryuk's approach (see [Fed93]) to the complex WKB theory, and in the Classification Theorem we give a complete topological classification of the *Stokes complexes* of all cubic potentials. As a consequence of the Classification Theorem, we obtain our second main result: all polynomials whose monodromy data, in the WKB approximation, are the monodromy data of the intégrale tritronquée have the same topological type of Stokes complex and satisfy a pair of *Bohr-Sommerfeld* quantization conditions, namely system (25). In particular, in this way we reproduce the conditions obtained by Boutroux, through a completely different approach, in his study of the asymptotic distributions of the poles of the intégrale tritronquée.

A priori, the WKB method is expected to give an approximation of poles $z = a$ for a sufficiently large. Surprisingly our approach proves to be numerically very efficient also for poles close to the origin, see Table 2 below. [1]

The paper is organized as follows. In Section 2 we derive the Schrödinger equation associate with P-I and study thoroughly its relations with poles of P-I transcendents. Section 3 is devoted to the topological classification of Stokes complexes. In Section 4 we calculate the monodromy data in the WKB approximation, we derive the correct Bohr-Sommerfeld conditions for the poles of tritronquée, and we introduce the "small parameter" of the approximation. In Section 5 we obtain an asymptotic description of poles of the *integrale tritronquée.* In Appendix A and Appendix B we prove some theorems regarding the WKB functions that are used in section 2 and 3.

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[1] In the subsequent paper [MB10] the author shows that eventually around any solution of the Bohr-Sommerfeld-Boutroux system there is one and only one pole of the intégrale tritronquée and the distance between a pole and its approximation vanishes asymptotically.

the theory of nonlinear waves and their applications".

2 Poles and Cubic Oscillators

We review some well-known facts regarding the isomonodromic approach to the P-I equation and analyze the isomonodromic deformation in a neighborhood of the singularities.

2.1 P-I as an Isomonodromic Deformation

P-I is equivalent to the compatibility condition of the following system of linear ODEs:

$$\overrightarrow{\Phi}_\lambda(\lambda,z) = \begin{pmatrix} y'(z) & 2\lambda^2+2\lambda y(z)-z+2y^2(z) \\ 2(\lambda-y(z)) & -y'(z) \end{pmatrix} \overrightarrow{\Phi}(\lambda,z) \quad (1)$$

$$\overrightarrow{\Phi}_z(\lambda,z) = -\begin{pmatrix} 0 & 2y(z)+\lambda \\ 1 & 0 \end{pmatrix} \overrightarrow{\Phi}(\lambda,z) \quad . \quad (2)$$

The precise meaning of the word *compatibility* is given by the following

Lemma 1. *Fix z_0, λ_0 and the Cauchy data $y(z_0)$, $y'(z_0)$, and $\overrightarrow{\Phi}(\lambda_0,z_0)$. Let U_{z_0} be any simply connected neighborhood of z_0. Then $y(z)$ satisfies the Painlevé first equation in U_{z_0} iff the system (1,2) has a solution $\forall(\lambda,z)\in\mathbb{C}\times U_{z_0}$. Moreover the solution is unique.*

Proof. See [Kap04]. □

In this subsection we suppose that we have fixed a solution y of P-I and a simply connected region U such that $y|_U$ is holomorphic.

We are now going to define the important concepts of *monodromy data* and *isomonodromic deformation* of equation (1). For this reason, we have to introduce some particular solutions of system (1,2), to be uniquely defined by the asymptotic behaviour for $\lambda\to\infty$.

Fix $k\in\mathbb{Z}_5=\{-2,\dots,2\}$ and the branch of $\lambda^{\frac12}$ in such a way that $\mathrm{Re}\lambda^{\frac52}\to+\infty$ as $|\lambda|\to\infty, \arg\lambda=\frac{2\pi k}{5}$. Then (see [Kap04]) for any y solution of P-I, there exists a unique solution $\overrightarrow{\Phi_k}(\lambda,z)$ of (1,2) such that

$$\lim_{\substack{\lambda\to\infty \\ \left|\arg\lambda-2\frac{\pi k}{5}\right|<\frac{3\pi}{5}-\varepsilon}} e^{+\frac45\lambda^{\frac52}-\frac12 z\lambda^{\frac12}} \begin{pmatrix} \lambda^{-\frac14} & 0 \\ 0 & \lambda^{+\frac14} \end{pmatrix} \overrightarrow{\Phi}_k(\lambda) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \forall z\in U \ , \quad (3)$$

where $\lambda^{\frac14}$ is defined globally on the complex plane minus the negative real axis, and is positive on the positive real axis. Notice that, depending on

$k \in \mathbb{Z}_5$, $\left(\lambda^{\frac{1}{4}}\right)^2$ may not be equal to $\lambda^{\frac{1}{2}}$. Here and in the following, if not otherwise stated, ε is an arbitrarily small positive number.

From the asymptotics (3) it follows that $\overrightarrow{\Phi}_k(\lambda, z)$ and $\overrightarrow{\Phi}_{k+1}(\lambda, z)$ are linearly independent for any $k \in \mathbb{Z}_5$ and the following equality holds true

$$\overrightarrow{\Phi}_{k-1}(\lambda) = \overrightarrow{\Phi}_{k+1}(\lambda) + \sigma_k(z)\overrightarrow{\Phi}_k(\lambda) \ , \tag{4}$$

where $\sigma_k(z)$ is an analytic function of z, for any $k \in \mathbb{Z}_5$.

Definition 1. *Fixed z, we call σ_k the k-th* Stokes multiplier *of equation (1) and the set of all five Stokes multipliers the* monodromy data *of (1). The problem of calculating the monodromy data is called the* direct monodromy problem*.*

Stokes multipliers are very important for our analysis and we list their main properties in the following

Lemma 2. *Let $\sigma_k(z), k \in \mathbb{Z}_5$ be defined as above. Then*

(i) equation (2) is an isomonodromic deformation of equation (1), i.e. $\frac{d\sigma_k(z)}{dz} = 0$.

(ii) The numbers $\sigma_k, k \in \mathbb{Z}_5$ satisfy the following system of algebraic equations

$$1 + \sigma_k \sigma_{k+1} = -i\,\sigma_{k+3} \ , \ k \in \mathbb{Z}_5 \ . \tag{5}$$

Proof. See [Kap04].

□

Observe that only 3 of the algebraic equations (5) are independent.

Definition 2. *We denote V the algebraic variety of quintuplets of complex numbers satisfying (5) and call admissible monodromy data the elements of V. Due to Lemma 2, equations (4) define the following map*

$$\mathcal{M} : \{\textit{P-I transcendents}\} \to V \ .$$

Lemma. *$\mathcal{M}$ is injective.*

Proof. See [Kap04]. □

We end the section with a result of Kapaev, which completely characterizes the intégrale tritronquée in term of Stokes multipliers.

Theorem 1. *(Kapaev)*

The image under $\mathcal{M}$ of the intégrale tritronquée are the monodromy data uniquely characterized by the following equalities

$$\sigma_2 = \sigma_{-2} = 0 \ . \tag{6}$$

Proof. See [Kap04]. □

2.2 Poles of y: cubic oscillator

So far we dealt with the system (1,2) in a region U which does not contain any pole of $y(z)$. Indeed, the situation at a pole is different, for equation (1) makes no sense.

However, we show that any solution $\overrightarrow{\Phi}(\lambda, z)$ of system (1, 2) is meromorphic in all the $z-plane$; moreover, a pole $z = a$ of $y(z)$ is also a pole of $\overrightarrow{\Phi}(\lambda, z)$ and the residue at the pole of its second component satisfies the scalar equation of Schrödinger type (8).

In order to be able to describe the local behavior of $\overrightarrow{\Phi}(\lambda, z)$ near a pole a of $y(z)$, we have to know the local behavior of $y(z)$ close to the same point a.

Lemma 3 (Painlevé)**.** *Let $a \in \mathbb{C}$ be a pole of y. Then in a neighborhood of* a*, y has the following convergent Laurent expansion*

$$y(z) = \frac{1}{(z-a)^2} + \frac{a(z-a)^2}{10} + \frac{(z-a)^3}{6} + b(z-a)^4 + \sum_{j\geq 5} c_j(a,b)(z-a)^j \quad (7)$$

where b is some complex number and $c_j(a, b)$ are real polynomials in a and b, not depending on the particular solution y.

Conversely, fixed arbitrary $a, b \in \mathbb{C}$, the above expansion has a non zero radius of convergence and solves P-I.

Proof. See [GLS00]. □

Definition 3. *We define the map*

$$\mathcal{L} : \mathbb{C}^2 \to \{\textit{P-I transcendents}\} .$$

$\mathcal{L}(a, b)$ is the unique analytic continuation of the Laurent expansion (7).

We have already collected all elements necessary to formulate the important

Theorem 2. *Fix a solution y of P-I and let $\Phi_k^{(i)}(\lambda, z)$, $i = 1, 2$ $k \in \mathbb{Z}_5$ be the i-th component of $\overrightarrow{\Phi}_k(\lambda, z)$. Then*

- *(i) $\overrightarrow{\Phi}_k(\lambda, z)$ is a meromorphic function of z. All the singularities are double poles. Moreover, $a \in \mathbb{C}$ is a pole of $\overrightarrow{\Phi}_k(\lambda, z)$ iff it is a pole of y.*
- *(ii) If $a \in \mathbb{C}$ is a pole of y then*

$$\Psi_k(\lambda) = \lim_{z\to a} (z-a)\, \Phi_k^{(2)}(\lambda, z)$$

is an entire function of λ. It satisfies the following Schrödinger equation with cubic potential

$$\frac{d^2\Psi_k(\lambda)}{d\lambda^2} = \left(4\lambda^3 - 2a\lambda - 28b\right) \Psi_k(\lambda) , \quad (8)$$

where $b \in \mathbb{C}$ is the coefficient entering into the Laurent expansion (7) of y around a.

(iii) If $\lambda^{\frac{1}{2}}$ and $\lambda^{\frac{1}{4}}$ are chosen as in asymptotics (3), then $\forall \varepsilon > 0$

$$\lim_{\lambda\to\infty, |\lambda - \frac{2\pi k}{5}| < \frac{3\pi}{5} - \varepsilon} \lambda^{\frac{3}{4}} e^{+\frac{4}{5}\lambda^{\frac{5}{2}} - \frac{1}{2} a \lambda^{\frac{1}{2}}} \Psi_k(\lambda) = i. \tag{9}$$

(iv) Equation (8) possesses the same monodromy data as equation (1), i.e.

$$\Psi_{k-1}(\lambda) = \Psi_{k+1}(\lambda) + \sigma_k \Psi_k(\lambda) \ .$$

Proof. (i) From the Laurent expansion (7), it is easily seen that a pole a of y is a fuchsian singularity with trivial monodromy of equation (2). In particular the following Laurent expansions of $\Phi_k^{(i)}(\lambda, z)$ are valid

$$\begin{aligned}
\Phi_k^{(2)}(\lambda, z) &= \frac{\psi_k(\lambda)}{(z-a)} \left(1 - \frac{\lambda}{2}(z-a)^2\right) + \varphi_k(\lambda)(z-a)^2 + O((z-a)^3) \ , \\
\Phi_k^{(1)}(\lambda, z) &= \frac{\psi_k(\lambda)}{(z-a)^2} \left(1 + \frac{\lambda}{2}(z-a)^2\right) - 2\varphi_k(\lambda)(z-a) + O((z-a)^2)
\end{aligned} \tag{10}$$

Expansions (10) show that $\overrightarrow{\Phi}_k(\lambda, z)$ is meromorphic in a neighborhood of the point a and this point is a pole of order not greater than 2.

(ii), (iii) The proof is in Appendix B.

(iv) Since the functions $(z-a)\Phi_k^{(2)}$ satisfy equations (4) for any z with constant Stokes multipliers, then their limit, i.e. the functions $\Psi_k(\lambda)$, satisfy the same equations.

□

Definition 4. *We call any cubic polynomial of the form $V(\lambda; a, b) = 4\lambda^3 - 2a\lambda - 28b$ a* cubic potential. *The above formula identifies the space of cubic potentials with $\mathbb{C}^2 \ni (a, b)$.*

We define the map

$$\mathcal{T} : \mathbb{C}^2 \to V \ .$$

$\mathcal{T}(a, b)$ is the monodromy data of equation (8).

Theorem 2 has the following

Corollary. *$\mathcal{M} \circ \mathcal{L} = \mathcal{T}$.*

The above corollary implies

Theorem 3. *Let y be any solution of P-I. Then $a \in \mathbb{C}$ is a pole of y iff there exists $b \in \mathbb{C}$ such that $\mathcal{M}(y) = \mathcal{T}(a, b)$.*

We finish this section with a theorem from Nevanlinna's theory [Nev70], which implies the surjectivity of the map $\mathcal{M}$.

Theorem 4. *The map $\mathcal{T}$ is surjective. The preimage of any admissible monodromy data is a countable infinite subset of the space of cubic potentials.*

Proof. See [Elf34]. □

As a consequence of the above theorem we have

Theorem 5 (stated in [KK93])**.** *The map $\mathcal{M}$ is bijective: solutions of P-I are in 1-to-1 correspondence with admissible monodromy data.*

Theorem 3 shows that the distribution of poles of P-I transcendents is a part of the theory of *anharmonic oscillators*, which has been object of intense study since the seminal papers [BW68] and [Sim70].

Remark. *In the theory of anharmonic oscillators a special importance is given to the vanishing of some Stokes multipliers. For a given $k \in \mathbb{Z}_5$, the problem is to find all $(a,b) \in \mathbb{C}^2$ such that the Stokes multiplier σ_k of equation (8) vanishes. This is called the k-th* lateral connection problem*. Since fixed a, there exists a discrete number of solutions to any lateral connection problem, equation $\sigma_k = 0$ is referred to as a quantization condition.*

As a consequence of Theorem 1 and Theorem 3, we have the following

Corollary. *The point $a \in \mathbb{C}$ is a pole of the intégrale tritronquée if and only if there exists $b \in \mathbb{C}$ such that the Schrödinger equation with the cubic potential $V(\lambda; a, b)$ admits the simultaneous solution of two different quantization conditions, namely $\sigma_{\pm 2} = 0$.*

2.3 Asymptotic values

As it was previously observed, Stokes multipliers are defined by particular normalized solutions of equations (1) and (8). Following Nevanlinna, we define the monodromy data of equation (8) in a more invariant way.

Definition 5. *Let $\{\varphi, \chi\}$ be a basis of solution of (8).*
We call

$$w_k(\varphi, \chi) = \lim_{\substack{\lambda \to \infty \\ \left|\arg \lambda - \frac{2\pi k}{5}\right| < \frac{\pi}{5} - \varepsilon}} \frac{\varphi(\lambda)}{\chi(\lambda)} \in \mathbb{C} \cup \infty\,,\ k \in \mathbb{Z}_5\,. \tag{11}$$

the k-th asymptotic value.

We collect the main properties of the asymptotic values in the following

Lemma 4. *(i) Let $\varphi' = a\,\varphi + b\,\chi$ and $\chi' = c\,\varphi + d\,\chi'$, $a, b, c, d \in \mathbb{C}$. Then*

$$w_k(\varphi', \chi') = \frac{a\,w_k(\varphi, \chi) + b}{c\,w_k(\varphi, \chi) + d}\,. \tag{12}$$

(ii) $w_{k-1}(\varphi,\chi) = w_{k+1}(\varphi,\chi)$ *iff* $\sigma_k = 0$.

(iii) $w_{k+1}(\varphi,\chi) \neq w_k(\varphi,\chi)$

Proof. See [Elf34]. □

Making use of equation (11), given the Stokes multipliers it is possible to calculate the asymptotic values. The converse is also true. In particular, the asymptotic values associated to the tritronquée intégrale can be chosen to be

$$w_0 = 0, w_1 = w_{-2} = 1, w_2 = w_{-1} = \infty \ . \tag{13}$$

3 Stokes Complexes

In the complex WKB method a prominent role is played by the *Stokes and anti-Stokes lines*, and in particular by the topology of the *Stokes complex*, which is the union of the Stokes lines.

The main result of this section is the Classification Theorem, where we show that the topological classification of Stokes complexes divides the space of cubic potentials into seven disjoint subsets.

Even though Stokes and anti-Stokes lines are well-known objects, there is no standard convention about their definitions, so that some authors call Stokes lines what others call anti-Stokes lines. We follow here the notation of Fedoryuk [Fed93].

Remark. *To simplify the notation and avoid repetitions, we study the Stokes lines only. Every single statement in the following section remains true if the word Stokes is replaced with the word anti-Stokes, provided in equation (14) the angles* φ_k *are replaced with the angles* $\varphi_k + \frac{\pi}{5}$.

Definition 6. *A simple (resp. double, resp. triple) zero* λ_i *of* $V(\lambda) = V(\lambda; a, b)$ *is called a simple (resp. double, resp. triple) turning point. All other points are called generic.*

Fix a generic point λ_0 *and a choice of the sign of* $\sqrt{V(\lambda_0)}$. *We call action the analytic function*

$$S(\lambda_0, \lambda) = \int_{\lambda_0}^{\lambda} \sqrt{V(u)} du$$

defined on the universal covering of λ*-plane minus the turning points.*

Let $\tilde{i}_{\lambda_0}$ be the level curve of the real part of the action passing through a lift of λ_0. Call its projection to the punctured plane i_{λ_0}. Since i_{λ_0} is a one dimensional manifold, it is diffeomorphic to a circle or to a line. If i_{λ_0} is diffeomorphic to the real line, we choose one diffeomorphism $i_{\lambda_0}(x), x \in \mathbb{R}$ in such a way that the continuation along the curve of the imaginary part of the action is a monotone increasing function of $x \in \mathbb{R}$.

Lemma 5. *Let λ_0 be a generic point. Then i_{λ_0} is diffeomorphic to the real line, the limit $\lim_{x\to+\infty} i_{\lambda_0}(x)$ exists (as a point in $\mathbb{C}\bigcup\infty$) and it satisfies the following dichotomy:*

(i) Either $\lim_{x\to+\infty} i_{\lambda_0}(x) = \infty$ and the curve is asymptotic to one of the following rays of the complex plane

$$\lambda = \rho e^{i\varphi_k}, \varphi_k = \frac{(2k+1)\pi}{5}, \rho \in \mathbb{R}^+, k \in \mathbb{Z}_5 , \tag{14}$$

(ii) or $\lim_{x\to+\infty} i_{\lambda_0}(x) = \lambda_i$, where λ_i is a turning point.

Furthermore,

(iii) if $\lim_{x\to\pm\infty} i_{\lambda_0}(x) = \infty$ then the asymptotic ray in the positive direction is different from the asymptotic ray in the negative direction.

(iv) Let $\varphi_k, k \in \mathbb{Z}_5$ be defined as in equation (14). Then $\forall \varepsilon > 0, \exists K \in \mathbb{R}^+$ such that if $\varphi_{k-1} + \varepsilon < \arg \lambda_0 < \varphi_k - \varepsilon$ and $|\lambda_0| > K$, then $\lim_{x\to\pm\infty} i_{\lambda_0}(x) = \infty$. Moreover the asymptotic rays of i_{λ_0} are the ones with arguments φ_k and φ_{k-1}.

Proof. See [Str84].

□

Definition 7. *We call Stokes line the trajectory of any curve i_{λ_0} such that there exists at least one turning point belonging to its boundary.*

We call a Stokes line internal if ∞ does not belong to its boundary.

We call Stokes complex the union of all the Stokes lines together with the turning points.

We state all important properties of the Stokes lines in the following

Theorem 6. *The following statements hold true*

(i) The Stokes complex is simply connected. In particular, the boundary of any internal Stokes line is the union of two different turning points.

(ii) Any simple (resp. double, resp. triple) turning point belongs to the boundary of 3 (resp. 4, resp 5) Stokes lines.

(iii) If a turning point belongs to the boundary of two different non-internal Stokes lines then these lines have different asymptotic rays.

(iv) For any ray with the argument φ_k as in equation (14), there exists a Stokes line asymptotic to it.

Proof. See [Str84]. □

3.1 Topology of Stokes complexes

In what follows, we give a complete classification of the Stokes complexes, with respect to the orientation preserving homeomorphisms of the plane.

We define the map L from the λ-plane to the interior of the unit disc as

$$\begin{aligned} L &: \mathbb{C} \to D_1 \\ L(\rho e^{i\varphi}) &= \frac{2}{\pi} e^{i\varphi} \arctan \rho. \end{aligned} \tag{15}$$

The image under the map L of the Stokes complex is naturally a decorated graph embedded in the closed unit disc. The vertices are the images of the turning points and the five points on the boundary of the unit disc with arguments φ_k, with φ_k as in equation (14). The bonds are obviously the images of the Stokes lines. We call the first set of vertices *internal* and the second set of vertices *external*. External vertices are decorated with the numbers $k \in \mathbb{Z}_5$. We denote $\mathcal{S}$ the decorated embedded graph just described. Notice that due to Theorem 6 (iii), there exists not more than one bond connecting two vertices.

The combinatorial properties of $\mathcal{S}$ are described in the following

Lemma 6. *$\mathcal{S}$ possesses the following properties*

(i) the sub-graph spanned by the internal vertices has no cycles.

(ii) Any simple (resp. double, resp. triple) turning point has valency 3 (resp. 4, resp. 5).

(iii) The valency of any external vertex is at least one.

Proof. (i) Theorem 6 part (i)

(ii) Theorem 6 part (ii)

(iii) Theorem 6 part (iv)

□

Definition 8. *We call an* admissible graph *any decorated simple graph embedded in the closure of the unit disc, with three internal vertices and five decorated external vertices, such that (i) the cyclic-order inherited from the decoration coincides with the one inherited from the counter-clockwise orientation of the boundary, and (ii) it satisfies all the properties of Lemma 6. We call two admissible graphs equivalent if there exists an orientation-preserving homeomorphism of the disk mapping one graph into the other.*

Theorem 7. *Classification Theorem*

All equivalence classes of admissible maps are, modulo a shift $k \to k + m, m \in \mathbb{Z}_5$ of the decoration, the ones depicted in Figure 1.

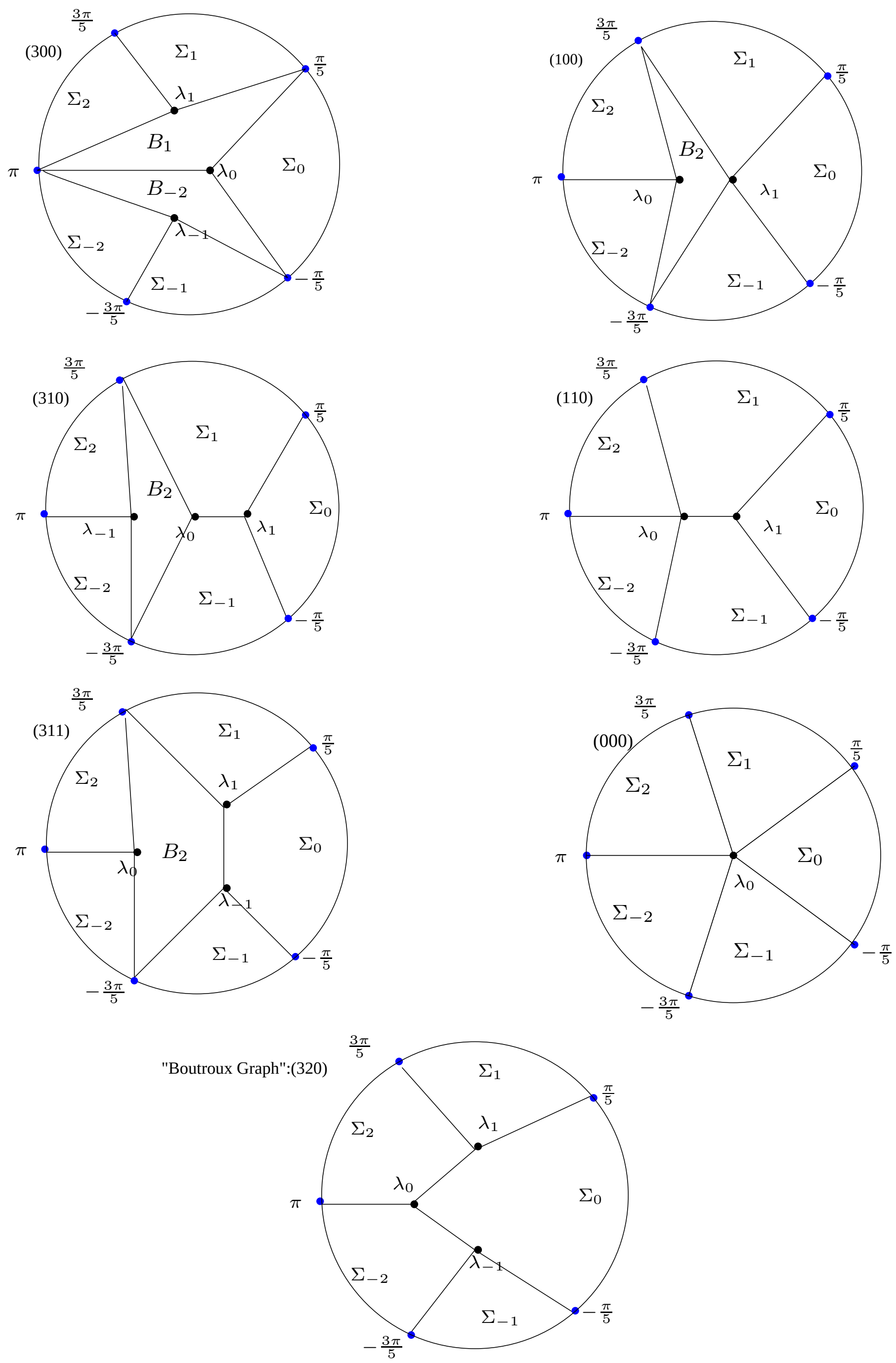

Figure 1: All the equivalence classes of admissible graphs.

Proof. Let us start analyzing the admissible graphs with three internal vertices and no internal edges.

Any internal vertex is adjacent to a triplet of external vertices. Due to the Jordan curve theorem, there exists an internal vertex, say λ_0, adjacent to a triplet of non consecutive external vertices. Performing a shift, they can be chosen to be the ones labelled by $0, 2, -1$. Call the respective edges e_0, e_{-1}, e_2.

The disk is cut in three disjoint domains by those three edges. No internal vertices can belong to the domain cut by e_0 and e_4, since it could be adjacent only to two external vertices, namely the ones labelled with 0 and -1. By similar reasoning it is easy to show that one and only one vertex belong to each remaining domains.

Such embedded graph is equivalent to the graph (300).

Classifications for all other cases may proved by similar methods.

□

The equivalence classes are encoded by a triplet of numbers (a b c): a is the number of simple turning points, b is the number of internal Stokes lines, while c is a progressive number, distinguishing non-equivalent graphs with same a and b. Some additional information shown Figure 1 will be explained in the next section.

Remark. *For any admissible graph there exists a real polynomial with an equivalent Stokes complex.*

Remark. *Notice that the automorphism group of every graph in Figure 1 is trivial. Therefore the unlabelled vertices can be labelled. In the following we will label the turning points as in figure 1. We denote "Boutroux graph" the graph* (320)

3.2 Stokes Sectors

In the λ-plane the complement of the Stokes complex is the disjoint union of a finite number of connected and simply-connected domains, each of them called a *sector*.

Combining Theorem 6 and the Classification Theorem we obtain the following

Lemma 7. *All the curves i_{λ_0}, with λ_0 belonging to a given sector, have the same two asymptotic rays. Moreover, two different sectors have different pairs of asymptotic rays.*

For any $k \in \mathbb{Z}_5$ there is a sector, called the k-th Stokes sectors, whose asymptotic rays have arguments φ_{k-1} and φ_k. This sector will be denoted Σ_k. The boundary $\partial\Sigma_k$ of each Σ_k is connected.

Any other sector has asymptotic rays with arguments φ_{k-1} and φ_{k+1}, for some k. We call such a sector the k-th sector of band type, and we denote it B_k. The boundary ∂B_k of each B_k has two connected components.

Choose a sector and a point λ_0 belonging to it. The function $S(\lambda_0, \lambda)$ is easily seen to be bi-holomorphic into the image of this sector. In particular, with one choice of the sign of $\sqrt{V}$ it maps a Stokes sector into the half plane $\mathrm{Re}S > c$, for some $-\infty < c < 0$ while it maps a B_k sector in the vertical strip $c < \mathrm{Re}S < d$, for some $-\infty < c < 0 < d < +\infty$.

Definition 9. *We call a differentiable curve $\gamma : [0,1] \to \mathbb{C}$ an admissible path provided γ is injective on $[0,1[$, $\lambda_i \notin \gamma([0,1])$, for all turning points λ_i, and $\mathrm{Re}S(\gamma(0), \gamma(t))$ is a monotone function of $t \in [0,1]$.*

We say that $\Sigma_j \leftrightarrows \Sigma_k$ if there exist $\mu_j \in \Sigma_j$, $\mu_k \in \Sigma_k$ and an admissible path such that $\gamma(0) = \mu_j, \gamma(1) = \mu_k$.

The relation $\leftrightarrows$ is obviously reflexive and symmetric but it is not in general transitive.

Notice that $\Sigma_j \leftrightarrows \Sigma_k$ *if and only if* for every point $\mu_j \in \Sigma_j$ and every point $\mu_k \in \Sigma_k$ an admissible path exists.

Lemma 8. *The relation $\leftrightarrows$ depends only on the equivalence class of the Stokes complex $\mathcal{S}$.*

Proof. Consider an admissible path from Σ_j to Σ_k, $j \neq k$. The path is naturally associated to the sequence of Stokes lines that it crosses. We denote the sequence $l_n, n = 0, \ldots, N$, for some $N \in \mathbb{N}$. We continue analytically $S(\mu_j, \cdot)$ to a covering of the union of the Stokes sectors crossed by the path together with the Stokes lines belonging to the sequence. Since $S(\mu_j, \cdot)$ is constant along each connected component of the boundary of every lift of a sector crossed by the path, then each of such connected components cannot be crossed twice by the path. Hence, due to the classification theorem no admissible path is a loop. Therefore, the union of the Stokes sectors crossed by the path together with the Stokes lines belonging to the sequence is simply connected.

Conversely, given any injective sequence of Stokes lines l_n, $n = 0 \ldots, N$ such that for any $0 \leq n \leq N-1$, l_n and l_{n+1} belong to two different connected components of the boundary of a same sector, there exists an admissible path with that associated sequence. This last observation implies that the relation $\leftrightarrows$ depends only on the topology of the graph $\mathcal{S}$. Moreover, if the sequence exists it is unique; indeed, if there existed two admissible paths, joining the same μ_j and μ_k but with different sequences, then there would be an admissible loop. □

With the help of Lemma 8 and of the Classification Theorem, relation $\leftrightarrows$ can be easily computed, as it is shown in Table 1. As it is evident from Figure 1, for any graph type we have that $\Sigma_k \leftrightarrows \Sigma_{k+1}$, $\forall k \in \mathbb{Z}_5$.

Map	Pairs of non consecutive Sectors not satisfying the relation $\leftrightarrows$
300	None
310	(Σ_0, Σ_2), (Σ_0, Σ_{-2})
311	(Σ_1, Σ_{-1})
320	(Σ_1, Σ_{-1}), (Σ_1, Σ_{-2}), (Σ_{-1}, Σ_2)
100	(Σ_1, Σ_{-1}), (Σ_0, Σ_{-2}), (Σ_0, Σ_2)
110	All but (Σ_1, Σ_{-1})
000	All

Table 1: Computation of the relation $\leftrightarrows$

4 Complex WKB Method and Asymptotic Values

In this section we introduce the *WKB functions* $j_k, k \in \mathbb{Z}_5$ and use them to evaluate the asymptotic values of equation (8). The topology of the Stokes complex will show all its importance in these computations.

On any Stokes sector Σ_k, we define the functions

$$
\begin{aligned}
S_k(\lambda) &= S(\lambda^*, \lambda) , && (16)\\
L_k(\lambda) &= -\frac{1}{4}\int_{\lambda^*}^{\lambda} \frac{V'(u)}{V(u)} du , && (17)\\
j_k(\lambda) &= e^{-S_k(\lambda)+L_k(\lambda)} . && (18)
\end{aligned}
$$

Here λ^* is an arbitrary point belonging to Σ_k and the branch of $\sqrt{V}$ is such that $\mathrm{Re} S_k(\lambda)$ is bounded from below.

We call j_k the k-th *WKB function*.

4.1 Maximal Domains

In this subsection we construct the k-th *maximal domain*, that we denote D_k. This is the domain of the complex plane where the k-th WKB function approximates a solution of equation (8).

The construction is done for any k in a few steps (see Figure 2 for the example of the Stokes complex of type (300)):

(i) for every Σ_l such that $\Sigma_l \leftrightarrows \Sigma_k$, denote $D_{k,l}$ the union of the sectors and of the Stokes lines crossed by any admissible path connecting Σ_l and Σ_k.

(ii) Let $\widehat{D_k} = \bigcup_l D_{l,k}$. Hence $\widehat{D_k}$ is a connected and simply connected subset of the complex plane whose boundary $\partial\widehat{D_k}$ is the union of some Stokes lines.

(iii) Remove a δ-tubular neighborhood of the boundary $\partial\widehat{D_k}$, for an arbitrarily small $\delta > 0$, such that the resulting domain is still connected.

(iii) For all $l \neq k, l \neq k-1$, remove from $\widehat{D_k}$ an angle $\lambda = \rho e^{i\varphi}, |\varphi - \varphi_l| < \epsilon, \rho > R$, for ε arbitrarily small and R arbitrarily big, in such a way that the resulting domain is still connected. The remaining domain is D_k.

4.2 Main Theorem of WKB Approximation

We can now state the main theorem of the WKB approximation.

Theorem 8 (G.D. Birkhoff [Bir33], Olver [Olv74])**.** *Continue the WKB function j_k to D_k. Then there exists a solution $\psi_k(\lambda)$ of (8), such that for all $\lambda \in D_k$*

$$\begin{aligned} \left|\frac{\psi_k(\lambda)}{j_k(\lambda)} - 1\right| &\leq g(\lambda)\left(e^{2\rho(\lambda)} - 1\right) \\ \left|\frac{\psi_k'(\lambda)}{j_k(\lambda)\sqrt{V(\lambda)}} + 1\right| &\leq \left|\frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}\right| + (1 + \left|\frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}\right|)g(\lambda)(e^{2\rho(\lambda)} - 1) \end{aligned}$$

Here ρ_k is a bounded positive continuous function, called the error function, satisfying

$$\lim_{\substack{\lambda\to\infty \\ \varphi_{k-1}<arg\lambda<\varphi_{k+1}}} \rho_k(\lambda) = 0 \ ,$$

and $g(\lambda)$ is a positive function such that $g(\lambda) \leq 1$ and

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in D_k\cap\Sigma_{k\pm2}}} g(\lambda) = \frac{1}{2} \ .$$

Proof. The proof is in the appendix A. □

Notice that j_k is *sub-dominant* (i.e. it decays exponentially) in Σ_k and *dominant* (i.e. it grows exponentially) in $\Sigma_l, \forall l \neq k$.

For the properties of the error function, ψ_k is subdominant in Σ_k and dominant in $\Sigma_{k\pm1}$. Therefore, in any Stokes sector Σ_k there exists a subdominant solution, which is defined uniquely up to multiplication by a non zero constant.

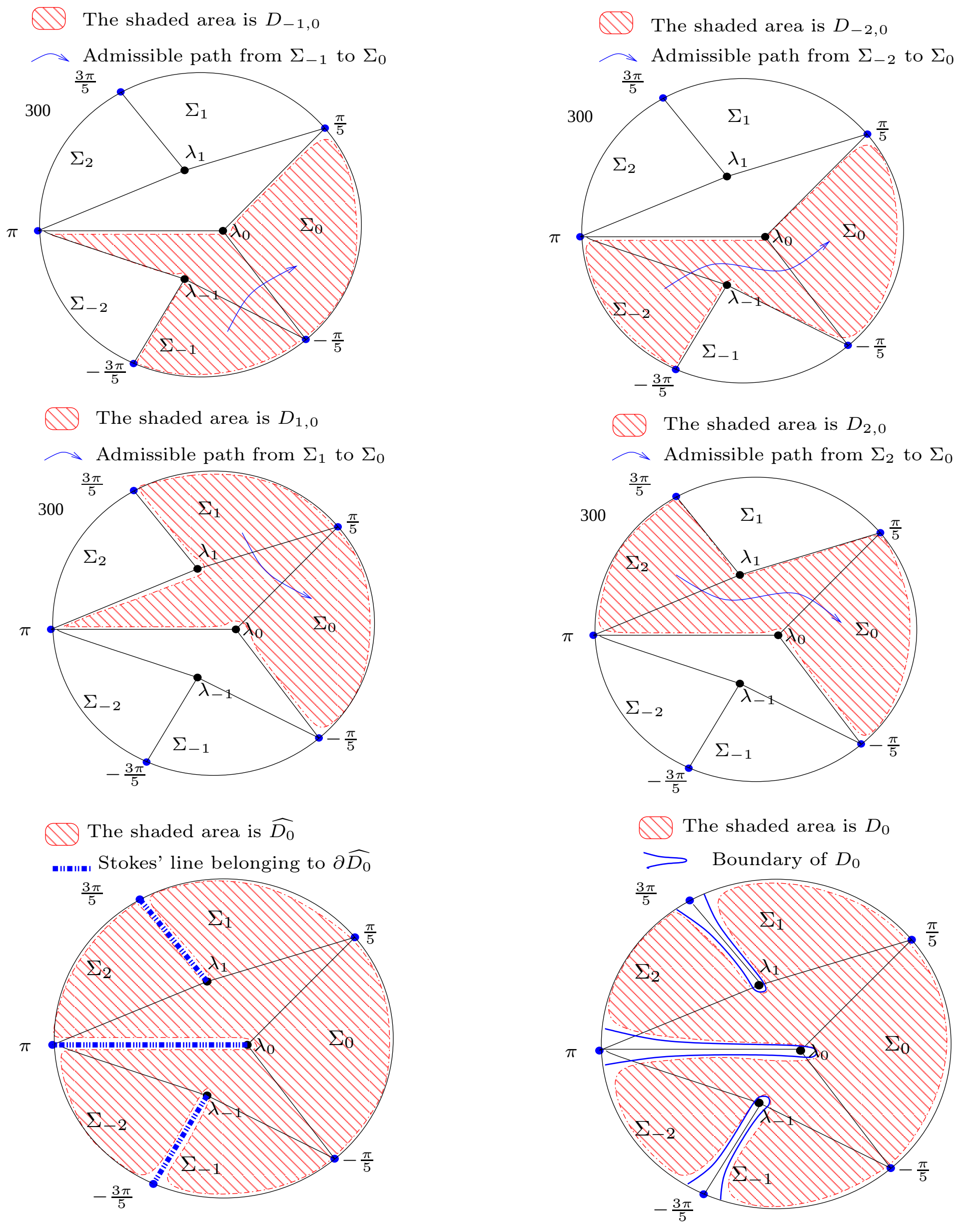

Figure 2: In the drawings, the construction of D_0 for a graph of type (300) is depicted.

4.3 Computations of Asymptotic Values in WKB Approximation

The aim of this paragraph is to compute the asymptotic values for the Schrödinger equation (8) in WKB approximation. We explicitely work out the example of the Stokes complex of type (320), relevant to the study of poles of the intégrale tritronquée.

Definition 10. *Define the relative errors*

$$\rho_l^k = \begin{cases} \lim\limits_{\substack{\lambda \to \infty \\ \lambda \in \Sigma_k \cap D_l}} \rho_l(\lambda), & \textit{if } \Sigma_l \leftrightarrows \Sigma_k \\ \infty, & \textit{otherwise} \end{cases}$$

and the asymptotic values

$$w_k(l,m) \stackrel{def}{=} w_k(\psi_l, \psi_m). \tag{19}$$

We say that $\Sigma_k \sim \Sigma_l$ *provided* $\rho_l^k < \frac{\log 3}{2}$*. The relation* $\sim$ *is a sub-relation of* $\leftrightarrows$*.*

Notice that $\rho_l^{l+1} = 0$ and $\rho_l^m = \rho_m^l$ (see Appendix A).

In order to compute the asymptotic value $w_k(l,m)$, we have to know the asymptotic behavior of ψ_l and ψ_m in Σ_k. By Theorem 8,

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \Sigma_k \cap D_l}} \frac{\psi_l(\lambda)}{j_l(\lambda)} \neq 0\,, \text{ if } \frac{1}{2}(e^{2\rho_k^l} - 1) < 1\,.$$

Hence the asymptotic behavior of ψ_l in Σ_k can be related to the asymptotic behavior of j_l in Σ_k if the relative error ρ_l^k is so small that the above inequality holds true, i.e. if $\Sigma_k \sim \Sigma_l$.

Remark. *Depending on the type of the graph* $\mathcal{S}$*, there may not exist two indices* $k \neq l$ *such that all the relative errors* $\rho_l^n, \rho_k^n, n \in \mathbb{Z}_5$ *are small. However it is often possible to compute an approximation of all the asymptotic values* $w_n(l,k)$ *using the strategy below.*

(i) We select a pair of non consecutive Stokes sectors Σ_l, Σ_{l+2}, with the hypothesis that the functions ψ_l and ψ_{l+2} are linearly independent, so that $w_l(l,l+2) = 0, w_{l+2}(l,l+2) = \infty$. Since $\rho_l^{l+1} = \rho_{l+2}^{l+1} = 0$ then

$$w_{l+1}(l,l+2) = \lim_{\substack{\lambda \to \infty \\ \lambda \in \Sigma_{l+1} \cap D_l \cap D_{l+2}}} \frac{j_l(\lambda)}{j_{l+2}(\lambda)}\,.$$

Therefore, we find three exact and distinct asymptotic values.

(ii) For any $k \neq l, l+1, l+2$ such that $\Sigma_l \sim \Sigma_k$ and $\Sigma_{l+2} \sim \Sigma_k$, we define the approximate asymptotic value

$$\widehat{w}_k(l,m) = \lim_{\substack{\lambda\to\infty \\ \lambda\in\Sigma_k\cap D_l\cap D_{l+2}}} \frac{j_l(z)}{j_m(z)} .$$

The spherical distance between $w_k(l,m)$ and $\widehat{w}_k(l,m)$ may be easily estimated from above knowing the relative errors ρ_k^l and ρ_k^{l+2}.

If for any $k \neq l, l+1, l+2$, $\Sigma_l \sim \Sigma_k$ and $\Sigma_{l+2} \sim \Sigma_k$, then the calculation is completed.

(iii) If not, we can use the fact that quintuplets of asymptotic values for different choices of ψ_l, ψ_{l+2} are related by a Möbius transformation (see formula (12)). If for some pair $(l, l+2)$ the assumption $\Sigma_l \sim \Sigma_k$, $\Sigma_{l+1} \sim \Sigma_k$ fails to be true for just one value of the index $k = k^*$, and, for another pair $(l', l'+2)$ the assumption $\Sigma_{l'} \sim \Sigma_{k'}$, $\Sigma_{l'+2} \sim \Sigma_{k'}$ fails to be true for just one valued of the index $k' = k'^*$, with $k'^* \neq k^*$, then there are three values of the index $m \in \mathbb{Z}_5$ such that an approximation of $w_m(l, l+2)$ and $w_m(l', l'+2)$ is computable. Since any Möbius transformation is fixed by the action on three values, then we can compute an approximation of the transformation relating the quintuplets $w_k(l, l+2)$ and $w_k(l', l'+2)$ for any $k \in \mathbb{Z}_5$. Hence, we can calculate an approximation of the whole quintuplets $w_k(l, l+2)$ and $w_{k'}(l', l'+2)$.

Remark. *As shown in Table 1, the relation* $\leftrightarrows$ *is uniquely characterized by the graph type. For the sake of computing the asymptotic values the important relation is* $\sim$ *and not* $\leftrightarrows$*. Indeed, the calculations for a given graph type, say* $(a\ b\ c)$*, are valid for (and only for) all the potentials whose relation* $\sim$ *is equivalent to the relation* $\leftrightarrows$ *characterizing the graph type* $(a\ b\ c)$*.*

Due to the above remark, in what follows we suppose that the relation $\sim$ is equivalent to the relation $\leftrightarrows$. We have the following

Lemma 9. *Let* $V(\lambda; a, b)$ *such that the type of the Stokes complex is* (300), (310), (311)*; moreover, suppose that the* $\sim$ *relation coincides with* $\leftrightarrows$*. Then all the asymptotic values of equation (8) are pairwise distinct, but for at most pair.*

Proof. For a graph of type (300) or (311) the thesis is trivial. For a graph of type (320), it may be that $w_0 = w_2$ or $w_0 = w_{-2}$. Since $w_2 \neq w_{-2}$ the thesis follows. □

We completely work out the case of Stokes complex of type (320), while for the other cases we present the results only. Due to Lemma 9, we omit the results for potentials whose graph type is (300), (310) and (311).

Boutroux Graph = 320 We suppose that $\Sigma_0 \sim \Sigma_{\pm 2}$.

Let us consider first the pair Σ_0 and Σ_{-2}. In Figure 3 the maximal domains D_0 ans D_{-2} are depicted by colouring the Stokes lines not belonging to them blue and red respecitvely. In particular S_0, L_0, j_0 (resp. S_{-2}, L_{-2}, j_{-2}) can be extended to all D_0 (resp. D_{-2}) along any curve that does not intersect any blue (resp. red) Stokes line.

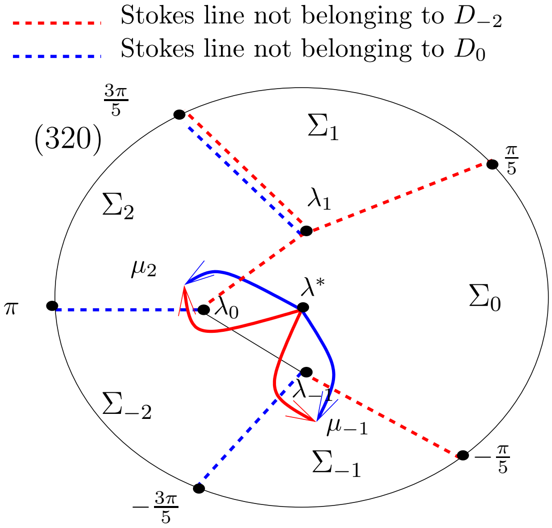

Figure 3: Calculation of $w_{-1}(0,-2)$ and of $\widehat{w}_2(0,-2)$

We fix a point $\lambda^* \in \Sigma_0$ such that $S_0(\lambda^*) = S_{-2}(\lambda^*) = L_0(\lambda^*) = L_{-2}(\lambda^*) = 0$.

By definition

$$\begin{aligned} \widehat{w}_k(0,-2) &= \lim_{\lambda\to\infty_k} \frac{j_0(\lambda)}{j_{-2}(\lambda)} \\ &= \lim_{\lambda\to\infty_k} e^{-S_0(\lambda)+S_{-2}(\lambda)} e^{L_0(\lambda)-L_{-2}(\lambda)} \quad , \end{aligned}$$

Here $\lambda \to \infty_k$ is a short-hand notation for $\lambda \to \infty, \lambda \in \Sigma_k \cap D_0 \cap D_{-2}$. We calculate $\widehat{w}_k(0,-2)$ for $k = -1, 2$.

We first calculate $\lim_{\lambda\to\infty_k} e^{-S_0(\lambda)+S_{-2}(\lambda)}$.

Notice that $\frac{\partial S_0}{\partial \lambda} = \frac{\partial S_{-2}}{\partial \lambda}$ in Σ_k. Hence

$$\lim_{\lambda\to\infty_k} -S_0(\lambda) + S_{-2}(\lambda) = -S_0(\mu_k) + S_{-2}(\mu_k)\,, k = -1, 2\,,$$

where μ_k is any point belonging to Σ_k (in Figure 3, the paths of integration defining $S_0(\mu_k)$ and $S_{-2}(\mu_k)$ are coloured blue and red respectively).

On the other hand, since $\frac{\partial S_0}{\partial \lambda} = -\frac{\partial S_{-2}}{\partial \lambda}$ in $\Sigma_0 \bigcup \Sigma_{-2}$, we have that

$$-S_0(\mu_k) + S_{-2}(\mu_k) = -2S_0(\lambda_s)\,, s = -1 \text{ if } k = -1 \text{ and } s = 0 \text{ if } k = 2\,.$$

We now compute $\lim_{\lambda \to \infty_k} e^{L_0(\lambda)-L_{-2}(\lambda)}$. Since $\frac{\partial L_0}{\partial \lambda} = \frac{\partial L_{-2}}{\partial \lambda}$ in $D_0 \bigcap D_{-2}$, we have that

$$\begin{aligned} \lim_{\lambda \to \infty_k} L_0(\lambda) - L_{-2}(\lambda) &= L_0(\mu_k) - L_{-2}(\mu_k) \ , \\ L_0(\mu_k) - L_{-2}(\mu_k) &= -\frac{1}{4} \oint_{c_k} \frac{V'(\mu)}{V(\mu)} d\mu \, , k = -1, 2 \ . \end{aligned}$$

Here c_k is the blue path connecting λ^* with μ_k composed with the inverse of the red path connecting λ^* with μ_k (see Figure 3).

Therefore, we have

$$\lim_{\lambda \to \infty_k} L_0(\lambda) - L_{-2}(\lambda) = -\sigma \frac{2\pi i}{4} \, , \sigma = -1 \text{ if } k = -1 \text{ and } \sigma = +1 \text{ if } k = 2 \ .$$

Combining the above computations, we get

$$w_{-1}(0,-2) = i \, e^{-2S_0(\lambda_{-1})}, \ \widehat{w}_2(0,-2) = -i \, e^{-2S_0(\lambda_0)}.$$

We stress that $w_{-1}(0,-2)$ is exact while $\widehat{w}_2(0,-2)$ is an approximation.

Performing the same computations for the pair Σ_0 and Σ_2, we obtain

$$w_1(0,2) = -i \, e^{-2S_0(\lambda_1)}, \ \widehat{w}_{-2}(0,2) = i \, e^{-2S_0(\lambda_0)}$$

Having calculated the triplet of asymptotic values w_0, w_2, w_{-2} for two different pairs of Stokes sectors, we can compute an approximation of the Möbius transformation relating all the asymptotic values for the two pairs:

$$\widehat{w}_k(0,-2) = -i \, e^{-2S_0(\lambda_0)} \frac{\widehat{w}_k(0,2)}{\widehat{w}_k(0,2) - i \, e^{-2S_0(\lambda_0)}} \ , \ k \in \mathbb{Z}_5 \quad .$$

We eventually compute the last asymptotic value for the pair Σ_0, Σ_{-2}, that is

$$\widehat{w}_1(0,-2) \quad = \quad -i \, \frac{e^{-2S_0(\lambda_1)}}{1 + e^{-2(S_0(\lambda_1) - S_0(\lambda_0))}} \quad .$$

Quantization Conditions The computations above provides us with the following quantization conditions:

$$\begin{aligned} \widehat{w}_1 &= w_{-2} \Leftrightarrow e^{-2(S_0(\lambda_1) - S_0(\lambda_0))} = -1 && (20) \\ \widehat{w}_2 &= w_{-1} \Leftrightarrow e^{-2(S_0(\lambda_{-1}) - S_0(\lambda_0))} = -1 && (21) \\ \widehat{w}_1 &= w_{-1} \Leftrightarrow e^{-2(S_0(\lambda_1) - S_0(\lambda_{-1}))} = -1 + e^{-2(S_0(\lambda_1) - S_0(\lambda_0))} && (22) \end{aligned}$$

We notice that equation (22) is incompatible both with (20) and (21). Equations (20) and (21) are Bohr-Sommerfeld quantizations.

As was shown in equation (13), the poles of the integràle tritronquèe are related to the polynomials such that $w_1 = w_{-2}$ and $w_{-1} = w_2$. Since equations (20) and (21) can be simultaneously solved, solutions of system (20,21) describe, in WKB approximation, polynomials related to the intégrale tritronquée. System (20,21) was found by Boutroux in [Bou13] (through a completely different analysis), to characterize the asymptotic distribution of the poles of the integràle tritronquèe. Therefore we call (20,21) the *Bohr-Sommerfeld-Boutroux system*.

Equation (22) will not be studied in this paper, even though is quite remarkable. Indeed, it describes the breaking of the PT symmetry (see [DT00] and [BBM+01]).

Case (100)

$$\begin{aligned} w_0(1,-1) &= -1 \\ \widehat{w}_{-2}(1,-1) &= \widehat{w}_2(1,-1) = 1 \end{aligned}$$

Since $w_0 \neq \widehat{w}_{\pm 2}$ and $w_2 \neq w_{-2}$, if the error ρ_1^{-2} or ρ_{-1}^2 is small enough, then all the asymptotic values are pairwise distinct.

Case (110)

$$\begin{aligned} \widehat{w}_{-1}(1,-2) &= 1 \\ w_2(1,-2) &= -1 \end{aligned}$$

In this case, it is impossible to calculate w_0 with the WKB method that has been here developed. Hence it may be that either $w_0 = w_2$ or $w_0 = w_{-2}$.

Notice, however, that (110) is the graph only of a very restricted class of potentials namely $V(\lambda) = (\lambda + \lambda_0)^2(\lambda - 2\lambda_0)$, where λ_0 is real and positive. Since the potential is real then $w_0 \neq w_{\pm 2}$.

Case (000) In this case, no asymptotic values can be calculated. Notice, however, that $V(\lambda) = \lambda^3$ is the only potential with graph (000). For this potential the asymptotic values can be computed exactly, simply using symmetry considerations. Indeed one can choose $w_k = e^{\frac{2k\pi}{5}i}$, $k \in \mathbb{Z}_5$.

4.4 Small Parameter

The WKB method normally applies to problem with an *external small parameter*, usually denoted $\hbar$ or ε. In the study of the distributions of poles of a given solution y of P-I there is no external small parameter and we have to explore the whole space of cubic potentials. The aim of this section is to introduce an *internal small parameter* in the space of cubic potentials, that greatly simplifies our study.

On the linear space of cubic potentials in canonical form

$$V(\lambda; a, b) = 4\lambda^3 - 2a\lambda - 28b,$$

we define the following action of the group $\mathbb{R}^+ \times \mathbb{Z}_5$ (similar to what is called Symanzik rescaling in [Sim70])

$$(x, m)[V(\lambda; a, b)] = V(\lambda; \Omega^{2m} x^2 a, \Omega^{3m} x^3 b),\ x \in \mathbb{R}^+,\ m \in \mathbb{Z}_5,\ \Omega = e^{\frac{2\pi}{5} i}. \quad (23)$$

The induced action on the graph $\mathcal{S}$, on the relative error ρ_l^m, and on the difference $S_i(\lambda_j) - S_i(\lambda_k)$ is described in the following

Lemma 10. *Let the action of the group $\mathbb{R}^+ \times \mathbb{Z}_5$ be defined as above. Then*

(i) (x, m) *leaves the graph S invariant, but for a shift of the labels $k \to k + m$ of the external vertices.*

(ii) $(x, m)[S_i(\lambda_j) - S_i(\lambda_k)] = x^{\frac{5}{2}} \left(S_i(\lambda_j) - S_i(\lambda_k)\right)$.

(iii) $(x, m)[\rho_l^k] = x^{-\frac{5}{2}} \rho_l^k$.

Proof. The proof of (i) and (ii) follows from the following equality

$$\sqrt{V(\lambda; \Omega^{2k} x^2 a, \Omega^{3k} x^3 b)} d\lambda = x^{\frac{5}{2}} \sqrt{V(\lambda'; a, b)} d\lambda' , \quad \lambda = x\lambda' .$$

The proof of point (iii) follows from a similar scaling law of the 1-form $\alpha(\lambda) d\lambda$ (see equation (31) in Appendix A). □

Due to Lemma 10(iii), $\varepsilon = \left|\frac{a}{b}\right|$ plays the role of the small parameter. Indeed, along any orbit of the action of the group $\mathbb{R} \times \mathbb{Z}_5$, all the (finite) relative errors go to zero uniformly as $\left|\frac{a}{b}\right| \to 0$.

Since all the relevant information is encoded in the quotient of the space of cubic potentials with respect to the group action, we define the following change of variable

$$\nu(a, b) = \frac{b}{a} , \qquad \mu(a, b) = \frac{b^2}{a^3} \quad . \quad (24)$$

The induced action on these coordinates is simple, namely

$$(x, m)[\mu(a, b)] = \mu(a, b) \text{ and } (x, m)[\nu(a, b)] = \Omega^m x \, \nu(a, b) .$$

Moreover, the orbit of the set $\left\{(\nu, \mu) \in \mathbb{C}^2 \text{ s.t. } |\nu| = 1, |\arg \nu| < \frac{\pi}{5} ,\ \mu \neq 0\right\}$ is a dense open subset of the space of cubic potentials.

5 Poles of Integrále Tritronquée

From Lemma 9 and the results of the computations in Section 4.3, it follows that equation (8) admits in WKB approximation the simultaneous solutions of the two quantization conditions $w_{\pm 1} = w_{\mp 2}$ only if the Stokes complex is of type (320). In particular, after our calculations the poles of the intégrale tritronquée are related, in WKB approximation, to the solutions of the *Bohr-Sommerfeld-Boutroux* system (20,21).

We rewrite this system in the following equivalent form:

$$
\begin{aligned}
\oint_{a_1} \sqrt{V(\lambda; a, b)} d\lambda &= i\pi(n - \frac{1}{2}) \\
\oint_{a_{-1}} \sqrt{V(\lambda; a, b)} d\lambda &= -i\pi(m - \frac{1}{2})
\end{aligned}
\tag{25}
$$

where m, n are positive natural numbers and the paths of integration are shown in figure 4.

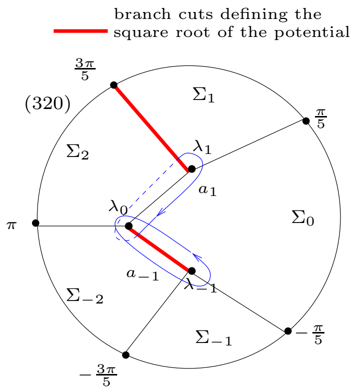

Figure 4: Riemann surface $\mu^2 = V(\lambda; a, b)$

System (25) is studied in detail in [KK93] where the following lemma is proven.

Lemma 11. *If a polynomial* $V(\lambda) = 4\lambda^3 - 2a\lambda - 28b$ *satisfies the system (25) then* $|\arg a| > \frac{4\pi}{5}$.

The Lemma above should be compared to the conjecture, to which we referred in the introduction.

Real Poles We compute all the real solutions of system (25) and compare them with some numerical results from [JK01]. We note that the accuracy of the WKB method is astonishing also for small a and b (see Table 2 below).

For the subset of real potentials, we have

$$\oint_{a_1} \sqrt{V(\lambda; a, b)} d\lambda = \overline{\oint_{a_{-1}} \sqrt{V(\lambda; a, b)} d\lambda} \quad ,$$

where $^{-}$ stands for complex conjugation.

Therefore system (25) reduces to one equation and the real poles of tritronquée are characterized, in WKB approximation, by one natural number.

Lemma 12. *Let μ and ν be defined as in equation (24). Then the real polynomials whose Stokes complex is of type* (320) *are the orbit of a single point of the $\mathbb{R}^+$ action, characterized by $\mu^* \cong -3158,92$ and $\nu > 0$.*

Moreover, if a real polynomial $V(\lambda; a, b)$ satisfies the Bohr-Sommerfeld-Boutroux conditions (25), then

$$a = a^*(n - \frac{1}{2})^{\frac{4}{5}}, \ b = b^*(n - \frac{1}{2})^{\frac{6}{5}} \tag{26}$$

for some $n \in \mathbb{N}^*$ and $a^* \cong -4,0874$, $b^* \cong -0,1470$.

Proof. All real cubic potentials whose Stokes complex has type (320) have one real turning point λ_0 and two complex conjugate turning points $\lambda_{\pm 1}$. For the subset of real potentials with two complex conjugate turning points the cycles $a_{\pm 1}$, as shown in Figure 4, are unambiguously defined. From the Classification Theorem, it follows that a Stokes complex has type (320) if and only if $\mathrm{Re} \oint_{a_1} \sqrt{V(\lambda; a, b)} d\lambda = 0, \mathrm{Im} \oint_{a_1} \sqrt{V(\lambda; a, b)} d\lambda \neq 0$. Since these conditions are invariant under the $\mathbb{R}^+$ action, if they are satisfied for a point of the space of cubic potentials, then they are satisfied on all its orbit. Moreover, it is easily seen that this orbit exists and is unique. With the help of a software of numeric calculus, we characterized numerically the orbit. Afterthat, using the scaling law in Lemma 10(ii) we calculated all the real solutions of the Bohr-Sommerfeld-Boutroux system. □

To the best of our knowledge, the asymptotic for the b coefficients has never been given.

In the paper [JK01], the authors showed that the intégrale tritronquée has no poles on the real positive axis. The real poles are a decreasing sequence of negative numbers a_n and some of them are evaluated numerically in the same paper.

In Table 2, we compare the first two real solutions to system (25) with the numerical evaluation of the first two poles of the intégrale tritronquée.

6 Concluding Remarks

We have studied the distribution of the poles of solutions to the Painlevé first equation using the theory of the cubic anharmonic oscillator. We have

	WKB	Numeric	Error %
a_1	$-2,34$	$-2,38$	$1,5$
b_1	$-0,064$	$-0,062$	2
μ_1	-3158	-3510	10
a_2	$-5,65$	$-5,66$	$0,2$
b_2	$-0,23$	unknown	unknown
μ_2	-3158	unknown	unknown

Table 2: Comparison between numerical and WKB evaluation of the first two real poles of the intégrale tritronquée.

applied a suitable version of the complex WKB method to analyze the distribution of poles of the intégrale tritronquée.

In subsequent publications we plan to pursue our study of poles of P-I transcendents in different directions.

In particular, we want to use the Nevanllina theory of the branched coverings of the sphere to complete the analysis of the poles of the intégrale tritronquée, showing that the developed WKB method yields a complete qualitative picture and efficient quantitative estimates of the distribution of the poles.

Since the quantization condition $\sigma_0 = 0$ characterizes the monodromy data of a family of special solutions of P-I, called *intégrale tronquée* (see [Kap04]), these solutions are strictly related to the spectral theory of PT symmetric anharmonic oscillators and to functional equations of Bethe Ansatz type (for what concerns the PT symmetric anharmonic oscillators and the Bethe Ansatz equations, see [DDT01] and references therein). We are going to investigate the consequences of this relation in a subsequent publication.

Remark. *After the main computations of the present paper had been completed, the author learned from B. Dubrovin about the results of V. Novokshenov presented at the conference NEEDS09 (May 2009). Novokshenov studied WKB solutions to the Schrödinger equation (8) with $b = 0$ and their connections to the distributions of poles of certain particular solutions to the Painlevé-I equation, including intégrales tronquée and intégrale tritronquée.*

Remark. *In the paper [MB10] written after this paper was published, the author proves that eventually (for big enough n and m) around any solution of the Bohr-Sommerfeld-Boutroux system (25) there is one and only one pole of the intégrale tritronquée. Moreover the distance between a pole and its approximation vanishes asymptotically.*

A Appendix

The aim of this appendix is to prove Theorem 8. Our approach is similar to the approach of Fedoryuk [Fed93].

Notations are as in sections 3 and 4, except for ∞_k. In what follows, we suppose to have fixed a certain cubic potential $V(\lambda; a, b)$ and a maximal domain D_k. To simplify the notation we write $V(\lambda)$ instead of $V(\lambda; a, b)$.

A.1 Gauge Transform to an L-Diagonal System

The strategy is to find a suitable gauge transform of equation (8) such that for large λ it simplifies. We rewrite the Schrödinger equation

$$-\psi''(\lambda) + V(\lambda)\psi(\lambda) = 0 \ , \tag{27}$$

in first order form:

$$\begin{aligned} \Psi'(\lambda) &= E(\lambda)\Psi(\lambda) \ , \\ E(\lambda) &= \begin{pmatrix} 0 & 1 \\ V(\lambda) & 0 \end{pmatrix} \ . \end{aligned} \tag{28}$$

Lemma 13 (Fedoryuk)**.** *In* D_k

(i) the gauge transform

$$\begin{aligned} Y(\lambda) &= A(\lambda)U(\lambda) \ , \\ A(\lambda) &= j_k(\lambda) \begin{pmatrix} 1 & 1 \\ \sqrt{V(\lambda)} - \frac{V'(\lambda)}{4V(\lambda)} & -\sqrt{V(\lambda)} - \frac{V'(\lambda)}{4V(\lambda)} \end{pmatrix} \ , \end{aligned} \tag{29}$$

is non singular and

(ii) the system (28) is transformed into the following one

$$\begin{aligned} U'(\lambda) &= F(\lambda)U(\lambda) = \left(A^{-1}EA - A^{-1}A'\right) U \ , \\ F(\lambda) &= 2\sqrt{V(\lambda)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha(z) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \ , \end{aligned} \tag{30}$$

$$\alpha(\lambda) = \frac{1}{32\sqrt{V(\lambda)}^5}\left(4V(\lambda)V''(\lambda) - 5V'^2(\lambda)\right) . \tag{31}$$

Proof. (i) Indeed $\det A(\lambda) = 2j_k^2(\lambda)\sqrt{V(\lambda)} \neq 0, \ \forall \lambda \in D_k$, by construction of j_k and D_k.

(ii) It is proven by a simple calculation.

□

A.2 Some Technical Lemmas

Before we can begin the proof of Theorem 8, we have to introduce the compactification of D_k and the preparatory Lemmas 14 and 15.

Compactification of D_K Since D_k is simply connected, it is conformally equivalent to the interior of the unit disk D. We denote U the uniformisation map, $U : D \to D_k$.

By construction, the boundary of D_k is the union of n free Jordan curves, all intersecting at ∞. Here n is equal to the number of sectors Σ_l such that $\Sigma_l \leftrightarrows \Sigma_k$ minus 2.

Due to an extension of Carathéodory's Theorem ([Car], §134-138), the map U extends to a continuous map from the closure of the unit circle to the closure of D_k. The map is injective on the closure of D minus the n counterimages of ∞. Hence, the uniformisation map realizes a n point compactification of D_k, that we call $\overline{D_k}$. In $\overline{D_k}$ there are n point at ∞. We denote ∞_k the point at ∞ belonging to the closure of $U(\Sigma_{k-1} \cup \Sigma_k \cup \Sigma_{k+1})$. Moreover, for $\lambda = k+2$ or $\lambda = k-2$, if $\Sigma_l \leftrightarrows \Sigma_k$ we denote ∞_l the point at ∞ belonging to the closure of $U(\Sigma_l)$.

Definition 11. *Let H be the space of function holomorphic in D_k and continuous in $\overline{D_k}$. H endowed with the sup norm is a Banach space $(H, \|\cdot\|_H)$.*

Let $\Gamma(\lambda), \lambda \in \overline{D_k} - \infty_k$ be the set of injective piecewise differentiable curves $\gamma : [0,1] \to \overline{D_k}$, such that

1. *$\gamma(0) = \lambda$, $\gamma(1) = \infty_k$,*
2. *$ReS_k(\gamma(0), \gamma(t))$ is eventually non decreasing,*
3. *there is an $\varepsilon > 0$ such that eventually $\left|\arg\gamma(t) - \frac{2\pi k}{5}\right| < \frac{\pi}{5} - \varepsilon$,*
4. *the length of the curve restricted to $[0,T]$ is $O\left(|\gamma(T)|\right)$, as $t \to 1$.*

Let $\tilde{\Gamma}(\lambda)$ be the subset of $\Gamma(\lambda)$ of the paths along which $ReS_k(\gamma(0), \gamma(t))$ is non decreasing.

Let $K_1 : H \to H$ and $K_2 : H \to H$ be defined (for the moment formally)

$$K_1[h](\lambda) = -\int_{\gamma\in\Gamma(\lambda)} e^{2S_k(\mu,\lambda)}\alpha(\mu)h(\mu)d\mu\ , \tag{32}$$

$$K_2[h](\lambda) = \int_{\gamma\in\Gamma(\lambda)} \alpha(\mu)h(\mu)d\mu\ . \tag{33}$$

Let $\rho : \overline{D_k} \to \overline{D_k}$:

$$\rho(\lambda) = \begin{cases} \inf_{\gamma\in\tilde{\Gamma}(\lambda)} \int_0^1 \left|\alpha(\gamma(t))\frac{d\gamma(t)}{dt}\right| dt, & \text{if } \lambda \neq \infty_k \\ 0 & , \text{if } \lambda = \infty_k\ . \end{cases}$$

Remark. *Since along rays of fixed argument φ, with $\left|\varphi - \frac{2\pi k}{5}\right| < \frac{\pi}{5} - \varepsilon$, ReS_k is eventually increasing, there are paths satisfying point (1) through (4) of the above definition. Moreover, by construction of D_k, $\tilde{\Gamma}(\lambda)$ is non empty for any λ.*

Before beginning the proof of the theorem, we need two preparatory lemmas.

Lemma 14. *Fix $\varepsilon > 0$, an angle $\left|\arg\varphi - \frac{2\pi l}{5}\right| < \frac{\pi}{5} - \varepsilon$, and let $\Omega = \Sigma_l \cap \left\{\lambda \in \mathbb{C}, \left|\lambda - \frac{2\pi l}{5}\right| < \frac{\pi}{5} - \varepsilon\right\}$. Denote $i(R) = i_{Re^{i\varphi}} \cap \Omega, R \in \mathbb{R}^+$, and let $L(R)$ be the length with respect to the euclidean metric of $i(R)$. Then $L(R) = O(R)$ and $\inf_{\lambda' \in i(R)} |\lambda'| = O(R)$.*

Let r be any level curve of $S_l(\lambda^*, \cdot)$ asymptotic to the ray of argument $\frac{2\pi l}{5}$, $\Omega(R) = \left\{\lambda \in \Omega, ReS_l(\lambda, Re^{i\varphi}) \geq 0\right\}$, and $M(R)$ be the length of $r \cap \Omega(R)$. Then $M(R) = O(R)$.

Proof. [Str84], chapter 3. ☐

Lemma 15. *(i) ρ is a continuous function.*

(ii) K_1 and K_2 are well-defined bounded operator. In particular

$$|K_i[h](\lambda)| \leq \rho(\lambda)\|h\|_H\ ,\ i = 1, 2 \tag{34}$$

(iii) $K_2[h](\infty_k) = K_1[h](\infty_k) = K_1(\infty_{k\pm 2}) = 0, \forall h \in H$

Proof. (i)Since $\alpha(\lambda)d\lambda = O(|\lambda|^{-\frac{7}{2}})$, then $\alpha(\lambda)d\lambda$ is integrable along any curve $\gamma \in \tilde{\Gamma}(\lambda)$. Therefore ρ is a continuous function on $\overline{D_k}$.

(ii)We first prove that (a) $K_i[h](\lambda)$ does not depend on the integration path for any $\lambda \in \overline{D_k}$ minus the points at infinity. A result that easily implies that $K_i[h](\cdot)$ is an analytic function on D_k, continuous on $\lambda \in \overline{D_k}$ minus the points at ∞. We then prove (b) the estimates (34) and (c) the existence of the limits $K_i[h](\infty_l)$, $l = \infty_k, \infty_{k\pm 2}$.

To simplify the notation, we prove the theorem for the operator K_1. The proof for K_2 is almost identical .

(a)Let $\gamma_a, \gamma_b \in \Gamma(\lambda)$. The curve $i_{\gamma_a(T)}$, where $T = 1 - \varepsilon$ for some small $\varepsilon > 0$ intersect γ_b at some $\gamma_b(T')$. Therefore we can decompose $-\gamma_b \circ \gamma_a$ into two different paths with the help of a segment of $i_{\gamma_a(T)}$, $\int_{-\gamma_b \circ \gamma_a} e^{2S_k(\mu,\lambda)}\alpha(\mu)h(\mu)d\mu = \int_{\gamma_1} + \int_{\gamma_2} e^{2S_k(\mu,\lambda)}\alpha(\mu)h(\mu)d\mu$. One path γ_1 is the loop based at λ and the other γ_2 is the loop based at ∞_k. Since $\gamma_1 \subset D_k$, then $\int_{\gamma_1} e^{2S_k(\mu,\lambda)}\alpha(\mu)h(\mu)d\mu = 0$. Along γ_2, $e^{2S_k(\gamma_2(t),\lambda)} \leq 1$ therefore the integrand can be estimated just by $|\alpha(\gamma_2(t))|$. Due to lemma 14, $\int_{\gamma_2} |\alpha(\mu)h(\mu)d\mu| = O(|\gamma_a(T)|^{-\frac{5}{2}})$. Since ε is arbitrary, then $K_1[h](\lambda)$ does not depend on the integration path.

(b)Clearly for any path $\gamma \in \tilde{\Gamma}(\lambda)$, $|K_1[h](\lambda)| \leq \int_0^1 |\alpha(\lambda)h(\lambda)d\lambda|\, dt$. Since $K_1[h](\lambda)$ does not depend on γ, then estimate (34) follows.

(c) Let λ_n be a sequence converging to $\infty_l, l = k+2$ or $l = k-2$; without losing any generality we suppose that the sequence is ordered such that $ReS_k(\lambda_n) \leq ReS_k(\lambda_{n+1})$. Fix a curve r, as defined in Lemma 14. By construction of D_k, it is always possible to connect two points λ_n and λ_{n+m} with a union of segments of the curves $i_{\lambda_n}, i_{\lambda_{n+m}}$ and of r. We denote by γ the union of this three segment. By construction of D_k (see Subsection 4.1 (iii)), there exists $\varepsilon > 0$ such that $\left|\arg\lambda_n - \frac{2\pi l}{5}\right| < \frac{\pi}{5} - \varepsilon, \forall n$. Therefore, due to Lemma 14, γ has length of order $|\lambda_n| + |\lambda_{n+m}|$. Hence $|K_1[h](\lambda_n) - K_1[h](\lambda_{n+m})| \leq \int_\gamma |h(\lambda)\alpha(\lambda)d\lambda| = O(|\lambda_n|^{-\frac{5}{2}})$. Then $K_1[h](\lambda_n)$ is a Cauchy sequence and the limit is well defined.

We now prove that this limit is zero by calculating it along a fixed ray $\lambda = xe^{i\varphi}$ inside $\Sigma_{k\pm2}$. Let us fix a point x^* on this ray in such a way that the function $ReS_k(x^*, x)$ is monotone decreasing in the interval $[x^*, +\infty[$. Along the ray we have

$$K_1[h](x) = -\frac{\int_x^{x^*} e^{2S_k(y,x^*)}\alpha(y)h(y)dy + g(x^*)}{e^{2S_k(x^*,x)}} ,$$

where $g(x^*)$ is a constant, namely $\int_{\gamma\in\Gamma(x^*)} e^{2S_k(\mu,x^*)}\alpha(\mu)h(\mu)d\mu$. Hence $\lim_{x\to\infty} K_1[h](x) = \lim_{x\to\infty} \frac{\alpha(x)h(x)}{\sqrt{V(x)}} = 0$.

With similar methods the reader can prove that the limit $K_1[h](\infty_k)$ exists and is zero.

□

We are now ready to prove Theorem 8.

Theorem 9. *Extend the WKB function j_k to D_k. There exists a unique solution ψ_k of (8) such that for all $\lambda \in D_k$*

$$\begin{aligned}\left|\frac{\psi_k(\lambda)}{j_k(\lambda)} - 1\right| &\leq g(\lambda)(e^{2\rho(\lambda)} - 1) , \\ \left|\frac{\psi_k'(\lambda)}{j_k(\lambda)\sqrt{V(\lambda)}} + 1\right| &\leq \left|\frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}\right| + (1 + \left|\frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}\right|)g(\lambda)(e^{2\rho(\lambda)} - 1) ,\end{aligned}$$

where $g(\lambda)$ is a positive function, $g(\lambda) \leq 1$ and $g(\infty_{k\pm2}) = \frac{1}{2}$.

Proof. We seek a particular solution to the linear system (30) via successive approximation.

If $U(\lambda) = U^{(1)} \oplus U^{(2)} \in H \oplus H$ satisfies the following integral equation of Volterra type

$$\begin{aligned}U(\lambda) &= U_0 + K[U](\lambda) \ , \ U_0 \equiv \begin{pmatrix}0\\1\end{pmatrix} , \\ K[U](\lambda) &= \begin{pmatrix}K_1[U^{(1)} + U^{(2)}](\lambda)\\K_2[U^{(1)} + U^{(2)}](\lambda)\end{pmatrix} ,\end{aligned} \tag{35}$$

then $U(\lambda)$ restricted to D_k satisfies (30).

We define the the *Neumann series* as follows

$$U_{n+1} = U^0 + K[U_n]\,, U_{n+1} = \sum_{i=0}^{n+1} K^i[U^0]\,. \tag{36}$$

More explicitly,

$$K^n[U_0](\lambda) = \left(\int_\lambda^{\infty_k} d\mu_1 \int_{\mu_1}^{\infty_k} d\mu_2 \dots \int_{\mu_{n-1}}^{\infty_k} d\mu_n \begin{matrix} -e^{2S(\mu_1,z)}\alpha(\mu_1) & \times \\ \alpha(\mu_1) & \times \end{matrix} \right.$$
$$\left. \begin{matrix} \alpha(\mu_2)(1-e^{2S(\mu_2,\mu_1)})\dots\alpha(\mu_n)(1-e^{2S(\mu_n,\mu_{n-1})}) \\ \alpha(\mu_2)(1-e^{2S(\mu_2,\mu_1)})\dots\alpha(\mu_n)(1-e^{2S(\mu_n,\mu_{n-1})}) \end{matrix} \right)\,.$$

Here the integration path γ belong to $\Gamma(\lambda)$. For any $\gamma \in \tilde{\Gamma}(\lambda)$ and any $n \geq 1$

$$\left|K^n[U_0]^{(i)}(\lambda)\right| \leq \frac{1}{2}\int_\lambda^{\infty_k}\int_{\mu_1}^{\infty_k}\dots\int_{\mu_{n-1}}^{\infty_k}\prod_{i=1}^{n}|2\alpha(\mu_i)d\mu_i| = \frac{2^{n-1}}{n!}\left(\int_\gamma d\mu_1|\alpha(\mu_1)|\right)^n,$$

where $K^n[U_0]^{(i)}$ is the i-th component of $K^n[U_0]$. Hence

$$|K^n[U]_i(\lambda)| \leq \frac{1}{2}\frac{1}{n!}\,(2\rho(\lambda))^n \tag{37}$$

Thus the sequence U^n converges in H and is a solution to (35); call U its limit. Due to Lemma 15, $U^{(1)}(\infty_{k\pm2}) = 0$.

Let Ψ_k be the solution to (28) whose gauge transform is U restricted to D_k; The first component ψ_k of Ψ_k satisfies equation (27).

From the gauge transform (29), we obtain

$$\begin{aligned} \frac{\psi_k(\lambda)}{j_k(\lambda)} - 1 &= U_1(\lambda) + U_2(\lambda) - 1\,, \\ \frac{\psi_k'(\lambda)}{j_k(\lambda)\sqrt{V(\lambda)}} + 1 &= U_1(\lambda)(1 - \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}) - (U_2(\lambda)-1)(1 + \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}) + \\ &- \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}\,, \end{aligned}$$

The thesis follows from these formulas, inequality (37) and from the fact that $U_1(\infty_{k\pm2}) = 0$.

□

Remark. *The solution $\psi_k(\lambda)$ of equation (8) described in Theorem 8 may be extended from D_k to the whole complex plane, since the equation is linear with entire coefficients. The continuation is constructed in the following Corollary.*

Corollary 1. *For any $\lambda \in \mathbb{C}$, λ not a turning point, we define $\Gamma(\lambda)$ as in Definition 11. Fixed any $\gamma \in \Gamma(\lambda)$ and h a continuous function on γ, we define the functionals $K_i[h](\lambda)$ as in equations (32) and (33). We define the Neumann series as in equations (35) and (36), and we continue j_k along γ.*

Then then Neumann series converges and we call $U^{(1)}(\lambda)$ and $U^{(2)}(\lambda)$ the first and second component of its limit.

Moreover, $\psi_k(\lambda) = \left(U^{(1)}(\lambda) + U^{(2)}(\lambda)\right) j_k(\lambda)$ solves equation (8) and for any $\varepsilon > 0$

$$\lim_{|\lambda| \to \infty \, , \, \left|arg\lambda - \frac{2\pi k}{5}\right| < \frac{3\pi}{5} - \varepsilon} \left(U^{(1)}(\lambda) + U^{(2)}(\lambda)\right) = 1$$

The reader should notice that if $\lambda \notin D_k$, then $\tilde{\Gamma}(\lambda)$ is empty and we cannot estimate $\frac{\psi_k(\lambda)}{j_k(\lambda)}$.

B Appendix

The aim of this Appendix is to prove Theorem 2 (ii) and (iii). The notation is, if not otherwise stated, as in the previous sections of the paper.

Next to a pole $z = a$ of a solution $y(z)$ of P-I, equation (1) becomes meaningless. To get rid of this singularity we perform a gauge transform of (1) such that the gauge-transformed equation has full meaning in the limit. In what follows, we suppose that z belongs to a punctured neighborhood of a, where $y(z)$ is holomorphic.

A gauge transform

Let z be a fixed regular value of $y(z)$. Let $\overrightarrow{\Phi}(\lambda, z) = G(\lambda, z)\overrightarrow{\Psi}(\lambda, z)$,

$$G(\lambda, z) = \begin{pmatrix} \frac{y'(z) + \frac{1}{2(\lambda - y(z))}}{\sqrt{2(\lambda - y(z))}} & \frac{1}{\sqrt{2(\lambda - y(z))}} \\ \sqrt{2(\lambda - y(z))} & 0 \end{pmatrix} . \tag{38}$$

Then $\overrightarrow{\Phi}(\lambda; z)$ satisfies (1) if and only if $\overrightarrow{\Psi}(\lambda; z)$ satisfies the following equation

$$\Psi_\lambda(\lambda, z) = \begin{pmatrix} 0 & 1 \\ Q(\lambda; z) & 0 \end{pmatrix} \Psi(\lambda, z)$$

where

$$Q(\lambda; z) = 4\lambda^3 - 2\lambda z + 2zy(z) - 4y^3(z) + y'^2(z) + \frac{y'(z)}{\lambda - y(z)} + \frac{3}{4(\lambda - y(z))^2} \quad (39)$$

We denote ψ the first component of $\overrightarrow{\Psi}$. The equation for $\overrightarrow{\Psi}$ is equivalent to the following second order scalar equation for ψ

$$\psi_{\lambda\lambda}(\lambda, z) = Q(\lambda; z)\psi(\lambda, z) \quad (40)$$

We summarize some property of the perturbed potential, which can be easily verified using the expansion (7).

Lemma 16. *Let $\varepsilon^2 = \frac{1}{y(z)} = (z-a)^2 + O((z-a)^6)$ then*

(i) $Q(\lambda; z)$ has a double pole at $\lambda = \frac{1}{\varepsilon^2}$. It is an apparent fuchsian singularity for equation (40): the local monodromy around it is -1.

(ii) $Q(\lambda; z)$ has two simple zeros at $\lambda = \frac{1}{\varepsilon^2} + O(\varepsilon^2)$

(iii) $Q(\lambda; z) = 4\lambda^3 - 2(a+\varepsilon)\lambda - 28b + O(\varepsilon) - \frac{2\lambda\varepsilon^{-1}}{\lambda - \varepsilon^{-2}} + \frac{3}{4(\lambda - \varepsilon^{-2})^2}$, where $O(\varepsilon)$ does not depend on λ.

Equation (40) is a perturbation of the cubic Schrödinger equation (8) and the asymptotic behaviours of solutions to the two equations are very similar. Indeed the local picture around the point at ∞ depends only on the terms $4\lambda^3$ and $-2z\lambda^3$.

More precisely, the equivalent of Corollary 1 in Appendix A is valid also for the perturbed Schrödinger equation.

Definition 12. *For any z, define a cut from $\lambda = \frac{1}{(z-a)^2}$ to ∞ such that it eventually does not belong to the the angular sector $\left|\arg\lambda - \frac{2\pi k}{5}\right| \leq \frac{3\pi}{5}$.*

Fix λ^* in the cut plane. $S_k(\lambda; z) = \int_{\lambda^*}^{\lambda} \sqrt{Q(\mu; z)}d\mu$ is well-defined for $\left|\arg\lambda - \frac{2k\pi}{5}\right| < \frac{3\pi}{5}$ and $\lambda >> 0$. Here the branch of $\sqrt{Q}$ is chosen such that $ReS_k(\lambda) \to +\infty$ as $|\lambda| \to \infty$, $\left|\arg\lambda - \frac{2\pi k}{5}\right| < \frac{\pi}{5} - \varepsilon$. We define $j_k(\lambda; z)$ as in equation (18) and $\alpha(\lambda; z)$ as in equation (31), but replacing $V(\lambda)$ with $Q(\lambda; z)$.

For any λ in the cut plane, let $\Gamma(\lambda)$ be the set of piecewise differentiable curves $\gamma : [0, 1]$ to the cut plane, $\gamma(0) = \lambda$, $\gamma(1) = \infty$, satisfying properties (2)(3) and (4) of Definition 11.

For any $\gamma \in \Gamma(\lambda)$, let H be the Banach space of continuous functions on γ that have a finite limit as $t \to 1$. Formulae (32) and (33) define two bounded functionals on H. We call such functionals $K_1(\lambda; z)$ and $K_2(\lambda; z)$.

Following the proof of Theorem 8, the reader can prove the following

Lemma 17. *Let λ belong to the cut plane, λ not a zero of $Q(\cdot;z)$. Fixed any $\gamma \in \Gamma(\lambda)$, we define the Neumann series as in equations (35) and (36), and we continue j_k along γ.*

Then the Neumann series converges and $\psi_k(\lambda) = (U_1(\lambda) + U_2(\lambda))\, j_k(\lambda)$ solves equation (40). Moreover, for any $\varepsilon > 0$

$$\lim_{|\lambda| \to \infty\,,\ \left|arg\lambda - \frac{2\pi k}{5}\right| < \frac{3\pi}{5} - \varepsilon} \left(U^{(1)}(\lambda) + U^{(2)}(\lambda) \right) = 1$$

Definition 13. *Let $\tilde{\psi_k}(\lambda, z)$ be the unique solution of equation (40) such that*

$$\frac{\tilde{\psi_k}(\lambda, z)}{\lambda^{-\frac{3}{4}} e^{-\frac{4}{5}\lambda^{\frac{5}{2}} + z\lambda^{\frac{1}{2}}}} \to 1\,,\ \ as\ \ |\lambda| \to \infty\,,\ arg\lambda = \frac{2\pi k}{5}\ . \tag{41}$$

Here the branch of $\lambda^{\frac{1}{4}}$ is fixed as $\lambda \to \infty$, $\arg \lambda = \frac{2\pi k}{5}$, and there it coincides with the branch chosen in equation (3). We define $\tilde{\psi_k}(\lambda, a)$ to be the unique solution of equation (8) with asymptotic (41), where $z = a$.

We denote $\psi_k(\lambda; z)$ the unordered pair $\left\{ \tilde{\psi_k}(\lambda, z), -\tilde{\psi_k}(\lambda, z) \right\}$.

Remark. *We notice that if the cuts are continuous in z, then $\tilde{\psi_k}(\lambda, z) = c(z)\psi_k(\lambda)$, where $\psi_k(\lambda)$ is the solution constructed in Lemma 17 and $c(z)$ is a bounded holomorphic function.*

Theorem 10. $\lim_{z \to a} \psi_k(\lambda, z) = \psi_k(\lambda, a)$, $\forall \lambda \in \mathbb{C}$.

Proof. Let λ be any point in the complex plane which is not a zero of $V(\lambda; a, b)$. For any sequence ε_n converging to zero, we choose two fixed rays r_1 and r_2 of different argument φ_1 and φ_2, $\left|\varphi_i - \frac{2k\pi}{5}\right| < \frac{\pi}{5}$. We denote $D_{R,\varepsilon}$ a disk of radius R with center $\lambda = \frac{1}{e^2}$ and we split the sequence ε_n into two subsequences ε_n^i such that $r_i \cap D_{R,\varepsilon_n^i} = \emptyset$ for any n big enough.

For $i = 1, 2$, we choose the cuts defined in Definition 12 in such a way that there exists a differentiable curve $\gamma_i : [0,1] \to \overline{\mathbb{C}}$, $\gamma_i(0) = \lambda$, $\gamma_i(1) = \infty$ with the following properties: (i)γ_i avoids the zeroes of $Q(\lambda, \varepsilon_n^i)$ and a fixed, arbitrarily small, neighborhood of the zeroes of $V(\lambda; a, b)$, (ii)γ_i does not intersect any cut, and (iv) γ_i eventually lies on r_i.

The proof of the thesis relies on the following estimates:

$$\begin{aligned}
&\sup\nolimits_{\lambda \in \mathbb{C} - D_{R,\varepsilon}} \left|\lambda^{-\delta}\right| |Q(\lambda; a+\varepsilon) - V(\lambda; a, b)| = O(\varepsilon^{2\delta - 3})\,, \qquad (42)\\
&\sup\nolimits_{\lambda \in \mathbb{C} - D_{R,\varepsilon}} \left|\lambda^{-\delta}\right| |Q_\lambda(\lambda; a+\varepsilon) - V_\lambda(\lambda; a, b)| = O(\varepsilon^{2\delta - 3})\,,\\
&\sup\nolimits_{\lambda \in \mathbb{C} - D_{R,\varepsilon}} \left|\lambda^{-\delta}\right| |Q_{\lambda\,\lambda}(\lambda; a+\varepsilon) - V_{\lambda\,\lambda}(\lambda; a, b)| = O(\varepsilon^{2\delta - 3})\,.
\end{aligned}$$

Due the above estimates it is easily seen that $\gamma_i \in \Gamma(\lambda)$, $\forall \varepsilon_n^i$. Due to Lemma 17 and Corollary 1, to prove the thesis it is sufficient to show that the functionals $K_1(\lambda; a+\varepsilon_n^i)$ and $K_2(\lambda; a+\varepsilon_n^i)$ converge in norm to $K_1(\lambda; a)$ and $K_2(\lambda; a)$. Here $K_i(\lambda; a)$, $i = 1, 2$ are defined as in Corollary 1. We notice that the norm of the functionals are just the $L^1(\gamma_i)$ norm of their integral kernels.

We first consider the functionals $K_2(\lambda; a + \varepsilon_n^i)$. Due to the above estimates

$$\lambda^{\frac{7}{2}}\alpha(\mu, \varepsilon_n^i) \to \lambda^{\frac{7}{2}}\alpha(\mu), \text{ uniformly on } \gamma_i([0,1]) \text{ as } n \to \infty .$$

Hence the sequence $\alpha(\mu, \varepsilon_n^i)$ converges in norm $L^1(\gamma_i)$ to $\alpha(\mu)$ and the sequence $K_2(\lambda; e_n^i)$ converges in operator norm to $K_2(\lambda; a)$.

We consider now the sequence $K_1(\lambda; a + \varepsilon_n^i)$.

To prove the convergence of the above sequence of operators, it is sufficient to prove that

$$e^{S_k(\lambda; a+\varepsilon_n^i) - S_k(\mu; a+\varepsilon_n^i)} \to e^{S_k(\lambda;0) - S_k(\mu;0)} \text{ uniformly on } \gamma_i([0,1]) \text{ as } n \to \infty .$$

We first note that

$$\begin{aligned} e^{S_k(\lambda;0)-S_k(\mu;0)} \quad & - e^{S_k(\lambda;a+\varepsilon_n^i)-S_k(\mu;a+\varepsilon_n^i)} = e^{S_k(\lambda;0)-S_k(\mu;0)}\left(1 - e^{g(\mu;\varepsilon)}\right) , \\ g(\mu, \varepsilon) \quad & = \int_{\lambda,\gamma_i}^{\mu} \frac{Q(\nu,\varepsilon) - V(\nu; a, b)}{\sqrt{Q(\nu,\varepsilon)} + \sqrt{P(\nu; a, b)}}\, d\nu . \end{aligned}$$

Using estimate (42), it is easy to show that $g(\mu; \varepsilon) = f(\varepsilon)O(\mu^{\delta})$, where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $0 < \delta << 1$. Therefore the difference of the exponential functions converges uniformly to 0.

□

We can prove Theorem 2 (ii) and (iii).

Indeed from (29), it is easily seen that (choosing one of the two branches of the gauge transform)

$$\overrightarrow{\Psi_k}(\lambda; z) = G(\lambda, z)^{-1}\overrightarrow{\Phi_k}(\lambda; z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{\psi}_k(\lambda, z) \\ \tilde{\psi}'_k(\lambda, z) \end{pmatrix} ,$$

if $\left|\arg\lambda - \frac{2\pi k}{5}\right| < \frac{3\pi}{5}$ and $|\lambda| >> 0$.

Moreover from (29), it follows that

$$\lim_{z\to a}(z - a)\Phi_k^{(2)}(\lambda, z) = i\sqrt{2}\Psi_k^{(1)}(\lambda; a) .$$

Hence Theorem 2 (ii) and (iii) follow from Theorem 10.

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