DEGENERATE RIEMANN-HILBERT-BIRKHOFF PROBLEMS, SEMISIMPLICITY, AND CONVERGENCE OF WDVV-POTENTIALS

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Abstract. In the first part of this paper, we give a new analytical proof of a theorem of C. Sabbah on integrable deformations of meromorphic connections on $\mathbb{P}^1$ with coalescing irregular singularities of Poincaré rank 1, and generalizing a previous result of B. Malgrange. In the second part of this paper, as an application, we prove that any semisimple formal Frobenius manifold (over $\mathbb{C}$), with unit and Euler field, is the completion of an analytic pointed germ of a Dubrovin-Frobenius manifold. In other words, any formal power series, which provides a quasi-homogenous solution of WDVV equations and defines a semisimple Frobenius algebra at the origin, is actually convergent under no further tameness assumptions.

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1. Introduction

In this paper, we address the problem of convergence of formal solutions, in the ring of formal power series, of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) associativity equations. This is the overdetermined system of non-linear partial differential equations, in a single scalar function $F(t^1,\dots,t^n)$, given by

$$\sum_{\mu,\nu}\frac{\partial^3 F}{\partial t^\alpha\partial t^\beta\partial t^\mu}\eta^{\mu\nu}\frac{\partial^3 F}{\partial t^\nu\partial t^\gamma\partial t^\delta}=\sum_{\mu,\nu}\frac{\partial^3 F}{\partial t^\delta\partial t^\beta\partial t^\mu}\eta^{\mu\nu}\frac{\partial^3 F}{\partial t^\nu\partial t^\gamma\partial t^\alpha},\qquad \alpha,\beta,\gamma,\delta=1,\dots,n,$$
$$\frac{\partial^3 F}{\partial t^1\partial t^\alpha\partial t^\beta}=\eta_{\alpha\beta}=\text{const.},\quad \eta=(\eta_{\alpha\beta})_{\alpha,\beta},\quad \eta^{-1}=(\eta^{\alpha\beta})_{\alpha,\beta}\qquad \alpha,\beta=1,\dots,n.$$

Introduced in the physics of topological field theories [Wit90, DVV91], the geometry of solutions F of WDVV equations, satisfying a further quasi-homogeneity condition (w.r.t. the variables $\boldsymbol{t}$, and up to quadratic terms), was firstly axiomatized by B. Dubrovin, with the notion of Frobenius manifolds [Dub92, Dub96, Dub98, Dub99].

It was soon realized that these quasi-homogeneous solutions of WDVV equations arise in areas of mathematics which are very apart from each other (singularity theory, algebraic and symplectic geometry, integrable systems, mirror symmetry, to name just a few), often leading to new and non-trivial relations between them, see [Dub96, Man99, Her02, Sab07].

Typically, the corresponding solutions $F(\boldsymbol{t})$ of WDVV equations are given as generating functions of numerical sequences of geometrical interest (e.g. Gromov-Witten invariants). Consequently, they can be handled just as formal power series in $k[\![\boldsymbol{t}]\!]$, where k is a commutative $\mathbb{Q}$-algebra, defining a *formal Frobenius manifold* structure on the formal spectrum $\mathrm{Spf}\, k[\![\boldsymbol{t}]\!]$, see [Man99, III.§1]. This defines a formal family of Frobenius algebras with structure constants given by $c^\gamma_{\alpha\beta}(\boldsymbol{t}):=\eta^{\lambda\gamma}\partial^3_{\alpha\beta\lambda}F(\boldsymbol{t})$.

The relevance of these formal structures is further highlighted by their deep relations with the cohomology of the Deligne-Mumford moduli stacks $\overline{\mathcal{M}}_{g,\mathfrak{n}}$ of $\mathfrak{n}$-pointed stable curves of genus g, [KM94, Man99]. Remarkably enough, any formal Frobenius manifold is equivalent to a *tree level*[1] *Cohomological Field Theory* (CohFT), i.e. the datum of a family of $\mathfrak{S}_\mathfrak{n}$-covariant

[1]A richer notion of *complete* CohFT on a given (H,η) is also available, in which the datum is enriched to a family $(\Omega_{g,\mathfrak{n}})_{g,\mathfrak{n}}$ of k-linear tensors $\Omega_{g,\mathfrak{n}}\in(H^*)^{\otimes\mathfrak{n}}\otimes_k H^\bullet(\overline{\mathcal{M}}_{g,\mathfrak{n}};k)$, satisfying further compatibility properties, for any pair $(g,\mathfrak{n})$ of non-negative integers in the stable range $2g-2+\mathfrak{n}>0$. The prototypical example of a complete CohFT is provided by the Gromov-Witten theory of a smooth projective variety X. The corresponding formal Frobenius manifold attached to its genus zero sector is called *quantum cohomology of* X. See [KM94, Man99] and Section 6 of this paper.

tensors $\Omega_{0,\mathfrak{n}} \in (H^*)^{\otimes \mathfrak{n}} \otimes_k H^\bullet(\overline{\mathcal{M}}_{0,\mathfrak{n}}; k)$, on a given free metric k-module (H, η) of finite rank, satisfying some compatibility conditions w.r.t. the natural forgetful morphisms $\overline{\mathcal{M}}_{0,\mathfrak{n}} \to \overline{\mathcal{M}}_{0,\mathfrak{n}-1}$, and gluing morphisms $\overline{\mathcal{M}}_{0,\mathfrak{n}_1} \times \overline{\mathcal{M}}_{0,\mathfrak{n}_n} \to \overline{\mathcal{M}}_{0,\mathfrak{n}_1+\mathfrak{n}_2}$. The corresponding WDVV-potential $F(\boldsymbol{t})$ is a generating power series for integrals of the form $\int_{\overline{\mathcal{M}}_{0,\mathfrak{n}}} \Omega_{0,\mathfrak{n}}(\bigotimes_{j=1}^{\mathfrak{n}} \Delta_{\alpha_j})$ for a k-basis $(\Delta_j)_j$ of H. See [KM94, Man99, Pan18] for more details.

One of the main point of the current paper is to find sufficient conditions ensuring the convergence of quasi-homogeneous solutions $F \in k[\![\boldsymbol{t}]\!]$ of WDVV equations, in the case $k = \mathbb{C}$. The convergence condition allows to jump from the formal category to the complex analytic category: formal Frobenius manifolds can be promoted to *analytic* Frobenius manifolds, the class of geometrical objects originally conceived by Dubrovin, and for this reason also called *Dubrovin-Frobenius manifolds*.

The new main result of this paper, Theorem 5.1, claims that any formal *semisimple* Frobenius manifold over $k = \mathbb{C}$ is actually the completion of a pointed germ of an analytic Dubrovin-Frobenius manifold. Alternatively stated, given a quasi-homogeneous formal solution $F \in \mathbb{C}[\![\boldsymbol{t}]\!]$ of WDVV equations whose corresponding Frobenius $\mathbb{C}$-algebra at the origin $\boldsymbol{t} = 0$ is semisimple, its domain of convergence is non-empty, and it thus carries a Dubrovin-Frobenius manifold structure. This statement is a refinement of a seemingly known result, referred to as a "general fact" in [Man99, III.§7.1, pag.135], and stated under stronger unnecessary *tameness* assumptions[2] (see below).

At the core of our proof there is the local identification of semisimple points $\boldsymbol{t}$ of a Dubrovin-Frobenius manifold with the parameters of isomonodromic deformations of ordinary differential equations with rational coefficients, of the form

$$\frac{d}{dz}Y(z, \boldsymbol{t}) = \left(\mathcal{U}(\boldsymbol{t}) + \frac{1}{z}\mu(\boldsymbol{t})\right) Y(z, \boldsymbol{t}), \tag{1.1}$$

where $\mathcal{U}, \mu$ are (matrices representing) suitably defined tensors on the Dubrovin-Frobenius manifold. This identification – one of the main point of the theory of Dubrovin– was originally established in [Dub96, Dub98, Dub99] at *tame semisimple points*, i.e. points $\boldsymbol{t}$ at which the leading term $\mathcal{U}(\boldsymbol{t})$ of the coefficient of (1.1) has simple spectrum. Subsequently, in [CG17, CG18, CDG19, CDG20] the isomonodromic approach to the Frobenius geometry was extended to *all* semisimple points, including points $\boldsymbol{t}$ at which some of the eigenvalues of $\mathcal{U}(\boldsymbol{t})$ coalesce.

The proof of Theorem 5.1 consists of two parts. Firstly, given a formal Frobenius manifold $F \in \mathbb{C}[\![\boldsymbol{t}]\!]$, it is constructed an *analytic family* (1.1) of ODEs specializing to the given one[3] for $\boldsymbol{t} = 0$, and defining a Dubrovin-Frobenius manifold. Secondly, it is proved that the underlying analytic WDVV-potential $F^{\rm an} \in \mathbb{C}\{\boldsymbol{t}\}$ coincides with the original formal one, i.e. $F = F^{\rm an}$. Having said, it is thus clear that the first step of the proof of Theorem 5.1 relies on the existence of solutions of *families* of Riemann-Hilbert-Birkhoff boundary value problems.

[2]I do not know any reference in literature where a complete proof is given. I thank Yu.I. Manin for a friendly e-mail correspondence on this point. The current paper both recovers a proof of this known fact, and it also removes the tameness assumption.

[3]Given a formal Frobenius manifold, the system (1.1) has coefficients in $M_n(\mathbb{C}[\![\boldsymbol{t}]\!])$. Hence, for $t = 0$, we have a well defined differential system with coefficients in $M_n(\mathbb{C})$.

In the case $\boldsymbol{t} = 0$ is a tame semisimple point of the given formal Frobenius manifold, a well-known result of B. Malgrange [Mal83a, Mal83b, Mal86], on the existence of universal integrable deformations of meromorphic connections on $\mathbb{P}^1$ with irregular singularities, can be applied. This leads to the already known result mentioned[4] in [Man99, III.§7, pag.135].

In [Sab18], C. Sabbah obtained an extension of the theorem of Malgrange, in order to include the case of meromorphic connections on $\mathbb{P}^1$ with coalescing irregular singularities. In the geometrical case attached to Frobenius manifolds, the assumptions of [Sab18, Th. 4.9] are satisfied. Sabbah Theorem can thus be applied in the first step of the proof of Theorem 5.1, in the case $\boldsymbol{t} = 0$ is a coalescing semisimple point for the given formal Frobenius manifold. Remarkably enough, the assumptions of [Sab18, Th. 4.9] exactly coincide with the sharp conditions, found in [CG17, CDG19], under which the resulting analytic family (1.1) of ODEs has a well-behaved deformation theory of both formal and genuine solutions.

The original proof of [Sab18, Th. 4.9] is actually only one of the outcomes of a more general study, invoking a mix of techniques, including properties of *good* and *very-good formal decompositions* of flat meromorphic bundles [Sab93, Sab00], and recent results on meromorphic connections in dimension $\geqslant 2$ due to K. Kedlaya (in the complex analytic case) and T. Mochizuki (in the algebraic case), see [Ked10, Ked11, Moc09, Moc11a, Moc11b, Moc14]. In Section 3, we give an alternative proof of [Sab18, Th. 4.9], with a more analytical perspective, closer to the one of [CG17, CDG19]. Our proof is uniquely based on properties of Fredholm-operator-valued holomorphic functions. In particular, a result due to B. Gramsch [Gra70] – an *analytical Fredholm alternative* w.r.t. several parameters– will be invoked to prove that the solvability of a family of Riemann-Hilbert-Birkhoff boundary value problems is an open property, in the same spirit of [Zho89]. This is a well-known strategy for proving the Painlevé property of solutions of the isomonodromy deformations equations, see e.g. [FZ92, FIKN06].

Many of the results of this paper can be extended to the case of flat F-manifolds [HM99, Man05]. These are slightly weaker structures than the Frobenius one, but whose geometry encompasses even more areas of modern mathematics, such as special solutions of the oriented associativity equations [LM04], quantum K-theory [Lee04], all Painlevé transcendents [AL15], open WDVV equations [BB19], F-cohomological field theories [ABLR20], and even information geometry [CM20]. We plan to give more analytical details in a future publication.

Structure of the paper. In Section 2 we review necessary background material on the Riemann-Hilbert-Birkhoff problems with a geometrical perspective. The main results of B. Malgrange on the existence of universal integrable deformation of meromorphic connection, as well as their generalization to degenerate cases due to C. Sabbah, are presented and summarized.

Section 3 is devoted to an analytical proof of Malgrange-Sabbah Theorem. After introducing the notion of admissible data, we formulate a Riemann-Hilbert-Birkhoff boundary value problem $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$, depending on parameters $\boldsymbol{u} \in \mathbb{C}^n$. We factorize its solutions via two auxiliary RHB problems, and we analyze its solvability with respect to $\boldsymbol{u}$.

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[4] We warn the reader that in the exposition of [Man99], the isomonodromic system (1.1) is replaced by a Fuchsian one obtained by applying a (formal) Laplace transform, see [Man99, Ch. II.§1-3].

In Section 4 basic notions in the theory of both formal and analytic Frobenius manifolds are given. We explain how to pass from the analytic to the formal category, and vice-versa under convergence assumption of the WDVV potential.

In Section 5, we review necessary results on the extended deformed connection on both formal and analytic Frobenius manifolds, properties of solutions of the Darboux-Egoroff system of partial differential equations, and the reconstruction procedure of the Frobenius potential. Consequently, we prove the main new result, Theorem 5.1.

In the last Section 6, reformulations and applications of Theorem 5.1 to cohomological field theories and quantum cohomology are given.

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2. Degenerations of Riemann-Hilbert-Birkhoff inverse problems

2.1. **Riemann-Hilbert-Birkhoff inverse problems.** Let D be a disc centered at $z=\infty$ in $\mathbb{P}^1$. Given a (trivial) vector bundle on D equipped with a meromorphic connection with a pole at $z=\infty$, the Riemann-Hilbert-Birkhoff (RHB) problem is the following:

Problem 2.1. Does it exist a *trivial* vector bundle E^o on $\mathbb{P}^1$ equipped with a meromorphic connection ∇^o, restricting to the given data on D, and with a further *logarithmic* pole only at $z=0$?

Assume that the pole at $z=\infty$ is of order 2: in a basis of sections on D, the meromorphic connection has connection matrix $\Omega=-A(z)dz$, where $A(z)$ is a $n\times n$ matrix of the form

$$A(z)=\sum_{k=0}^{\infty}A_kz^{-k},\quad A_0\neq 0.$$

Denote by $\mathcal{O}\{\frac{1}{z}\}$ the ring of convergent power series in $\frac{1}{z}$. The RHB problem 2.1 is then equivalent to find a so-called *Birkhoff normal form*: does it exist a matrix $G\in GL_n(\mathcal{O}\{\frac{1}{z}\})$ such that $B(z)=G^{-1}AG-zG^{-1}\frac{d}{dz}G$ is of the form

$$B(z)=B_0+\frac{B_1}{z},\quad B_0,B_1\in M_n(\mathbb{C})?$$

2.2. **Universal integrable deformations of Birkhoff normal forms: Malgrange Theorems.** In this paper we consider *families* of RHB problems parametrized by a parameter space X, see [Mal83a, Mal83b, Mal86][Sab07, Ch.VI].

Definition 2.2. Let (E^o,∇^o) be a trivial vector bundle on a disc D equipped with a meromorphic connection with a pole of order 2 at $z=\infty$. An *integrable deformation* of (E^o,∇^o) parametrized by X is the datum (E,∇) of

- a trivial vector bundle E on $D \times X$,
- a *flat* connection ∇ on E with poles of order 2 along $\{\infty\} \times X$,

such that (E, ∇) restricts to (E^o, ∇^o) at a point $x_o \in X$. The integrable deformation is called *versal* if any other deformation with base space X' is induced by the previous one via pull-back by a holomorphic map $\varphi\colon (X', x'_o) \to (X, x_o)$. It is *universal* if the germ at x'_o of the base-change φ is uniquely determined.

Let (E^o, ∇^o) be a solution of a RHB problem 2.1, i.e. a trivial vector bundle on $\mathbb{P}^1$ with meromorphic connection with matrix (in a suitable basis of sections) of the form

$$\Omega = -\left(\Lambda_o + \frac{B_o}{z}\right) dz. \tag{2.1}$$

Recall that a matrix $A \in M_n(\mathbb{C})$ is said to be *regular* if any (and hence all) of the following equivalent conditions is satisfied:

(1) the characteristic polynomial of A equals its minimal polynomial,
(2) the commutator of A in $M_n(\mathbb{C})$ is of minimal dimension (i.e. it equals n),
(3) the commutator of A in $M_n(\mathbb{C})$ is $\mathbb{C}[M]$.

Theorem 2.3 ([Mal83a, Mal86])**.** *Assume that the matrix Λ_o is regular. The connection ∇^o matrix with connection* (2.1) *has a germ of universal deformation.*

This results can be refined to a global statement, under the further *semisimplicity* assumption on Λ_o. Let us then assume that $\Lambda_o = \mathrm{diag}(u_o^1, \dots, u_o^n)$ with $u_o^i \neq u_o^j$ for $i \neq j$.

Let Δ be the *big diagonal* in $\mathbb{C}^n$ defined by the equations

$$\Delta := \bigcup_{i<j} \{\boldsymbol{u} \in \mathbb{C}^n \colon u^i = u^j\},$$

let X_n be the complement $\mathbb{C}^n \backslash \Delta$, with base point $\boldsymbol{u}_o := (u_o^1, \dots, u_o^n)$. Denote by $\pi\colon (\widetilde{X}_n, \tilde{\boldsymbol{u}}_o) \to (X_n, \boldsymbol{u}_o)$ the universal cover of X_n, equipped with fixed base points $\tilde{\boldsymbol{u}}_o$ and $\boldsymbol{u}_o$, respectively. The space X_n is identified with the space of semisimple regular $n \times n$ matrices.

Theorem 2.4 ([JMU81, Mal83b])**.** *There exists on $\mathbb{P}^1 \times \widetilde{X}_n$ a vector bundle (E, ∇) such that*

(1) *the meromorphic connection ∇ is flat with a pole of order 2 along $\{\infty\} \times \widetilde{X}_n$ and a logarithmic pole along $\{0\} \times \widetilde{X}_n$,*
(2) *it restricts to (E^o, ∇^o) at $\tilde{\boldsymbol{u}}_o$,*
(3) *for any $\tilde{\boldsymbol{u}} \in \widetilde{X}_n$, the eigenvalues of the residue of ∇ at the point $(\infty, \tilde{\boldsymbol{u}})$ equal (up to permutation) the n-tuple $\pi(\tilde{\boldsymbol{u}})$.*

Let $\Theta \subseteq \widetilde{X}_n$ be the hypersurface of points $\tilde{\boldsymbol{u}} \in \widetilde{X}_n$ such that $E|_{\mathbb{P}^1 \times \{\tilde{\boldsymbol{u}}\}}$ is not trivial. The coefficients of ∇ have poles along Θ. Moreover, for any $\tilde{\boldsymbol{u}} \in \widetilde{X}_n \setminus \Theta$, the bundle with meromorphic connection (E, ∇) induces a universal deformation of its restriction $(E, \nabla)|_{\mathbb{P}^1 \times \{\tilde{\boldsymbol{u}}\}}$.

It is possible to explicitly describe the connection matrix of the universal deformation of Theorem 2.4.

For $\boldsymbol{u} \in \mathbb{C}^n$, denote $\Lambda(\boldsymbol{u}) := \mathrm{diag}(u^1, \dots, u^n)$, so that $\Lambda(\boldsymbol{u}_o) = \Lambda_o$. Given a matrix A denote by A' its diagonal part, and by A'' its off-diagonal part.

For $\boldsymbol{u}_o \notin \Delta$, there exists an off-diagonal matrix $F''(\boldsymbol{u})$, holomorphic near $\boldsymbol{u}_o$, such that the flat connection ∇ of Theorem 2.4 has connection matrix

$$- d\left(z\Lambda(\boldsymbol{u})\right) - \left([\Lambda(u), F''(\boldsymbol{u})] + B'_o\right)\frac{dz}{z} - [d\Lambda(\boldsymbol{u}), F''(\boldsymbol{u})], \tag{2.2}$$

e.g. see [Sab07, VI.§3.f, eq. (3.12)]. Notice that the dz-component of (2.2) restricts to (2.1) at $\boldsymbol{u} = \boldsymbol{u}_o$. Moreover, there exists a z^{-1}-formal base change which transforms (2.2) into

$$- d\left(z\Lambda(\boldsymbol{u})\right) - B'_o\frac{dz}{z}. \tag{2.3}$$

2.3. **Integrable deformations of degenerate Birkhoff normal forms: Sabbah Theorem.** In the notations of the previous section, assume $\boldsymbol{u}_o \in \Delta$. Define the partition $\{1, \dots, n\} = \dot{\cup}_{r\in R} I_r$ such that for any $r \in R$ we have

$$\{i, j\} \subseteq I_r \quad \text{if and only if} \quad u^i_o = u^j_o.$$

In [Sab18], C. Sabbah addressed the following problem.

Question: Is it possible to find an integrable deformation of the form (2.2) of the Birkhoff normal form (2.1) with z^{-1}-formal normal form (2.3)?

Remarkably, in [Sab18, Section 4] it is shown that the answer is positive, under (sharp) sufficient conditions on the coefficient B_o of the normal form (2.1).

Theorem 2.5 ([Sab18, Th. 4.9])**.** *Let $\boldsymbol{u}_o \in \Delta$, and $\mathcal{V}$ a neighborhood of $\boldsymbol{u}_o$ in $\mathbb{C}^n$. Assume that*

(1) $B''_o \in \mathrm{Im}\ \mathrm{ad}(\Lambda(\boldsymbol{u}^o))$,
(2) B'_o *is* partially non-resonant, *i.e.*

$$\forall\, r \in R, \quad \forall\, i, j \in I_r, \quad (B'_o)_{ii} - (B'_o)_{jj} \notin \mathbb{Z} \setminus \{0\}.$$

If $\mathcal{V}$ is sufficiently small, there exists a holomorphic hypersurface Θ in $\mathcal{V} \setminus \{\boldsymbol{u}_o\}$ and a holomorphic off-diagonal matrix $F''(\boldsymbol{u})$ on $\mathcal{V} \setminus \Theta$, such that the meromorphic connection, on the trivial vector bundle on $\mathbb{P}^1 \times (\mathcal{V} \setminus \Theta)$, with matrix (2.2) is integrable, restricts to (2.1) at $\boldsymbol{u}_o$, and it is formally equivalent at $z = \infty$ to the matrix connection (2.3).

3. An analytical proof of Sabbah Theorem

In this section we provide an analytical proof of Sabbah Theorem 2.5, based on properties of holomorphic Fredholm-operator-valued functions. General references for this section are [AB94, Bot20, CG81, CG18, CDG19, DZ02a, DZ02b, FIKN06, Its03, Its11, MP80, TO16, Vek67, Zho89].

3.1. **Admissible data and Riemann-Hilbert-Birkhoff boundary value problem.** Denote by $\mathrm{Arg}(z) \in]-\pi, \pi]$ the principal branch of the argument of the complex number z. Let $\boldsymbol{u} \in \mathbb{C}^n$, and set

$$\mathscr{S}(\boldsymbol{u}) := \left\{ \mathrm{Arg}\left(-\sqrt{-1}(\overline{u^i} - \overline{u^j}\right) + 2\pi k \colon k \in \mathbb{Z},\ i, j \text{ are s.t. } u^i \neq u^j \right\}.$$

Any element $\tau \in \mathbb{R} \setminus \mathscr{S}(\boldsymbol{u})$ will be said to be *admissible at $\boldsymbol{u}$*.

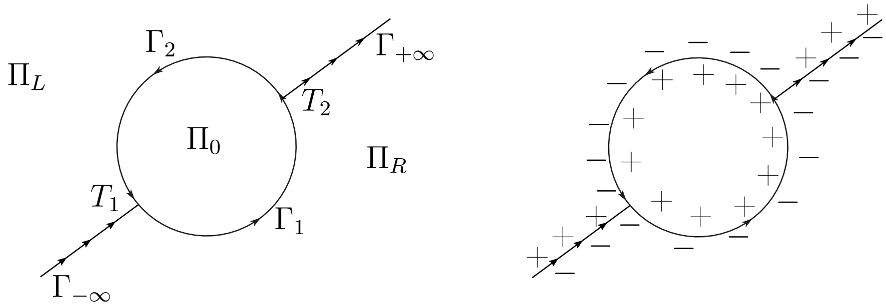

FIGURE 1. Contour Γ, paths $\Gamma_{\pm\infty}, \Gamma_1, \Gamma_2$, domains Π_0, Π_L, Π_R, and $\pm$ sides of Γ.

Definition 3.1. Let $\boldsymbol{u} \in \mathbb{C}^n$ and τ admissible at $\boldsymbol{u}$. A $(\boldsymbol{u}, \tau)$-*admissible datum* is a 6-tuple $\mathfrak{M} := (B, D, L, S_1, S_2, C)$ of matrices in $M_n(\mathbb{C})$ such that:

(1) the matrix B is diagonal, i.e. $B = B'$,
(2) D is a diagonal matrix of integers,
(3) we have
$$\operatorname{tr} B = \operatorname{tr} D + \operatorname{tr} L. \tag{3.1}$$
(4) the matrices S_1, S_2, C are invertible, with $\det S_1 = \det S_2 = 1$,
(5) $(S_1)_{ii} = (S_2)_{ii} = 1$,
(6) if $i \neq j$, then $(S_1^{-1})_{ij} = 0$ if $\operatorname{Re}\left(e^{\sqrt{-1}(\tau-\pi)}(u^i - u^j)\right) > 0$,
(7) if $i \neq j$, then $(S_2)_{ij} = 0$ if $\operatorname{Re}\left(e^{\sqrt{-1}\tau}(u^i - u^j)\right) > 0$,
(8) we have
$$S_1^{-1} e^{2\pi\sqrt{-1}B} S_2^{-1} = C^{-1} e^{2\pi\sqrt{-1}L} C. \tag{3.2}$$

If $\boldsymbol{u} \in \Delta$, define the partition $\{1, \dots, n\} = \dot\cup_{r \in R} I_r$ such that for any $r \in R$ we have $\{i, j\} \subseteq I_r$ if and only if $u^i = u^j$. We then require the further vanishing condition

(9) $(S_1^{-1})_{ij} = (S_2)_{ij} = 0$ if $i, j \in I_r$ for some $r \in R$.

Lemma 3.2. *Let $\boldsymbol{u}_o \in \mathbb{C}^n$ and τ admissible at $\boldsymbol{u}_o$. If $\mathfrak{M}$ is $(\boldsymbol{u}_o, \tau)$-admissible, then there exists a sufficiently small neighborhood $\mathcal{V}$ of $\boldsymbol{u}_o$ such*

(1) *τ is admissible at $\boldsymbol{u}$, for all $\boldsymbol{u} \in \mathcal{V}$,*
(2) *$\mathfrak{M}$ is $(\boldsymbol{u}, \tau)$-admissible for all $\boldsymbol{u} \in \mathcal{V}$.* □

Let $\boldsymbol{u} \in \mathbb{C}^n$ and τ admissible at $\boldsymbol{u}$. Consider the complex z-plane with a branch cut from 0 to ∞:
$$\tau - \pi < \arg z < \tau + \pi.$$
Let $r > 0$ and denote by $\Gamma = \Gamma(\tau, r)$ the union of the following oriented paths, see Figure 1:

(1) the half-line $\Gamma_{-\infty}$ defined by $\arg z = \tau \pm \pi$, $|z| > r$, originating from ∞;
(2) the half-line $\Gamma_{+\infty}$ defined by $\arg z = \tau$, $|z| > r$, ending to ∞;
(3) the half-circle Γ_1 defined by $\tau - \pi < \arg z < \tau$, $|z| = r$, counterclockwise oriented;
(4) the half-circle Γ_2 defined by $\tau < \arg z < \tau + \pi$, $|z| = r$, counterclockwise oriented.

The orientations uniquely define the + and - side for each path $\Gamma_{\pm\infty}, \Gamma_1, \Gamma_2$. For $z \in \Gamma_{-\infty}$ we use the symbol $z_\pm$ if $\arg z = \tau \pm \pi$. Set Π_0, Π_L, Π_R to be the components of complement $\mathbb{C} \setminus \Gamma$, and T_1, T_2 to be the two nodes of Γ, as in Figure 1.

Let $\mathfrak{M} := (B, D, L, S_1, S_2, C)$ be a $(\boldsymbol{u}, \tau)$-admissible datum. Define two functions

$$Q(-;\boldsymbol{u}), H(-;\boldsymbol{u}) \colon \Gamma \to GL(n, \mathbb{C}),$$

by

$$Q(z;\boldsymbol{u}) := \Lambda(\boldsymbol{u})z + B\log z, \quad \Lambda(\boldsymbol{u}) := \operatorname{diag}(u^1, \dots, u^n),$$

$$H(z;\boldsymbol{u}) := \begin{cases} e^{Q(z_-;\boldsymbol{u})}S_1^{-1}e^{-Q(z_-;\boldsymbol{u})}, & \text{along } \Gamma_{-\infty},\\ e^{Q(z;\boldsymbol{u})}S_2e^{-Q(z;\boldsymbol{u})}, & \text{along } \Gamma_{+\infty},\\ e^{Q(z;\boldsymbol{u})}C^{-1}z^{-L}z^{-D}, & \text{along } \Gamma_1,\\ e^{Q(z;\boldsymbol{u})}S_1^{-1}C^{-1}z^{-L}z^{-D}, & \text{along } \Gamma_2. \end{cases}$$

We denote by $H_{\pm\infty}, H_1, H_2$ the restrictions of H at $\Gamma_{\pm\infty}, \Gamma_1, \Gamma_2$.

Problem 3.3 (Problem $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$)**.** Find an analytic function $G \colon \mathbb{C}\setminus\Gamma \to M_n(\mathbb{C})$ such that

1. $G|_{\Pi_\nu}$ extends continuously to $\overline{\Pi_\nu}$ for $\nu = 0, L, R$;
2. the non-tangential limits $G_\pm \colon \Gamma \to M_n(\mathbb{C})$ of G from the - and + sides of Γ exist, and are continuous;
3. they are related by
$$G_+(z) = G_-(z)H(z;\boldsymbol{u});$$
4. $G(z)$ tends to the identity matrix I as $z \to \infty$.

Proposition 3.4. *The following smooth conditions at points T_1 and T_2 hold true:*

$$H_{-\infty}(z_-;\boldsymbol{u})H_2(z;\boldsymbol{u})H_1(z_-;\boldsymbol{u})^{-1} = I, \quad \text{at } T_1, \tag{3.3}$$

$$H_1(z;\boldsymbol{u})H_2(z;\boldsymbol{u})^{-1}H_{+\infty}(z;\boldsymbol{u})^{-1} = I, \quad \text{at } T_2. \tag{3.4}$$

Proof. A simple computation shows that (3.3) follows from (3.2). Equation (3.4) is easily checked. □

3.2. **Factorization of solutions.** We factorize solutions of the problem $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$ via two auxiliary RHB boundary value problems, $P_1[\boldsymbol{u}, \tau, \mathfrak{M}]$ and $\mathcal{P}_2[\boldsymbol{u}, \tau, \mathfrak{M}]$. Let us firstly describe the contours for both problems.

Let $P_1 \in \Gamma_{-\infty}$ preceding T_1, and $P_2 \in \Gamma_{+\infty}$ preceding T_2. Set

- $\ell_1 \subseteq \Gamma_{-\infty}$ to be the half-line contained from ∞ to P_1,
- $\ell_2 \subseteq \Gamma_{+\infty}$ to be the half-line contained from P_2 to ∞.

Define

- Γ' to be the union $\ell_1 \cup \ell_2$,
- Γ'' to be a circle of radius $R > \max\{|P_1|, |P_2|\}$.

See Figure 2.

Problem 3.5 (Problem $\mathcal{P}_1[\boldsymbol{u}, \tau, \mathfrak{M}]$)**.** Find an analytic function $\Psi \colon \mathbb{C} \setminus \Gamma' \to M_n(\mathbb{C})$ such that

1. the non-tangential limits $\Psi_\pm \colon \Gamma' \to M_n(\mathbb{C})$ of Ψ from the - and + sides of Γ' exist,
2. they are related by
$$\Psi_+(z) = \Psi_-(z)H(z;\boldsymbol{u}), \tag{3.5}$$
3. $\Psi(z)$ tends to the identity matrix I as $z \to \infty$.

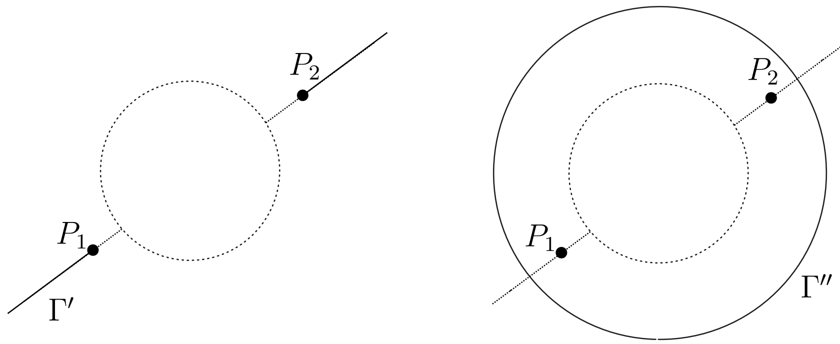

FIGURE 2. Contours Γ' and Γ''.

Lemma 3.6. *We have $H(\zeta;\boldsymbol{u})-I\to 0$ exponentially fast for $\zeta\to\infty$ along Γ'. In particular, $H(-;\boldsymbol{u})-I\in L^2(\Gamma';|d\zeta|)$.*

Proof. The (i,j)-entry of $H(\zeta;\boldsymbol{u})-I$ equals

$$c_{ij}\exp\{(u^i-u^j)\zeta+(B_{ii}-B_{jj})\log\zeta\}-\delta_{ij}, \tag{3.6}$$

where

$$c_{ij}=\begin{cases}(S_1^{-1})_{ij}, & \text{along } \ell_1,\\ (S_2)_{ij}, & \text{along } \ell_2.\end{cases}$$

By conditions (5), (6), (7) (and (9) if $\boldsymbol{u}\in\Delta$) of Definition 3.1, we deduce that (3.6) goes to zero exponentially fast for $\zeta\to\infty$ along ℓ_1 and ℓ_2. ☐

Theorem 3.7. *If $\min\{|P_1|,|P_2|\}$ is sufficiently big, then there exists a unique solution Ψ of the problem $\mathcal{P}_1[\boldsymbol{u},\tau,\mathfrak{M}]$, holomorphically depending on $\boldsymbol{u}$. Moreover, $\det\Psi\equiv 1$.*

Proof. If Ψ is a solution of $\mathcal{P}_1[\boldsymbol{u},\tau,\mathfrak{M}]$, then we have

$$\Psi(z;\boldsymbol{u})=I+\int_{\Gamma'}\frac{\Psi_-(\zeta)\,(H(\zeta;\boldsymbol{u})-I)}{\zeta-z}\frac{d\zeta}{2\pi\sqrt{-1}},$$

by the jump condition (3.5) and Cauchy Theorem, see e.g. [FIKN06, Ch. 3][Its11, §5.1.3] [TO16, Ch. 2]. Set $\delta\Psi:=\Psi-I$ and $\delta H:=H-I$. The previous equation can be written as

$$\delta\Psi(z)=\int_{\Gamma'}\frac{\delta\Psi_-(\zeta)\delta H(\zeta;\boldsymbol{u})}{\zeta-z}\frac{d\zeta}{2\pi\sqrt{-1}}+\int_{\Gamma'}\frac{\delta H(\zeta;\boldsymbol{u})}{\zeta-z}\frac{d\zeta}{2\pi\sqrt{-1}}. \tag{3.7}$$

Given a function f defined on Γ', introduce the functions $\mathcal{C}^{\pm}_{\Gamma'}[f]$ on Γ' defined by the Cauchy integrals

$$\mathcal{C}^{\pm}_{\Gamma'}[f](p):=\lim_{z\to p_\pm}\int_{\Gamma'}\frac{f(\zeta)}{\zeta-z}\frac{d\zeta}{2\pi\sqrt{-1}},\quad p\in\Gamma',$$

whenever the integral is finite. General results ensure that if $f\in L^p(\Gamma';|d\zeta|)$, with $1\leqslant p<\infty$, then $\mathcal{C}^{\pm}_{\Gamma'}[f]$ exists for $p\in\Gamma'$ a.e.. Moreover, the Cauchy operators $\mathcal{C}^{\pm}_{\Gamma'}$ are bounded in $L^p(\Gamma';|d\zeta|)$, with $1<p<\infty$, i.e. there exists a constant $k_p>0$ such that

$$\|\mathcal{C}^{\pm}_{\Gamma'}[f]\|_{L^p(\Gamma')}\leqslant k_p\|f\|_{L^p(\Gamma')},\quad \text{if } f\in L^p(\Gamma';|d\zeta|).$$

See [MP80, Zho89, DZ02a, DZ02b, TO16] for details and proofs. Taking the limit $z \to z_-$ in (3.7), we obtain the following integral equation for $\delta\Psi_-$:

$$\mathcal{C}[\delta H; \Gamma']\delta\Psi_- = \mathcal{C}^-_{\Gamma'}[\delta H], \tag{3.8}$$

where

$$\mathcal{C}[\delta H; \Gamma']f := f - \mathcal{C}^-_{\Gamma'}[f \cdot \delta H].$$

Notice that $\mathcal{C}^-_{\Gamma'}[\delta H] \in L^2(\Gamma'; |d\zeta|)$ by Lemma 3.6. Moreover, if $f \in L^2(\Gamma'; |d\zeta|)$, we have

$$\|\mathcal{C}^-_{\Gamma'}[f \cdot \delta H]\|_{L^2(\Gamma')} \leqslant k_2 \|f \cdot \delta H\|_{L^2(\Gamma')} \leqslant k_2 \cdot \sup_{\zeta \in \Gamma'} \|\delta H(\zeta; \boldsymbol{u})\| \cdot \|f\|_{L^2(\Gamma')}.$$

By Lemma 3.6, we can assume that $\min\{|P_1|, |P_2|\}$ is so big that

$$\sup_{\zeta \in \Gamma'} \|\delta H(\zeta; \boldsymbol{u})\| < \frac{1}{1 + k_2},$$

then the operator $\mathcal{C}[\delta H; \Gamma'] \colon L^2(\Gamma'; |d\zeta|) \to L^2(\Gamma'; |d\zeta|)$ is invertible with inverse

$$\mathcal{C}[\delta H; \Gamma']^{-1} = \sum_{m=0}^{\infty} \mathcal{C}^-_{\Gamma'}[(-) \cdot \delta H]^m.$$

Equation (3.8) can be uniquely solved in $\delta\Psi_-$, and the formula (3.7) gives the unique solution Ψ of the RHB boundary value problem. Notice that the Cauchy operator $\mathcal{C}^-_{\Gamma'}[(-) \cdot \delta H]$ depends holomorphically on $\boldsymbol{u}$, so that $\Psi(z; \boldsymbol{u})$ is holomorphic in $\boldsymbol{u}$. Finally, notice that the jump condition (3.5) implies

$$\det \Psi_+ = \det \Psi_- \det H = \det \Psi_-,$$

since $\det H(\zeta; \boldsymbol{u}) \equiv 1$ along Γ'. Hence, $\det \Psi$ is an entire function, and from the asymptotic condition $\Psi \to I$ for $z \to \infty$, we deduce $\det \Psi \equiv 1$ by Liouville Theorem. □

Define the function $\mathfrak{S}(-; \boldsymbol{u}) \colon \mathbb{C} \setminus \Gamma \to GL(n, \mathbb{C})$ by

$$\mathfrak{S}(z; \boldsymbol{u}) := \begin{cases} I, & \text{for } z \in \Pi_0, \\ H_1(z; \boldsymbol{u})^{-1}, & \text{for } z \in \Pi_1, \\ H_1(z; \boldsymbol{u})^{-1} H_{+\infty}(z; \boldsymbol{u}), & \text{for } z \in \Pi_2. \end{cases}$$

Lemma 3.8. *The function $\mathfrak{S}(-; \boldsymbol{u})$ is* naive *solution of $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$: it satisfies conditions (1),(2),(3), but not (4).*

Proof. This is easily checked, by invoking the cyclic relations (3.3)-(3.4). □

Consider the function $\widetilde{H}(-; \boldsymbol{u}) \colon \mathbb{C} \setminus \Gamma \to GL(n, \mathbb{C})$ defined by

$$\widetilde{H}(z; \boldsymbol{u}) := \Psi(z, \boldsymbol{u}) \mathfrak{S}(z; \boldsymbol{u})^{-1},$$

where Ψ is the unique piecewise analytic solution of $\mathcal{P}_1[\boldsymbol{u}, \tau, \mathfrak{M}]$, as in Theorem 3.7.

Lemma 3.9.

(1) *The function $\widetilde{H}(-; \boldsymbol{u})$ is continuous along Γ''.*

(2) *The function* $\det \widetilde{H}(-;\boldsymbol{u})$ *has zero index across* Γ'', *i.e.*
$$\mathrm{ind}_{\Gamma''} \det \widetilde{H}(-;\boldsymbol{u}) := \frac{1}{2\pi\sqrt{-1}} \oint_{\Gamma''} d \log \det \widetilde{H}(\zeta;\boldsymbol{u}) = 0.$$

Proof. Point (1) is obvious. For point (2) notice that, for $\zeta \in \Gamma''$, we have
$$\begin{aligned}\log \det \widetilde{H}(\zeta;\boldsymbol{u}) &= \log \det \mathfrak{S}(\zeta;\boldsymbol{u})^{-1}\\ &= -\operatorname{tr} Q(\zeta;\boldsymbol{u}) + \log \det C + (\operatorname{tr} L + \operatorname{tr} D) \log \zeta\\ &= -\zeta \sum_{i=1}^{n} u^i + \log \det C + \underbrace{(-\operatorname{tr} B + \operatorname{tr} L + \operatorname{tr} D)}_{0 \text{ by } (3.1)} \log \zeta.\end{aligned}$$
This completes the proof. □

We can now introduce a second auxiliary RHB boundary value problem, with continuous coefficients on the simple closed contour Γ''.

Problem 3.10 (Problem $\mathcal{P}_2[\boldsymbol{u}, \tau, \mathfrak{M}]$)**.** Find an analytic function $\Upsilon\colon \mathbb{C} \setminus \Gamma'' \to M_n(\mathbb{C})$ such that

(1) the non-tangential limits $\Upsilon_\pm\colon \Gamma'' \to M_n(\mathbb{C})$ of Ψ from the - and + sides of Γ'' exist,
(2) they are related by
$$\Upsilon_+(z) = \Upsilon_-(z)\widetilde{H}(z;\boldsymbol{u}), \tag{3.9}$$
(3) $\Upsilon(z)$ tends to the identity matrix I as $z \to \infty$.

Theorem 3.11. *The solvability of* $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$ *is equivalent to the solvability of* $\mathcal{P}_2[\boldsymbol{u}, \tau, \mathfrak{M}]$.

Proof. If G is the solution of $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$, then
$$\Upsilon(z;\boldsymbol{u}) := \begin{cases} G(z;\boldsymbol{u})\Psi(z;\boldsymbol{u})^{-1}, & \text{for } z \text{ outside } \Gamma'',\\ G(z;\boldsymbol{u})\mathfrak{S}(z;\boldsymbol{u})^{-1}, & \text{for } z \text{ inside } \Gamma'',\end{cases}$$
is the solution of $\mathcal{P}_2[\boldsymbol{u}, \tau, \mathfrak{M}]$. Vice-versa, if Υ is the solution of $\mathcal{P}_2[\boldsymbol{u}, \tau, \mathfrak{M}]$, then the solution G of $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$ is obtained by inverting the equations above. □

3.3. **Solvability as an open property.** If Υ is a solution of $\mathcal{P}_2[\boldsymbol{u}, \tau, \mathfrak{M}]$, then we have
$$\Upsilon(z) = I + \int_{\Gamma''} \frac{\Upsilon_-(\zeta)(\widetilde{H}(\zeta;\boldsymbol{u}) - I)}{\zeta - z} \frac{d\zeta}{2\pi\sqrt{-1}}. \tag{3.10}$$
In the limit $z \mapsto z_-$, we obtain the integral equation
$$\Upsilon_- = I + \mathcal{C}^-_{\Gamma''}\left[\Upsilon_- \delta\widetilde{H}\right], \quad \delta\widetilde{H} := \widetilde{H} - I, \tag{3.11}$$
where $\mathcal{C}^\pm_{\Gamma''}$ denotes the Cauchy integrals w.r.t. the contour Γ''. Conversely, if Υ_- is a solution of (3.11), then (3.10) gives the solution of $\mathcal{P}_2[\boldsymbol{u}, \tau, \mathfrak{M}]$, see [FIKN06, Ch. 3][Its11, §5.1.3][TO16, Ch. 2].

Theorem 3.12. *The operator*
$$T(\boldsymbol{u})\colon L^2(\Gamma''; |d\zeta|) \to L^2(\Gamma''; |d\zeta|), \quad f \mapsto f - \mathcal{C}^-_{\Gamma''}[f \cdot \delta\widetilde{H}(\boldsymbol{u})]$$
is a Fredholm operator with index 0.

Proof. Assume we are given a factorization $\widetilde{H}(\zeta;\boldsymbol{u}) = (I - W^-(\zeta;\boldsymbol{u}))^{-1}(I + W^+(\zeta;\boldsymbol{u}))$ with $(I \pm W^\pm(\zeta;\boldsymbol{u}))^{\pm 1} - I \in L^\infty(\Gamma'') \cap L^2(\Gamma'')$. Define the Cauchy type operator

$$\mathcal{C}_W\colon L^2(\Gamma'') \to L^2(\Gamma''), \quad f \mapsto \mathcal{C}^+_{\Gamma''}[fW^-] + \mathcal{C}^-_{\Gamma''}[fW^+].$$

Standard results imply that the operator $f \mapsto f - \mathcal{C}_W[f]$ is Fredholm, and its index is given by

$$\operatorname{ind}(Id - \mathcal{C}_W) = n \operatorname{ind}_{\Gamma''} \det \widetilde{H} = 0,$$

by Lemma 3.9 point (2), see [Zho89, TO16]. In our case, we can take $W^- = 0$ and $W^+ = \delta\widetilde{H}$ by Lemma 3.9 point (1). This completes the proof. □

Theorem 3.13. *Let $\boldsymbol{u}_o \in \mathbb{C}^n$. Assume that the pair $(\tau, \mathfrak{M})$ is admissible at each point of a sufficiently small open neighborhood $\mathcal{V}$ of $\boldsymbol{u}_o$. If $\mathcal{P}[\boldsymbol{u}_o, \tau, \mathfrak{M}]$ is solvable, there exists an analytic set $\Theta \subseteq \mathcal{V} \setminus \{\boldsymbol{u}_o\}$ such that $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$ is solvable for all $\boldsymbol{u} \in \mathcal{V} \setminus \Theta$. Moreover, the solution $G(z;\boldsymbol{u})$ is holomorphic w.r.t. $\boldsymbol{u} \in \mathcal{V} \setminus \Theta$.*

For the proof we firstly invoke the following Lemma.

Lemma 3.14 ([Gra70, Lemma 10])**.** *Let X be a Banach space and $\mathfrak{F}(X)$ be the set of its Fredholm operators. Let $\Omega \subseteq \mathbb{C}^n$ be a connected domain, and $T\colon \Omega \to \mathfrak{F}(X)$ a holomorphic function. If $T(\lambda_o)^{-1}$ exists for some $\lambda_o \in \Omega$, then $T(\lambda)^{-1}$ exists on the complement $\Omega \setminus \Theta$ of an analytic set (zero locus of a scalar analytic function), and T^{-1} is meromorphic on Ω.* □

Remark 3.15. Lemma 3.14 was originally due to I. Gohberg and E. Sigal in the case $n = 1$, [GS70]. The general case was proved by B. Gramsch, though special cases were previously obtained by several authors. For a sketch of a proof, based on arguments of [GS70] and [GGK90, XI.8], see [Kab12, Sec. 2].

Proof of Theorem 3.13. By assumption, the $\mathcal{P}_2[\boldsymbol{u}_o, \tau, \mathfrak{M}]$ is solvable. We claim that the solution Υ is unique. The function $\det \Upsilon(z;\boldsymbol{u}_o)$ solves the scalar RH problem

$$\det \Upsilon_+(z;\boldsymbol{u}_o) = \det \Upsilon_-(z;\boldsymbol{u}_o) \det \widetilde{H}(z;\boldsymbol{u}_o).$$

Since the function $\det \widetilde{H}(-;\boldsymbol{u}_o)$ has zero index along Γ'', this scalar equation can be uniquely solved: the solution is given by

$$\det \Upsilon(z;\boldsymbol{u}_o) = \exp \int_{\Gamma''} \frac{\log \det \widetilde{H}(\mu;\boldsymbol{u}_o)}{\mu - \lambda} \frac{d\mu}{2\pi\sqrt{-1}},$$

see e.g. [TO16, §2.3.1]. In particular, $\Upsilon(z;\boldsymbol{u}_o)$ is invertible. Assume that $\Upsilon(z;\boldsymbol{u}_o), \tilde{\Upsilon}(z;\boldsymbol{u}_o)$ are two solutions of $\mathcal{P}_2[\boldsymbol{u}_o, \tau, \mathfrak{M}]$. Put $X(z) := \Upsilon(z;\boldsymbol{u}_o)\tilde{\Upsilon}(z;\boldsymbol{u}_o)^{-1}$. For $z \in \Gamma''$ we have

$$X_+(z) = \Upsilon_+(z;\boldsymbol{u}_o)\tilde{\Upsilon}_+(z;\boldsymbol{u}_o) = \Upsilon_-(z;\boldsymbol{u}_o)\widetilde{H}(\mu;\boldsymbol{u}_o)\widetilde{H}(\mu;\boldsymbol{u}_o)^{-1}\tilde{\Upsilon}_-(z;\boldsymbol{u}_o) = X_-(z).$$

Hence $X(z)$ is analytic, and moreover $X(z) \to I$ for $z \to \infty$. By Liouville Theorem we have $X(z) \equiv I$, and $\Upsilon = \tilde{\Upsilon}$. It follows that the Fredholm operator $T(\boldsymbol{u}_o)$ has both trivial kernel and index zero. Hence $T(\boldsymbol{u}_o)^{-1}$ exists, Lemma 3.14 applies, and the problem $\mathcal{P}_2[\boldsymbol{u}, \tau, \mathfrak{M}]$ is solvable on the complement of an analytic set $\Theta \subseteq \mathcal{V} \setminus \{\boldsymbol{u}_o\}$. By Theorem 3.11 one concludes. □

3.4. **Proof of Sabbah Theorem.** Let (E_o, ∇_o) to be in Birkhoff normal form (2.1) with $\Lambda_o = \mathrm{diag}(u_o^1, \dots, u_o^n)$ and $\boldsymbol{u}_o \in \Delta$. Consider the differential system defining ∇_o-flat sections

$$\frac{dY}{dz} = \left(\Lambda_o + \frac{1}{z}B_o\right)Y, \tag{3.12}$$

where Y is a matrix-valued function.

Proposition 3.16 ([AB94][CG18][CDG19, Section 16])**.** *The differential system* (3.12) *has a fundamental system of solutions in* Birkhoff-Levelt normal form

$$Y_0(z) = \mathcal{G}_0(z) z^{\mathcal{D}} z^{S+R}, \quad \mathcal{G}_0(z) = K\left(I + \sum_{j=1}^{\infty} A_j z^j\right),$$

where

- *K puts B_o in Jordan form $J = K^{-1}B_oK$,*
- *$\mathcal{D}$ is a diagonal matrix of integers (called* valuations*),*
- *S is a Jordan matrix whose eigenvalues have real part in $[0, 1[$,*
- *R is a nilpotent matrix, with non-vanishing entries only if some of the eigenvalues the matrix B_o differ by a non-zero integer.*

Moreover, we have

$$J = \mathcal{D} + S.$$

□

Proposition 3.17 ([CDG19, Prop. 4.2])**.** *Assume that*

(1) *$B_o'' \in \mathrm{Im}\ \mathrm{ad}(\Lambda(\boldsymbol{u}_o))$,*
(2) *B_o' is partially non-resonant.*

Then, the differential system (3.12) *has a unique formal solution of the form*

$$Y_F(z) = \left(I + \sum_{k=1}^{\infty} F_k z^{-k}\right) z^{B_o'} e^{\Lambda_o z}.$$

If τ is admissible at $\boldsymbol{u}_o$, then there exists three unique fundamental systems of solutions Y_1, Y_2, Y_3 of (3.12) *such that*

$$Y_h(z) \sim Y_F(z), \quad |z| \to +\infty, \quad \tau - (3-h)\pi < \arg z < \tau + (h-2)\pi, \quad h = 1, 2, 3. \tag{3.13}$$

□

Remark 3.18. For $h = 1, 2, 3$, set

$$\mathcal{G}_h(z) := Y_h(z) e^{-\Lambda_o z} z^{-B_o'},$$

$$\mathcal{V}_{\tau,h} := \left\{ z \in \widehat{\mathbb{C}^*} \colon \tau - (3-h)\pi < \arg z < \tau + (h-2)\pi \right\}.$$

The precise meaning of the asymptotic relation (3.13) is the following:

$$\forall h \in \{1,2,3\},\ \forall \ell \in \mathbb{N},\ \forall \overline{\mathcal{V}} \subsetneq \mathcal{V}_{\tau,h},\ \exists C_{h,\ell,\overline{\mathcal{V}}} > 0\colon\ \text{if } z \in \overline{\mathcal{V}} \setminus \{0\} \text{ then}$$

$$\left\| \mathcal{G}_h(z) - \left(I + \sum_{m=1}^{\ell-1} \frac{F_m}{z^m}\right) \right\| < \frac{C_{h,\ell,\overline{\mathcal{V}}}}{|z|^\ell}.$$

Here $\overline{\mathcal{V}}$ denotes any unbounded closed sector of $\widehat{\mathbb{C}^*}$ with vertex at 0.

In the notations of Propositions 3.16, 3.17, consider the 6-tuples $\mathfrak{M} = (B, D, L, S_1, S_2, C)$ where

$$B := B_o', \quad D := \mathcal{D}, \quad L := S + R,$$

and the matrices S_1, S_2, C are defined by

$$Y_2(z) = Y_1(z)S_1, \quad Y_3(z) = Y_2(z)S_2, \quad Y_2(z) = Y_0(z)C.$$

Proposition 3.19. *The 6-tuple $\mathfrak{M}$ is a $(\boldsymbol{u}_o, \tau)$-admissible datum. The RHB boundary value problem $\mathcal{P}[\boldsymbol{u}_o, \tau, \mathfrak{M}]$ is solvable, with unique solution*

$$G(z; \boldsymbol{u}_o) = \begin{cases} \mathcal{G}_0(z), & z \in \Pi_0, \\ \mathcal{G}_2(z), & z \in \Pi_1, \\ \mathcal{G}_3(z), & z \in \Pi_2. \end{cases}$$

Proof. Conditions (1),(2),(3) of Definition 3.1 are trivially satisfied. The proof of conditions (4),(5),(6),(7) for the Stokes matrices S_1, S_2 are standard, see e.g. [CDG19, Section 6.3]. Notice that

$$Y_3(ze^{2\sqrt{-1}\pi}) = Y_1(z)e^{2\sqrt{-1}\pi B_o'}, \quad z \in \widehat{\mathbb{C}^*},$$

both sides having the same asymptotic expansion $Y_F(ze^{2\sqrt{-1}\pi})$ for $|z| \to +\infty$, and $\tau - 2\pi < \arg z < \tau - \pi$. We deduce that

$$Y_0(ze^{2\sqrt{-1}\pi})CS_2 = Y_0(z)CS_1^{-1}e^{2\sqrt{-1}\pi B_o'},$$

so that

$$e^{2\sqrt{-1}\pi L} = CS_1^{-1}e^{2\sqrt{-1}\pi B_o'}S_2^{-1}C^{-1}.$$

This proves condition (8) of Definition 3.1. Finally, condition (9) follows from [CG18, Th. 2.1], [CDG19, Prop. 6.1]. □

By Theorem 3.13, there exist an open neighborhood $\mathcal{V}$ of $\boldsymbol{u}_o$, an analytic set $\Theta \subsetneq \mathcal{V} \setminus \{\boldsymbol{u}_o\}$ on which the RHB problem $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$ is solvable, with unique solution $G(z; \boldsymbol{u})$ holomorphic w.r.t. $\boldsymbol{u} \in \mathcal{V} \setminus \Theta$. Define the functions

$$\begin{aligned} Y_{L/R}(z; \boldsymbol{u}) &:= G(z; \boldsymbol{u})z^{B_o'}e^{\Lambda(\boldsymbol{u})z}, && z \in \Pi_{L/R}, \\ Y_0(z; \boldsymbol{u}) &:= G(z; \boldsymbol{u})z^{\mathcal{D}}z^L, && z \in \Pi_0. \end{aligned}$$

We have $G(z; \boldsymbol{u}) = I + \frac{F_1(\boldsymbol{u})}{z} + O\left(\frac{1}{z^2}\right)$ in $z \to \infty$ in $\Pi_{L/R}$, so that

$$\begin{aligned} \frac{\partial Y_{L/R}}{\partial z} \cdot Y_{L/R}^{-1} &= \partial_z G \cdot G^{-1} + \frac{1}{z}GB_o'G^{-1} + G\Lambda G^{-1} \\ &= \Lambda(\boldsymbol{u}) + \frac{1}{z}\left([F_1(\boldsymbol{u}), \Lambda(\boldsymbol{u})] + B_o'\right) + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \\ \frac{\partial Y_0}{\partial z} \cdot Y_0^{-1} &= \partial_z G \cdot G^{-1} + \frac{1}{z}\left(GDG^{-1} + Gz^D L z^{-D} G^{-1}\right) \\ &= \frac{1}{z}K\left(D + L\right)K^{-1} + O(1), \quad z \to 0. \end{aligned}$$

The matrices S, C are constant w.r.t. both $\boldsymbol{u}$ and z: we deduce that the r.h.s. of the two equalities above are equal. This implies that $Y_{L/R}$ and Y_0 are solutions of the differential equation

$$\frac{\partial}{\partial z}Y=\left[\Lambda(\boldsymbol{u})+\frac{1}{z}V(\boldsymbol{u})\right]Y,\quad V(\boldsymbol{u}):=[F_1(\boldsymbol{u}),\Lambda(\boldsymbol{u})]+B_o'. \tag{3.14}$$

Similarly, we have

$$\begin{aligned}\frac{\partial Y_{L/R}}{\partial u^i}\cdot Y_{L/R}^{-1}&=\frac{\partial G}{\partial u^i}\cdot G^{-1}+zGE_iG^{-1}=zE_i+[F_1,E_i]+O\left(\frac{1}{z}\right),\\ \frac{\partial Y_0}{\partial u^i}\cdot Y_0^{-1}&=\frac{\partial G}{\partial u^i}\cdot G^{-1}=\frac{\partial G_0}{\partial u^i}\cdot G_0^{-1}+O(z),\end{aligned}$$

where $(E_i)_{ab}=\delta_{ai}\delta_{bi}$ and $G(z;\boldsymbol{u})=G_0(\boldsymbol{u})+O(z)$ for $z\to 0$ (and in particular $G_0(\boldsymbol{u}_o)=K$). The matrices S, C being constant, we deduce that the r.h.s. of the two equalities above are equal. Hence $Y_{L/R}$ and Y_0 are solutions of the differential equation

$$\frac{\partial}{\partial u^i}Y=(zE_i+V_i(\boldsymbol{u}))\,Y,\quad V_i(\boldsymbol{u}):=[F_1(\boldsymbol{u}),E_i]=\frac{\partial G_0}{\partial u^i}\cdot G_0^{-1},\quad i=1,\dots,n. \tag{3.15}$$

The datum of the compatible joint system of differential equations (3.14) and (3.15), for $\boldsymbol{u}\in\mathcal{V}\setminus\Theta$, proves the statement of Sabbah Theorem 2.5.

Remark 3.20. Note that in equations (3.14) and (3.15) we can replace F_1 with its off-diagonal part F_1'', since both $\Lambda(\boldsymbol{u})$ and E_i are diagonal.

Remark 3.21. Propositions 3.16 and 3.17 also hold true for $\boldsymbol{u}_o\in\mathbb{C}^n\setminus\Delta$, these are standard results. All the subsequent arguments can be applied, giving an analytical proof of Theorem 2.4.

4. Formal Frobenius and Dubrovin-Frobenius manifolds

We briefly review basic notions of the theory of Frobenius manifolds, in both formal and analytic frameworks. General references are [Dub96, Dub98, Dub99, Man99, Her02, Sab07].

4.1. **Formal Frobenius manifolds.** Let

- k be a commutative $\mathbb{Q}$-algebra,
- H be a free k-module of finite rank,
- $\eta\colon H\otimes H\to k$ be a symmetric pairing, inducing an isomorphism $\eta'\colon H\to H^T$, where H^T is the dual module,
- $K:=k[\![H^T]\!]$ be the completed symmetric algebra of H^T.

Fix a basis $(\Delta_1,\dots,\Delta_n)$ of H, and denote by $\boldsymbol{t}=(t^1,\dots,t^n)$ the dual coordinates. The algebra K is then identified with the algebra of formal power series $k[\![\boldsymbol{t}]\!]$. Denote by $\mathrm{Der}_k(K)$ the K-module of k-linear derivations of K. Put $\partial_\alpha=\frac{\partial}{\partial t^\alpha}\colon K\to K$. It is well known that $\mathrm{Der}_k(K)$ is a free K-module with basis $(\partial_1,\dots,\partial_n)$. We will write Φ_α for $\partial_\alpha\Phi$ for $\Phi\in K$.

Elements of $H_K:=K\otimes_k H$ will be identified with derivations on K, by $\Delta_\alpha\mapsto\partial_\alpha$.

For $\alpha,\beta=1,\dots,n$, set $\eta_{\alpha\beta}:=\eta(\Delta_\alpha,\Delta_\beta)$. The matrix $(\eta^{\alpha\beta})$ will denote the inverse of the Gram matrix $(\eta_{\alpha\beta})$ of η. Einstein summation rule will be used over repeated Greek indices.

Definition 4.1. A *formal Frobenius manifold structure* on (H, η) is given by a formal power series $\Phi \in K$, called *WDVV potential*, such that

$$\Phi_{\alpha\beta\gamma}\eta^{\gamma\delta}\Phi_{\delta\varepsilon\varphi} = \Phi_{\varphi\beta\gamma}\eta^{\gamma\delta}\Phi_{\delta\varepsilon\alpha}, \quad \alpha, \beta, \varepsilon, \varphi = 1, \dots, n. \tag{4.1}$$

Define the K-linear multiplication $\circ$ on H_K by

$$\Delta_\alpha \circ \Delta_\beta := c^\gamma_{\alpha\beta}\Delta_\gamma, \quad \alpha, \beta = 1, \dots, n,$$

where $c^\gamma_{\alpha\beta} := \Phi_{\alpha\beta\delta}\eta^{\delta\gamma}$. The WDVV equations (4.1) are equivalent to the associativity of $\circ$.

An element $e \in H_K$ is called *identity* if it is the identity for $\circ$. It is called *flat identity* if $e \in H$. An element $E \in H_K$ is called *Euler* if

$$\mathfrak{L}_E\eta = D\eta, \quad D \in k, \tag{4.2}$$

$$\mathfrak{L}_E c = c. \tag{4.3}$$

Here η is bilinearly extended to H_K. Here $\mathfrak{L}_E$ denotes the Lie derivative along E, and it is extended to the whole tensor algebra of H_K in the usual way (i.e. $\mathfrak{L}_E f = Ef$ for $f \in K$, and commutation with contractions). In this paper we always consider Frobenius manifolds equipped with a flat identity $e = \Delta_1$, and an Euler vector field.

4.2. **Dubrovin-Frobenius manifolds.** Given a complex analytic manifold M, we denote by TM, T^*M its holomorphic tangent and cotangent bundles.

A *Dubrovin-Frobenius manifold* structure on a complex manifold M of dimension n is defined by giving

(FM1) a symmetric $\mathcal{O}(M)$-bilinear metric tensor $\eta \in \Gamma\left(\bigodot^2 T^*M\right)$, whose corresponding Levi-Civita connection ∇ is flat;

(FM2) a $(1,2)$-tensor $c \in \Gamma\left(TM \otimes \bigodot^2 T^*M\right)$ such that
- (a) the induced multiplication of vector fields $X \circ Y := c(-, X, Y)$, for $X, Y \in \Gamma(TM)$, is *associative*,
- (b) $c^\flat \in \Gamma\left(\bigodot^3 T^*M\right)$,
- (c) $\nabla c^\flat \in \Gamma\left(\bigodot^4 T^*M\right)$;

(FM3) a vector field $e \in \Gamma(TM)$, called the *unity vector field*, such that
- (a) the bundle morphism $c(-, e, -)\colon TM \to TM$ is the identity morphism,
- (b) $\nabla e = 0$;

(FM4) a vector field $E \in \Gamma(TM)$, called the *Euler vector field*, such that
- (a) $\mathfrak{L}_E c = c$,
- (b) $\mathfrak{L}_E \eta = (2 - d) \cdot \eta$, where $d \in \mathbb{C}$ is called the *charge* of the Frobenius manifold.

Dubrovin-Frobenius manifolds will be also called *analytic* Frobenius manifolds.

By axiom (FM1), there exist system of *flat coordinates* $\boldsymbol{t} = (t^1, \dots, t^n)$, w.r.t. which the Levi-Civita connection ∇ coincides with partial derivatives $\partial_\alpha := \frac{\partial}{\partial t^\alpha}$, for $i = 1, \dots, n$. Without loss of generality, we assume that the coordinate t^1 is such that $\partial_1 = e$.

A *pointed Dubrovin-Frobenius manifold* is a pair (M, p), where M is a Dubrovin-Frobenius manifold, and $p \in M$ is a fixed *base point*. Given (M, p) we will always consider flat coordinates $\boldsymbol{t} = (t^1, \dots, t^n)$ vanishing at p.

4.3. **From Dubrovin-Frobenius to formal Frobenius structures, and vice-versa.** Given a pointed Dubrovin-Frobenius manifold (M,p), we can associate to it a formal Frobenius structure (H,η,Φ) over $k=\mathbb{C}$. Choose flat coordinates $\boldsymbol{t}$ vanishing at p, and set $H:=T_pM$ equipped with the metric $\eta|_p$. By axiom (FM2-c), the tensor $\partial_\alpha c_{\beta\gamma\delta}$ is completely symmetric: hence we deduce the local existence of a function F such that $\partial^3_{\alpha\beta\gamma}F=c_{\alpha\beta\gamma}$. By axioms (FM2-a), (FM2-b), we deduce that F is a solution of WDVV equations, i.e.

$$\partial^3_{\alpha\beta\gamma}F\ \eta^{\gamma\delta}\ \partial^3_{\delta\varepsilon\varphi}F=\partial^3_{\varphi\beta\gamma}F\ \eta^{\gamma\delta}\ \partial^3_{\delta\varepsilon\alpha}F,\quad \alpha,\beta,\varepsilon,\varphi=1,\dots,n.$$

Let $\mathcal{O}_{M,p}$ be the local ring of germs at p, and $\mathfrak{m}$ be its maximal ideal. The formal potential Φ is given by the image of F in the completion $\widehat{\mathcal{O}_{M,p}}:=\varprojlim\left(\mathcal{O}_{M,p}/\mathfrak{m}^\ell\right)$ of the local ring $\mathcal{O}_{M,p}$: this means that Φ is defined by the Taylor series expansion of F at p in coordinates $\boldsymbol{t}$. Moreover, the formal Frobenius structure (H,η,Φ) is also equipped with a flat unit $e|_p$ and an Euler vector field $E|_p$. We will say that the formal Frobenius structure constructed in this way, starting from a pointed Dubrovin-Frobenius manifold, is *convergent*.

Vice-versa, let us assume that (H,η,Φ) is a formal Frobenius structure over $k=\mathbb{C}$, with Euler vector field E. If the domain of convergence $\Omega\subseteq H$ of the power series $\Phi\in k[\![\boldsymbol{t}]\!]$ is non-empty, it is easily seen that Ω is equipped with a Dubrovin-Frobenius manifold structure.

4.4. **Semisimplicity of Frobenius structures.** In this Section we collect main results and properties which hold true for a wide class of Frobenius structures (both formal and analytic), namely *semisimple* Frobenius structures. We begin our exposition with the formal case.

Let (H,η,Φ) be a formal Frobenius manifold, and denote by $\circ_0$ the product on H with structure constants $\Phi^\gamma_{\alpha,\beta}(0)$. We say that (H,η,Φ) is

- *semisimple at the origin* if the k-algebra $(H,\circ_0)$ is isomorphic to k^n;
- *formally semisimple* if the K-algebra $(H_K,\circ)$ is isomorphic to K^n.

In the first (resp. second) case there exist an idempotent basis $(\pi_1,\dots,\pi_n)$ of H (resp. H_K) such that

$$\pi_i\circ\pi_j=\pi_i\delta_{ij},\quad \eta(\pi_i,\pi_j)=0,\quad i\neq j. \tag{4.4}$$

Notice that the idempotent vectors π_i are uniquely defined up to re-ordering.

Lemma 4.2. *A formal Frobenius manifold (H,η,Φ) is formally semisimple if and only if it is semisimple at the origin.*

Proof. Formal semisimplicity clearly implies semisimplicity at the origin. Let us prove the converse. Denote by $\mathfrak{m}:=(t^1,\dots,t^n)$ the maximal ideal of K. We will denote by $O(\mathfrak{m}^p)$ an arbitrary sum of elements of $\mathfrak{m}^p\cdot H_K$. For any fixed $h\in\mathbb{N}$ we call a *h-order idempotent basis* of H_K a basis $(\pi^h_1,\dots,\pi^h_n)$ such that

$$\pi^h_i\circ\pi^h_i=\pi^h_i+O(\mathfrak{m}^{h+1}),\qquad \pi^h_i\circ\pi^h_j=O(\mathfrak{m}^{h+1}),$$

for $i,j=1,\dots,n$ and $i\neq j$. Assume that (H,η,Φ) is semisimple at the origin. We claim there exist a h-order idempotent basis of H_K for any $h\in\mathbb{N}$. We prove it by induction on h. For $h=0$, it is trivial: if $(\pi^0_1,\dots,\pi^0_n)$ is an idempotent basis of $(H,\circ_0)$, then it is a 0-order idempotent basis of H_K. Assume that $(\pi^h_1,\dots,\pi^h_n)$ is a h-order idempotent basis of H_K: we have

$$\pi_i^h \circ \pi_i^h = \pi_i^h + \sum_k a_{ik}\pi_k^h, \quad a_{ij} \in \mathfrak{m}^{h+1}, \qquad \pi_i^h \circ \pi_j^h = \sum_k b_{ijk}\pi_k^h, \quad b_{ijk} \in \mathfrak{m}^{h+1},$$

for $i, j = 1, \dots, n$ and $i \neq j$. By commutativity and associativity, one deduces the following constraints on a_{ij}, b_{ijk}:

$$b_{ijk} = b_{jik}, \qquad i, j, k = 1, \dots, n, \tag{4.5}$$

$$b_{ijk} = 0, \qquad i, j, k = 1, \dots, n, \text{ distinct}, \tag{4.6}$$

$$b_{ijj} + a_{ij} = 0, \qquad i, j = 1, \dots, n, \quad i \neq j. \tag{4.7}$$

Set

$$\pi_i' := \pi_i^h + \sum_j w_{ij}\pi_j^h, \quad i = 1, \dots, n,$$

with arbitrary coefficients $w_{ij} \in \mathfrak{m}^{h+1}$. The n-tuple $(\pi_1', \dots, \pi_n')$ is a $(h+1)$-oder idempotent basis of H_K if and only if

$$w_{ii} = -a_{ii}, \qquad w_{ij} = a_{ij},$$

for $i, j = 1, \dots, n$ and $i \neq j$. This easily follows from (4.5)-(4.7). □

In the analytic case, we will say that a Dubrovin-Frobenius manifold M is *(generically) semisimple* if the set $M_{ss} := \{p \in M : (T_pM, \circ_p) \cong \mathbb{C}^n\}$ is non-empty. In such a case, it can be proved that M_{ss} is an open dense subset of M. At each point $p \in M_{ss}$ there exists tangent vectors $\pi_1|_p, \dots, \pi_n|_p$ satisfying the relations

$$\pi_i|_p \circ_p \pi_j|_p = \pi_i|_p\delta_{ij}, \quad \eta_p(\pi_i|_p, \pi_j|_p) = 0, \quad i \neq j.$$

It can be proved that, on sufficiently small open subsets M_{ss}, a coherent labeling of the idempotent tangent vectors can be chosen so that the resulting local vector fields are holomorphic.

Remark 4.3. In both the formal and analytic case we have $e = \sum_i \pi_i$.

Proposition 4.4 ([Dub92, Dub96, Man99])**.** *For both formal and analytic semisimple Frobenius manifolds, the idempotents vector fields $\pi_1, \dots, \pi_n$ are pairwise commuting, i.e. $[\pi_i, \pi_j] = 0$. Equivalently, the dual differential forms $\pi_i^\flat$, defined by $\langle \pi_i^\flat, \pi_j \rangle = \delta_{ij}$, are closed.* □

In both the formal and analytic cases, this result implies the existence of a local system of coordinates $\boldsymbol{u} := (u_1, \dots, u_n)$ such that

$$du_i = \pi_i^\flat, \quad \frac{\partial}{\partial u_i} = \pi_i.$$

We will refer to $\boldsymbol{u}$ as the *formal/analytic canonical coordinates*. These functions are defined up to re-ordering and shifts by constants. In the formal case, the functions u_i's are just formal functions, i.e. elements of $k[\![\boldsymbol{t}]\!]$.

Proposition 4.5 ([Dub92, Dub96, Man99])**.** *The formal/analytic canonical coordinates can be uniquely chosen (up to re-ordering) so that $E = \sum_{i=1}^n u_i \frac{\partial}{\partial u_i}$.* □

In all the subsequent part of the paper, we will reserve Latin indices for canonical coordinates $u_1, \dots, n_n$ and their vector fields $\partial_i := \frac{\partial}{\partial u_i}$. Einstein summation rule will be used only for repeated Greek indices.

5. Convergence of semisimple formal Frobenius manifolds

In this Section we prove the main result of the second part of this paper.

Theorem 5.1. *Let (H, Φ, η, e, E) be a semisimple formal Frobenius manifold over $\mathbb{C}$. Then the domain of convergence of Φ is non-empty.*

For the proof, we require some preliminary material.

5.1. **Extended deformed connection.**

We introduce one of the main object attached to Frobenius structures, namely an integrable connection. It can be introduced in both formal and analytic frameworks.

Formal case. Let k be a commutative $\mathbb{Q}$-algebra and (H, η, Φ) a formal Frobenius manifold as in Section 4.1. Denote by $k(\!(z)\!)$ the k-algebra of formal Laurent series in an auxiliary indeterminate z. Set $K(\!(z)\!) := k[\![\boldsymbol{t}]\!](\!(z)\!)$ and $H_{K(\!(z)\!)} := H \otimes_k K(\!(z)\!)$. In the following paragraphs we will define two connections on the modules H_K and $H_{K(\!(z)\!)}$ respectively. We firstly recall some basic notions.

5.1.1. *Algebraic connections on modules.* Let A be a commutative and unital k-algebra, and P an A-module. Denote by $\mathrm{Diff}_1(P, P)$ the set of first order differential operators on P, i.e. the k-linear morphisms $\mathscr{D} \in \mathrm{Hom}_k(P, P)$ such that

$$ab\mathscr{D}(p) - b\mathscr{D}(ap) - a\mathscr{D}(bp) + \mathscr{D}(abp) = 0, \quad a, b \in A, \quad p \in P.$$

Both $\mathrm{Der}_k(A)$ and $\mathrm{Diff}_1(P, P)$ are naturally equipped with an A-module structure. A connection ∇ on P is defined by an A-linear morphism $\nabla\colon \mathrm{Der}_k(A) \to \mathrm{Diff}_1(P, P)$, $u \mapsto \nabla_u$ satisfying the Leibniz rule

$$\nabla_u(ap) = u(a)p + a\nabla_u p, \quad a \in A, \quad p \in P.$$

The curvature of ∇ is the A-bilinear morphism $R\colon \mathrm{Der}_k(A) \times \mathrm{Der}_k(A) \to \mathrm{Hom}_A(P, P)$ defined by

$$R(u, v) := [\nabla_u, \nabla_v] - \nabla_{[u,v]}, \quad u, v \in \mathrm{Der}_k(A).$$

Given a connection on P we can induce connections on all the tensor products (over A) $P^{\otimes p} \otimes \mathrm{Hom}_A(P, A)^{\otimes q}$ by requiring that

(1) ∇ commutes with contractions,
(2) on A (i.e. $p = q = 0$) the morphism $\nabla\colon \mathrm{Der}_k(A) \to \mathrm{Diff}_1(A, A)$ is just the inclusion.

5.1.2. *Deformed connections on H_K.* Consider the case $(A, P) = (K, H_K)$. Define a one-parameter family of connections $\nabla^z\colon \mathrm{Der}_k(K) \to \mathrm{Diff}_1(H_K, H_K)$, with $z \in \mathbb{C}$, on the module H_K by the formula

$$\nabla^z_X Y := zX \circ Y, \quad X, Y \in \mathrm{Der}_k(K) \cong H_K.$$

Theorem 5.2 ([Dub92, Man99])**.** *WDVV equations (4.1) are equivalent to the flatness of ∇^z, for any $z \in \mathbb{C}$.* ☐

Remark 5.3. The connection $\nabla := \nabla^0$ is the (formal) Levi-Civita connection for η, i.e. the unique torsion-free connection satisfying $\nabla\eta = 0$. If $(e_1, \dots, e_n)$ is a basis of H_K, set $\nabla_{e_i} e_j = \sum_k \Gamma^k_{ij} e_k$. One can show that

$$\Gamma^k_{ij} = \frac{1}{2}\sum_\ell \eta^{\ell k}\left(e_i\eta_{jk} + e_j\eta_{ik} - e_k\eta_{ij}\right).$$

The standard differential-geometrical proof works verbatim in this formal framework.

Remark 5.4. The Euler vector field is an affine vector field, i.e. $\nabla\nabla E = 0$. This follows from flatness of ∇ and the Killing conformal condition (4.2).

5.1.3. *Extended deformed connection on $H_{K((z))}$.* We consider now the case $(A, P) = (K((z)), H_{K((z))})$. In what follows we assume that the K-linear operator $\nabla^0 E\colon \mathrm{Der}_k(K) \cong H_K \to H_K$ is (diagonalizable and) in diagonal form in the basis $(\Delta_1, \dots, \Delta_n)$. Define two new K-linear operators $\mathcal{U}, \mu$ by the formulae

$$\begin{aligned} \mathcal{U}\colon H_K \to H_K, &\qquad X \mapsto E \circ X,\\ \mu\colon \mathrm{Der}_k(K) \cong H_K \to H_K, &\qquad X \mapsto \frac{D}{2} - \nabla_X E, \end{aligned}$$

where $D \in k$ is as in (4.2). All the tensors $\eta, \circ, \mathcal{U}, \mu$ can be $K((z))$-linearly extended to $H_{K((z))}$. We will denote such an extension by the same symbols.

The extended deformed connection $\widehat{\nabla}\colon \mathrm{Der}_k(K((z))) \to \mathrm{Diff}_1(H_{K((z))}, H_{K((z))})$ is defined by the formulae

$$\widehat{\nabla}_{\frac{\partial}{\partial t^\alpha}} X = \nabla^z_{\frac{\partial}{\partial t^\alpha}} X, \qquad \widehat{\nabla}_{\frac{\partial}{\partial z}} X = \frac{\partial}{\partial z} X + \mathcal{U}(Y) - \frac{1}{z}\mu(X),$$

where $Y \in H_{K((z))}$.

Theorem 5.5 ([Dub96, Dub98, Dub99])**.** *The connection $\widehat{\nabla}$ is flat.*

Proof. The flatness of $\widehat{\nabla}$ is equivalent to the following conditions: $\partial_\delta\Phi_{\alpha\beta\gamma}$ is completely symmetric in $(\alpha, \beta, \gamma, \delta)$, the product $\circ$ is associative, $\nabla\nabla E = 0$, and $\mathfrak{L}_E c = c$. This can be easily checked by a straightforward computation. ☐

Analytic case. Let M be a Dubrovin-Frobenius manifold. Introduce the $(1,1)$-tensors $\mathcal{U}, \mu \in \Gamma(\mathrm{End}(TM))$ by the formulae

$$\mathcal{U}(X) = E \circ X, \quad \mu(X) := \frac{2-d}{2}X - \nabla_X E, \quad X \in \Gamma(TM),$$

where d is the charge of the Dubrovin-Frobenius structure, and ∇ is the Levi-Civita connection of η. We assume that μ is (diagonalizable and) in diagonal form in the frame $(\partial_{t^1}, \dots, \partial_{t^n})$.

Denote by $\pi\colon M \times \mathbb{C}^* \to M$ the canonical projection on the first factor. If $\mathscr{T}_M$ denotes the tangent sheaf of M, then $\pi^*\mathscr{T}_M$ is the sheaf of sections of π^*TM, and $\pi^{-1}\mathscr{T}_M$ is the sheaf of sections of π^*TM constant along the fibers of π. All the tensors $\eta, c, e, E, \mathcal{U}, \mu$ can be lifted to the pulled-back bundle π^*TM, and we denote these lifts with the same symbols. Consequently, also the Levi-Civita connection ∇ can be uniquely lifted on π^*TM in such a way that $\nabla_{\frac{\partial}{\partial z}} Y = 0$ for $Y \in \pi^{-1}\mathscr{T}_M$.

The extended deformed connection $\widehat{\nabla}$ is the connection on π^*TM defined by the formulae

$$\widehat{\nabla}_{\frac{\partial}{\partial t^\alpha}}Y=\nabla_{\frac{\partial}{\partial t^\alpha}}Y+z\frac{\partial}{\partial t^\alpha}\circ Y,\qquad \widehat{\nabla}_{\frac{\partial}{\partial z}}Y=\nabla_{\frac{\partial}{\partial z}}Y+\mathcal{U}(Y)-\frac{1}{z}\mu(Y),\tag{5.1}$$

where $Y\in\pi^*\mathscr{T}_M$.

Remark 5.6. If we consider a formal Frobenius manifold associated to a pointed Dubrovin-Frobenius manifold (M,p) as in Section 4.3, the Christoffel symbols of the formal connection $\widehat{\nabla}$ constructed in Section 5.1.3 are germs of the Christoffel symbols of (5.1) at the point p.

Theorem 5.5 and its proof hold verbatim for the connection $\widehat{\nabla}$ defined by (5.1).

Remark 5.7. In both the formal and analytic case, the operator $\mathcal{U}$ is η-self-adjoint, and μ is η-skew-symmetric: for arbitrary $X,Y\in H_K$ (resp. sections of TM), we have

$$\eta(\mathcal{U}(X),Y)=\eta(X,\mathcal{U}(Y)),\quad \eta(\mu(X),Y)=-\eta(X,\mu(Y)).\tag{5.2}$$

5.2. **Darboux-Egoroff equations.** Given a formal/analytic semisimple Frobenius manifold with idempotent vectors $\pi_1,\dots,\pi_n$ define the formal functions $\eta_{ii},\gamma_{ij}\in k[\![\boldsymbol{u}]\!]$ by

$$\begin{aligned}\eta_{ii}(\boldsymbol{u})&:=\eta(\pi_i(\boldsymbol{u}),\pi_i(\boldsymbol{u})), && i=1,\dots,n,\\ \gamma_{ij}(\boldsymbol{u})&:=\frac{\partial_i\sqrt{\eta_{ii}(\boldsymbol{u})}}{\sqrt{\eta_{jj}(\boldsymbol{u})}}=\frac{1}{2}\frac{\partial_i\partial_j t_1(\boldsymbol{u})}{\sqrt{\partial_i t_1(\boldsymbol{u})\partial_j t_1(\boldsymbol{u})}}, && i,j=1,\dots,n.\end{aligned}$$

Notice that $\gamma_{ij}=\gamma_{ji}$.

Theorem 5.8. *The flatness of η is equivalent to the following equations on $\gamma_{ij}(\boldsymbol{u})$:*

$$\partial_k\gamma_{ij}=\gamma_{ik}\gamma_{kj},\qquad i,j,k \text{ distinct},\tag{5.3}$$

$$\textstyle\sum_{k=1}^n\partial_k\gamma_{ij}=0,\qquad i\neq j\tag{5.4}$$

$$\textstyle\sum_{k=1}^n u_k\partial_k\gamma_{ij}=-\gamma_{ij},\qquad i\neq j.\tag{5.5}$$

Proof. The proofs of [Man99, Prop. 3.4.1, Th. 3.7.2] apply verbatim also to the formal case. □

Corollary 5.9. *For $i\neq j$, we have*

$$(u_i-u_j)\partial_i\gamma_{ij}=\sum_{k\neq i,j}(u_j-u_k)\gamma_{ik}\gamma_{kj}-\gamma_{ij}.\tag{5.6}$$

Proof. An easy consequence of (5.4) and (5.5). □

5.3. **$\widehat{\nabla}$-flatness in canonical coordinates.** Given a formal (resp. analytic) semisimple Frobenius manifold with idempotent vectors $\pi_1,\dots,\pi_n$ define the vectors

$$f_i(\boldsymbol{u}):=\eta_{ii}(\boldsymbol{u})^{-\frac{1}{2}}\pi_i(\boldsymbol{u}),\quad i=1,\dots,n,$$

for some choices of the square roots, and introduce the matrix $\Psi\in GL(n,k[\![\boldsymbol{u}]\!])$ (resp. $GL(n,k\{\boldsymbol{u}\})$) defined by

$$\Psi=(\Psi_{i\alpha})_{i,\alpha},\quad \frac{\partial}{\partial t^\alpha}=\sum_{i=1}^n\Psi_{i\alpha}f_i,\quad i=1,\dots,n.$$

Lemma 5.10. *We have*

$$\Psi^T\Psi=\eta,\quad \Psi_{i1}=\sqrt{\eta_{ii}},\quad f_i=\sum_{\alpha,\beta=1}^n\Psi_{i1}\Psi_{i\beta}\eta^{\alpha\beta}\partial_\alpha,\quad c_{\alpha\beta\gamma}=\sum_{i=1}^n\frac{\Psi_{i\alpha}\Psi_{i\beta}\Psi_{i\gamma}}{\Psi_{i1}}.\qquad\square$$

Lemma 5.11. *We have* $\mu(f_i)=\sum_{j\neq i}(u_j-u_i)\gamma_{ij}f_j$.

Proof. Set $\nabla_{\pi_i}\pi_j=\sum_k\Gamma^k_{ij}\pi_k$. The only nonzero Christoffel symbols are

$$\Gamma^i_{ii}=\frac{1}{2}\eta_{ii}^{-1}\frac{\partial\eta_{ii}}{\partial u_i},\quad \Gamma^j_{ii}=-\frac{1}{2}\eta_{jj}^{-1}\frac{\partial\eta_{ii}}{\partial u_j},\quad \Gamma^i_{ij}=\Gamma^i_{ji}=\frac{1}{2}\eta_{ii}^{-1}\frac{\partial\eta_{ii}}{\partial u_j},\quad i\neq j,$$

see Remark 5.3. The claim follows by straightforward computations. $\square$

The connection $\widehat{\nabla}$ induces a dual connection[5] $\widehat{\nabla}^T$ on $H^T_{K(\!(z)\!)}$ (resp. π^*T^*M). Consider the equation $\widehat{\nabla}^T\xi=0$: if $\xi^\sharp$ is η-dual to ξ, then we have

$$\frac{\partial}{\partial t^\alpha}\xi^\sharp=z\frac{\partial}{\partial t^\alpha}\circ\xi^\sharp,\qquad \frac{\partial}{\partial z}\xi^\sharp=\left(\mathcal{U}+\frac{1}{z}\mu\right)\xi^\sharp.\tag{5.7}$$

Set $\xi^\sharp(\boldsymbol{t},z)=\sum_{i=1}^n y_i(\boldsymbol{u}(\boldsymbol{t}),z)f_i$ for some formal (resp. analytic) functions y_i of the formal canonical coordinates $\boldsymbol{u}$ and z. Then, equations (5.7) are equivalent to

$$\frac{\partial}{\partial u_i}y=(zE_i+V_i(\boldsymbol{u}))\,y,\tag{5.8}$$

$$\frac{\partial}{\partial z}y=\left(U(\boldsymbol{u})+\frac{1}{z}V(\boldsymbol{u})\right)y,\tag{5.9}$$

where $y=(y_1,\dots,y_n)^T$ and

$$U=\operatorname{diag}(u_1,\dots,u_n),\quad V:=\Psi\mu\Psi^{-1},\quad V_i:=\frac{\partial\Psi}{\partial u_i}\Psi^{-1},\quad (E_i)_{ab}=\delta_{ai}\delta_{bi}.$$

The compatibility of the system (5.8),(5.9) is equivalent to the equations

$$\frac{\partial V}{\partial u_i}=[V_i,V],\tag{5.10}$$

$$[U,V_i]=[E_i,V].\tag{5.11}$$

Lemma 5.12. *Set* $\Gamma=(\gamma_{ab})$. *We have*

$$V^T+V=0,\quad V=[\Gamma,U],\quad V_i^T+V_i=0,\quad V_i=[\Gamma,E_i],\quad i=1,\dots,n,$$

Equation (5.10) *follows from Darboux-Egoroff equations* (5.3),(5.4),(5.5) *on* Γ.

Proof. The identity $V=[\Gamma,U]$ is Lemma 5.11. The identity $\Psi(\boldsymbol{u})^T\Psi(\boldsymbol{u})=\eta$ implies $\partial_i\Psi^T\Psi+\Psi^T\partial_i\Psi=0$, so that $V_i^T+V_i=0$. We have $[U,V_i]=[U,[\Gamma,E_i]]$, by (5.11) and Jacobi identity. The nucleus of the operator $[U,-]\colon M_n(k[\![\boldsymbol{u}]\!])\to M_n(k[\![\boldsymbol{u}]\!])$ consists of diagonal matrices: if $A\in M_n(k[\![\boldsymbol{u}]\!])$ is such that $[U,A]=0$, then $(u_a-u_b)A_{ab}(\boldsymbol{u})=0$ for any $a,b=1,\dots,n$ with $a\neq b$. We deduce that $A_{ab}(\boldsymbol{u})=0$, with $a\neq b$, since $k[\![\boldsymbol{u}]\!]$ is an integral domain. Hence $V_i=\mathcal{D}+[\Gamma,E_i]$, where $\mathcal{D}$ is a diagonal matrix. The skew-symmetry of V_i implies that $\mathcal{D}=0$. A simple computation shows that (5.3),(5.4),(5.5) and (5.6) imply (5.11). $\square$

[5]In B. Dubrovin's papers this is denoted by the same symbol $\widehat{\nabla}$.

5.4. **Reconstruction of the Frobenius structure.** By Theorem 5.2 we can look for formal functions $\tilde{\boldsymbol{t}} := (\tilde{t}_1, \dots, \tilde{t}_n)$ of the form

$$\tilde{t}_\alpha(\boldsymbol{t}, z) := \sum_{p=0}^{\infty} h_{\alpha,p}(\boldsymbol{t}) z^p \in k[\![\boldsymbol{t}, z]\!], \quad h_{\alpha,0}(\boldsymbol{t}) = t_\alpha \equiv t^\beta \eta_{\alpha\beta},$$

such that $\nabla^z d\tilde{t}_\alpha = 0$ for $\alpha = 1, \dots, n$.

Lemma 5.13. *The functions $h_{\alpha,p}$ satisfy the recursive equations*

$$h_{\alpha,0}(\boldsymbol{t}) = t_\alpha \equiv t^\beta \eta_{\alpha\beta}, \qquad \partial_\beta \partial_\gamma h_{\alpha,p+1} = c^\varepsilon_{\beta\gamma} \partial_\varepsilon h_{\alpha,p}, \quad p \in \mathbb{N}. \qquad \square$$

For $f, g \in K$ write $f \approx g$ iff $f - g$ is a (at most) quadratic polynomial in $\boldsymbol{t}$.

Lemma 5.14. *We have*

$$h_{\alpha,1} \approx \partial_\alpha \Phi, \qquad \alpha = 1, \dots, n, \tag{5.12}$$

$$h_{1,2} \approx t^\alpha \partial_\alpha \Phi - 2\Phi. \tag{5.13}$$

Proof. We have $\partial_\beta h_{\alpha,0} = \eta_{\alpha\beta}$, so that $\partial_\gamma \partial_\beta h_{\alpha,1} = c_{\alpha\beta\gamma}$. Equation (5.12) follows. We have $\partial_1 \Phi = \frac{1}{2}\eta_{\alpha\beta} t^\alpha t^\beta$, so that $\partial_\alpha \partial_\beta h_{1,2} = c^\gamma_{\alpha\beta} \partial_\gamma h_{1,1} = c^\gamma_{\alpha\beta} \partial_\gamma \partial_1 \Phi = c^\gamma_{\alpha\beta} \eta_{\gamma\nu} t^\nu$. We also have $\partial_\alpha \partial_\beta \left(t^\lambda \partial_\lambda \Phi - 2\Phi \right) = c_{\alpha\beta\lambda} t^\lambda$, and (5.13) follows. $\square$

Given a function $f \in K$, we denote by $\mathrm{gr} f \in H_K$ the η-gradient of f, defined by $\mathrm{gr} f := \sum_\alpha \eta^{\alpha\beta} \partial_\beta f \Delta_\alpha$. The following result allows to reconstruct the potential Φ (up to quadratic terms) from the first coefficients $h_{\alpha,p}$, with $p \leqslant 3$.

Theorem 5.15 ([Dub96, Dub99])**.** *We have*

$$\Phi \approx \frac{1}{2} \left[\eta(\mathrm{gr} h_{\alpha,1}, \mathrm{gr} h_{1,1}) \eta^{\alpha\beta} \eta(\mathrm{gr} h_{\beta,0}, \mathrm{gr} h_{1,1}) - \eta(\mathrm{gr} h_{1,1}, \mathrm{gr} h_{1,2}) - \eta(\mathrm{gr} h_{1,3}, \mathrm{gr} h_{1,0}) \right]. \tag{5.14}$$

Proof. The expression in square brackets in the r.h.s. of (5.14) equals

$$\begin{aligned} &\eta^{\nu\lambda} \partial_\nu h_{\alpha,1} \partial_\lambda h_{1,1} \eta^{\alpha\beta} \partial_\beta h_{1,1} - \eta^{\tau\varepsilon} \partial_\tau h_{1,1} \partial_\varepsilon h_{1,2} - \partial_1 h_{1,3} \\ \approx \quad &\eta^{\nu\lambda} \eta^{\alpha\beta} \partial^2_{\nu\alpha} \Phi \, \partial^2_{\lambda 1} \Phi \, \partial^2_{\beta 1} \Phi - \eta^{\tau\varepsilon} \partial^2_{\tau 1} \Phi \left(t^\lambda \partial^2_{\varepsilon\lambda} \Phi - \partial_\varepsilon \Phi \right) - \partial_1 h_{1,3}. \end{aligned} \tag{5.15}$$

We have $\partial^2_{\nu 1} \Phi = \eta_{\nu\alpha} t^\alpha$, and $\partial_\alpha \partial_1 h_{1,3} = c^\beta_{1\alpha} \partial_\beta h_{1,2} = \partial_\alpha h_{1,2}$ so that $\partial_1 h_{1,3} \approx h_{1,2}$. Hence (5.15) equals 2Φ up to quadratic terms. $\square$

5.5. **Proof of Theorem 5.1.** Let (H, η, Φ, e, E) be a formal Frobenius manifold. Fix one ordering $\boldsymbol{u}_o \in \mathbb{C}^n$ of the eigenvalues of $\mathcal{U}(\boldsymbol{t})$ specialized at the origin $\boldsymbol{t} = 0$. We have $n \times n$ matrix-valued (a priori) formal power series in $\boldsymbol{u}$

$$V(\boldsymbol{u}) = V_o + \sum_{k=1}^{\infty} \sum_{\ell_1, \dots, \ell_k = 1}^{n} \frac{1}{k!} V^{(\boldsymbol{\ell})} \prod_{j=1}^{k} \overline{u}_{\ell_j}, \qquad V_i(\boldsymbol{u}) = V_{i,o} + \sum_{k=1}^{\infty} \sum_{\ell_1, \dots, \ell_k = 1}^{n} \frac{1}{k!} V_i^{(\boldsymbol{\ell})} \prod_{j=1}^{k} \overline{u}_{\ell_j},$$

$$\Psi(\boldsymbol{u}) = \Psi_o + \sum_{k=1}^{\infty} \sum_{\ell_1, \dots, \ell_k = 1}^{n} \frac{1}{k!} \Psi^{(\boldsymbol{\ell})} \prod_{j=1}^{k} \overline{u}_{\ell_j}, \qquad \Gamma(\boldsymbol{u}) = \Gamma_o + \sum_{k=1}^{\infty} \sum_{\ell_1, \dots, \ell_k = 1}^{n} \frac{1}{k!} \Gamma^{(\boldsymbol{\ell})} \prod_{j=1}^{k} \overline{u}_{\ell_j},$$

where $\overline{u}_i := u_i - u_{o,i}$ for $i = 1, \dots, n$. These power series are well defined by the semisimplicity assumption, and they satisfy properties described in Theorem 5.8, and Lemmata 5.10, 5.11

and 5.12. We subdivide the proof in two parts. In the first part, we construct a pointed germ (M,p) of a Dubrovin-Frobenius manifold M starting from the datum of $\boldsymbol{u}_o, V_o, \Psi_o, \Gamma_o$. In the second part, we prove that the original formal structure (H,η,Φ,e,E) is the completion of the pointed analytic germ (M,p).

Part I. The system (5.9) specialized at $\boldsymbol{u}_o$, namely $\frac{\partial Y}{\partial z}=(U_o+\frac{1}{z}V_o)Y$, can be identified with equation (3.12) (in the special case $B_o'=0$). The arguments of Section 3.4 can be applied, in both cases $\boldsymbol{u}_o\in\mathbb{C}^n\setminus\Delta$ and $\boldsymbol{u}_o\in\Delta$. We can fix an admissible τ at $\boldsymbol{u}_o$, the $(\boldsymbol{u}_o,\tau)$-admissible datum $\mathfrak{M}$ is well-defined, and we can set the RHB problem $\mathcal{P}[\boldsymbol{u},\tau,\mathfrak{M}]$. This problem is solvable w.r.t. $\boldsymbol{u}$ on an open neighborhood $\mathcal{V}\setminus\Theta$ of $\boldsymbol{u}_o$, by Theorem 3.13. The unique solution $G(z;\boldsymbol{u})$ is holomorphic in $\boldsymbol{u}\in\mathcal{V}\setminus\Theta$, and with expansion

$$G(z;\boldsymbol{u})=I+\tfrac{1}{z}F_1^{\rm an}(\boldsymbol{u})+O\left(\tfrac{1}{z^2}\right),\qquad z\to\infty,\quad z\in\Pi_{L/R},$$
$$G(z;\boldsymbol{u})=G_0(\boldsymbol{u})+G_1(\boldsymbol{u})z+G_2(\boldsymbol{u})z^2+G_3(\boldsymbol{u})z^3+O(z^4),\qquad z\to 0.$$

Here the superscript "an" stands for *analytic*. As output of Section 3.4, we also obtain a compatible joint system of differential equations (with analytic coefficients in $\boldsymbol{u}$, not just formal) of the form

$$\frac{\partial Y}{\partial u_i}=(zE_i+V_i^{\rm an}(\boldsymbol{u}))Y,\quad \frac{\partial Y}{\partial z}=\left(U+\frac{1}{z}V^{\rm an}(\boldsymbol{u})\right)Y,\tag{5.16}$$

where $V^{\rm an}(\boldsymbol{u}):=[F_1^{\rm an}(\boldsymbol{u}),U]$, and $V_i^{\rm an}(\boldsymbol{u}):=[F_1^{\rm an}(\boldsymbol{u}),E_i]$. Moreover, we have

$$V^{\rm an}(\boldsymbol{u}_o)=V_o,\quad G_0(\boldsymbol{u}_o)=\Psi_o,\quad \partial_iG_0=V_i^{\rm an}G_0,\quad i=1,\dots,n.$$

From the datum of $G_i(\boldsymbol{u})$, with $i=0,1,2,3$, we can construct a Dubrovin-Frobenius manifold as follows: set

$$t^\alpha(\boldsymbol{u}) := \eta^{\alpha\beta}\sum_{i=1}^n G_{0,i\beta}(\boldsymbol{u})G_{1,i1}(\boldsymbol{u}),\quad \alpha=1,\dots,n,$$
$$F(\boldsymbol{u}) := \frac{1}{2}\left[t^\alpha(\boldsymbol{u})t^\beta(\boldsymbol{u})\sum_{i=1}^n G_{0,i\alpha}(\boldsymbol{u})G_{1,i\beta}(\boldsymbol{u})-\sum_{i=1}^n\left(G_{1,i1}(\boldsymbol{u})G_{2,i1}(\boldsymbol{u})+G_{0,i1}(\boldsymbol{u})G_{3,i1}(\boldsymbol{u})\right)\right].$$

Invert the first series expansions, to obtain $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{t})$. The function $F(\boldsymbol{u}(\boldsymbol{t}))$ gives a solution of WDVV equations, and defines an analytic Dubrovin-Frobenius manifold on an open subset of H. The formulae above are, in their essence, re-writing of formulae of Lemma 5.10 and formula (5.14). See [Dub99, Guz01].

Part II. We need to prove that the series expansion $F(\boldsymbol{u}(\boldsymbol{t}))$ obtained in Part I equals (up to quadratic terms) the original potential $\Phi(\boldsymbol{t})$. For that, it is sufficient to prove that $F_1^{\rm an}(\boldsymbol{u})''=\Gamma(\boldsymbol{u})''$.

Lemma 5.16. *We have* $F_1^{\rm an}(\boldsymbol{u}_o)''=\Gamma_o''$.

Proof. By Proposition 3.17, the system (5.9) specialized at $\boldsymbol{u}_o$, namely $\frac{\partial Y}{\partial z}=(U_o+\frac{1}{z}V_o)Y$, admits a unique formal solution $Y_F(z)=\left(I+A_1z^{-1}+A_2z^{-2}+O(z^{-3})\right)e^{zU}$. Let us recall how to compute A_1. It is uniquely determined by the two equations

$$[A_1,U_o]=V_o,\quad [A_2,U_o]=A_1+V_oA_1.$$

The first equation uniquely determines all the entries $(A_1)_{ab}$ for indices $a \neq b$ such that $u_{o,a} \neq u_{o,b}$:

$$(A_1)_{ab} = \frac{V_{o,ab}}{u_{o,b} - u_{o,a}} = \Gamma_{o,ab},$$

by Lemma 5.11. All the remaining entries $(A_1)_{ab}$, with $a \neq b$ such that $u_{o,a} = u_{o,b}$, are uniquely determined by the second equation:

$$(A_1)_{ab} = -\sum_{\ell} V_{o,a\ell}(A_1)_{\ell b} = -\sum_{\ell}(u_{o,\ell} - u_{o,a})\Gamma_{o,a\ell}\Gamma_{\ell b} = \Gamma_{o,ab}.$$

The last equality follows by specializing equation (5.6) to $\boldsymbol{u} = \boldsymbol{u}_o$. This prove that $A_1'' = \Gamma_o''$. By uniqueness of the formal solution we clearly have $F_1^{\mathrm{an}}(\boldsymbol{u}_o) = A_1$. $\square$

Lemma 5.17. *The off-diagonal entries of $F_1^{\mathrm{an}}(\boldsymbol{u})$ satisfy the Darboux-Egoroff system (5.3), (5.4), (5.5), (5.6).*

Proof. From the compatibility conditions $\partial_i\partial_j = \partial_j\partial_i$ of the system (5.16), we have

$$[E_j, \partial_i F_1^{\mathrm{an}}] - [E_i, \partial_j F_1^{\mathrm{an}}] + [[E_i, F_1^{\mathrm{an}}], [E_j, F_1^{\mathrm{an}}]] = 0.$$

This coincides with equations (5.3) and (5.4). Let $\kappa \in \mathbb{C}^*$. The piecewise analytic function $\widetilde{G}\colon (\Pi_0 \cup \Pi_L \cup \Pi_R) \times (\mathcal{V} \setminus \kappa\Theta) \to \mathbb{C}$ defined by

$$\widetilde{G}(z;\boldsymbol{u}) := G(\kappa z; \kappa^{-1}\boldsymbol{u})\kappa^D z^D \kappa^L z^{-D}, \qquad z \in \Pi_0,$$
$$\widetilde{G}(z;\boldsymbol{u}) := G(\kappa z; \kappa^{-1}\boldsymbol{u}), \qquad z \in \Pi_{L/R},$$

solves the same RHB problem $\mathcal{P}[\boldsymbol{u}, \tau, \mathfrak{M}]$ as G. By uniqueness of solution we have $\widetilde{G} = G$. This implies that $F_1^{\mathrm{an}}(\kappa^{-1}\boldsymbol{u}) = \kappa F_1^{\mathrm{an}}(\boldsymbol{u})$, and (5.5) follows. $\square$

Lemma 5.18. *Let*

$$F(\boldsymbol{u}) = F_o + \sum_{k=1}^{\infty} \sum_{\ell_1,\dots,\ell_k=1}^{n} \frac{1}{k!} F^{(\boldsymbol{\ell})} \prod_{j=1}^{k} \overline{u}_{\ell_j}, \quad \overline{u}_i := u_i - u_{o,i},$$

be a matrix-valued formal power series, with $F(\boldsymbol{u})^T = F(\boldsymbol{u})$, and whose off-diagonal entries F_{ij} are formal solutions of the Darboux-Egoroff system (5.3), (5.4), (5.5), (5.6). The off-diagonal entries of the coefficients $F^{(\ell)}$ can be uniquely reconstructed from the off-diagonal entries of F_o.

Proof. We have to show that the derivatives $\partial_{i_1} \dots \partial_{i_N} F_{ij}(\boldsymbol{u}_o)$ can be computed from the only knowledge of the numbers $F_{ij}(\boldsymbol{u}_o)$. We proceed by induction on N. Let us start with the case $N = 1$.

Step 1. For i, j, k distinct, by expanding both sides of $\partial_k F_{ij} = F_{ik}F_{kj}$ in power series, and equating the coefficients, one reconstructs the coefficients of $\partial_k F_{ij}(\boldsymbol{u}_o)$.

Step 2. From the identity (5.6) for F_{ij}, one can compute $\partial_i F_{ij}(\boldsymbol{u}_o)$ provided that $u_{o,i} \neq u_{o,j}$.

Step 3. Assume that $u_{o,i} = u_{o,j}$. By taking the ∂_i-derivative of both sides of (5.6) we obtain

$$2\partial_i F_{ij}(\boldsymbol{u}) + (u_i - u_j)\partial_i\partial_i F_{ij}(\boldsymbol{u}) = \sum_{k \neq i,j} (u_j - u_k)\left[\partial_i F_{ik}(\boldsymbol{u})F_{kj}(\boldsymbol{u}) + F_{ik}(\boldsymbol{u})\partial_i F_{kj}(\boldsymbol{u})\right]. \quad (5.17)$$

By evaluating (5.17) at $\boldsymbol{u} = \boldsymbol{u}_o$ we can compute all the numbers $\partial_i F_{ij}(\boldsymbol{u}_o)$, namely

$$\partial_i F_{ij}(\boldsymbol{u}_o) = \frac{1}{2} \sum_{k\neq i,j} (u_{o,j} - u_{o,k}) \left[\partial_i F_{ik}(\boldsymbol{u}_o) F_{kj}(\boldsymbol{u}_o) + F_{ik}(\boldsymbol{u}_o)^2 F_{ij}(\boldsymbol{u}_o)\right].$$

Notice that the only terms $\partial_i F_{ik}(\boldsymbol{u}_o)$ appearing in this sum are those computed in Step 2.
Step 4. By the symmetry condition $F(\boldsymbol{u})^T = F(\boldsymbol{u})$, we have $\partial_j F_{ij}(\boldsymbol{u}_o) = \partial_j F_{ji}(\boldsymbol{u}_o)$, and these numbers can be computed as in Steps 2 and 3.
This proves that all the first derivatives $\partial_k F_{ij}(\boldsymbol{u}_o)$ can be computed.

Inductive step. Assume to know all the N-th derivatives $\partial_{i_1} \dots \partial_{i_N} F_{ij}(\boldsymbol{u}_o)$. We show how to compute the number $\partial_{h_1} \dots \partial_{h_{N+1}} F_{ij}(\boldsymbol{u}_o)$ for any $(N+1)$-tuple $(h_1, \dots, h_{N+1})$.
Step 1. Assume that there exists $\ell \in \{1, \dots, N+1\}$ such that $h_\ell \neq i, j$. We have

$$\partial_{h_1} \dots \partial_{h_{N+1}} F_{ij} = \partial_{h_1} \dots \partial_{h_{\ell-1}} \partial_{h_{\ell+1}} \dots \partial_{h_{N+1}} [\partial_{h_\ell} F_{ij}] = \partial_{h_1} \dots \partial_{h_{\ell-1}} \partial_{h_{\ell+1}} \dots \partial_{h_{N+1}} [F_{ih_\ell} F_{h_\ell j}].$$

By evaluation at $\boldsymbol{u} = \boldsymbol{u}_o$, we can compute all the numbers $\partial_{h_1} \dots \partial_{h_{N+1}} F_{ij}(\boldsymbol{u}_o)$.
Step 2. Assume that $(h_1, \dots, h_{N+1}) = (i, i, \dots, i)$. Take the ∂_i^N-derivative of (5.6): by evaluation at $\boldsymbol{u} = \boldsymbol{u}_o$ we can reconstruct the numbers $\partial_i^{N+1} F_{ij}(\boldsymbol{u}_o)$ provided that $u_{o,i} \neq u_{o,j}$.
Step 3. Assume that $u_{o,i} = u_{o,j}$. Take the ∂_i^N-derivative of both sides of (5.17), to obtain

$$(N+2)\partial_i^{N+1} F_{ij} + (u_i - u_j)\partial_i^{N+2} F_{ij} = \sum_{k\neq i,j} (u_j - u_k) \partial_i^N [\partial_i F_{ik} F_{kj} + F_{ik}^2 F_{ij}]. \tag{5.18}$$

By evaluation $\boldsymbol{u} = \boldsymbol{u}_o$, one can compute the number $\partial_i^{N+1} F_{ij}(\boldsymbol{u}_o)$.
Step 4. Assume that $(h_1, \dots, h_{N+1}) = (j, j, \dots, j)$. By symmetry of $F(\boldsymbol{u})$, we have $\partial_j^{N+1} F_{ij}(\boldsymbol{u}_o) = \partial_j^{N+1} F_{ji}(\boldsymbol{u}_o)$, and we can proceed as in Steps 2 and 3.
This proves that all the $(N+1)$-th derivatives $\partial_{h_1} \dots \partial_{h_{N+1}} F_{ij}(\boldsymbol{u}_0)$ can be computed. $\square$

This proves that $F_1^{\mathrm{an}}(\boldsymbol{u})'' = \Gamma(\boldsymbol{u})''$. It follows that $V^{\mathrm{an}}(\boldsymbol{u}) = V(\boldsymbol{u}), V_i^{\mathrm{an}}(\boldsymbol{u}) = V_i(\boldsymbol{u})$ and so that $G_0(\boldsymbol{u}) = \Psi(\boldsymbol{u})$. Formula (5.14) then implies Theorem 5.1.

6. Application to CohFT's and Gromov-Witten theory

6.1. **Cohomological field theories.** Let k and (H, η, e) be as in Section 4.1. For a pair of nonnegative integers $(g, \mathfrak{n})$ in the stable range $2g - 2 + \mathfrak{n} > 0$, denote by $\overline{\mathcal{M}}_{g,\mathfrak{n}}$ the Deligne-Mumford moduli space of stable $\mathfrak{n}$-pointed curves of genus g. Denote by $\pi\colon \overline{\mathcal{M}}_{g,\mathfrak{n}+1} \to \overline{\mathcal{M}}_{g,\mathfrak{n}}$ the morphism forgetting the last puncture, by $\sigma\colon \overline{\mathcal{M}}_{g_1,\mathfrak{n}_1+1} \times \overline{\mathcal{M}}_{g_2,\mathfrak{n}_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,\mathfrak{n}_1+\mathfrak{n}_2}$ the morphism which identifies the last markings, and by $\tau\colon \overline{\mathcal{M}}_{g,\mathfrak{n}+2} \to \overline{\mathcal{M}}_{g+1,\mathfrak{n}}$ the morphism identifying the last two punctures of a same curve.

A *Cohomological field theory* (CohFT) on (H, η, e) is the datum of a system $(\Omega_{g,\mathfrak{n}})_{2g-2+\mathfrak{n}>0}$ of k-multilinear maps $\Omega_{g,\mathfrak{n}}\colon H^{\otimes \mathfrak{n}} \to H^\bullet(\overline{\mathcal{M}}_{g,\mathfrak{n}}, k)$ satisfying the following axioms:

1. each tensor $\Omega_{g,\mathfrak{n}}$ is $\mathfrak{S}_\mathfrak{n}$-covariant w.r.t. the natural actions of the symmetric group $\mathfrak{S}_\mathfrak{n}$ on both $H^{\otimes\mathfrak{n}}$ and $H^\bullet(\overline{\mathcal{M}}_{g,\mathfrak{n}}, k)$,
2. $\Omega_{0,3}(e \otimes \Delta_\alpha \otimes \Delta_\beta) = \eta_{\alpha\beta}$,
3. $\pi^*\Omega_{g,\mathfrak{n}}(\bigotimes_{i=1}^{\mathfrak{n}} v_{\alpha_i}) = \Omega_{g,\mathfrak{n}}(\bigotimes_{i=1}^{\mathfrak{n}} v_{\alpha_i} \otimes e)$,
4. $\sigma^*\Omega_{g_1+g_2,\mathfrak{n}_1+\mathfrak{n}_2}(\bigotimes_{i=1}^{\mathfrak{n}_1+\mathfrak{n}_2} v_{\alpha_i}) = \eta^{\mu\nu}\Omega_{g_1,\mathfrak{n}_1+1}(\bigotimes_{i=1}^{\mathfrak{n}_1} v_{\alpha_i} \otimes \Delta_\mu)\Omega_{g_2,\mathfrak{n}_2+1}(\bigotimes_{i=1}^{\mathfrak{n}_2} v_{\alpha_i} \otimes \Delta_\nu)$,
5. $\tau^*\Omega_{g+1,\mathfrak{n}}(\bigotimes_{i=1}^{\mathfrak{n}} v_{\alpha_i}) = \eta^{\mu\nu}\Omega_{g,\mathfrak{n}+2}(\bigotimes_{i=1}^{\mathfrak{n}} v_{\alpha_i} \otimes \Delta_\mu \otimes \Delta_\nu)$.

Given a CohFT, we may introduce generating functions, in infinitely many variables $\boldsymbol{t}^{\bullet}_{\bullet} = (t^\alpha_d)_{\substack{\alpha=1,\dots,n\\ d\in\mathbb{N}}}$ of intersection numbers with psi-classes,

$$\mathcal{F}_g(\boldsymbol{t}^{\bullet}_{\bullet}) := \sum_{\substack{\mathfrak{n}\geqslant 0\\ 2g-2+\mathfrak{n}>0}} \frac{1}{\mathfrak{n}!} \sum_{\substack{\alpha_1,\dots,\alpha_{\mathfrak{n}}=1,\dots,n\\ d_1,\dots,d_{\mathfrak{n}}\geqslant 0}} \left\langle \prod_{i=1}^{\mathfrak{n}} \tau_{d_i}\Delta_{\alpha_i}\right\rangle_g \prod_{i=1}^{\mathfrak{n}} t^{\alpha_i}_{d_i}, \tag{6.1}$$

$$\left\langle \prod_{i=1}^{\mathfrak{n}} \tau_{d_i}\Delta_{\alpha_i}\right\rangle_g := \int_{\overline{\mathcal{M}}_{g,\mathfrak{n}}} \Omega_{g,\mathfrak{n}}\left(\bigotimes_{i=1}^{\mathfrak{n}}\Delta_{\alpha_i}\right)\prod_{i=1}^{\mathfrak{n}}\psi_i^{d_i}. \tag{6.2}$$

In the genus zero sector and restricting to the small phase space, i.e. by setting $t^\alpha_d = 0$ for $d > 0$ and $t^\alpha_0 = t^\alpha$ for $\alpha = 1,\dots,n$, the expression above simplifies to

$$\mathcal{F}_0(\boldsymbol{t}) = \sum_{\mathfrak{n}>2}\sum_{\alpha_1,\dots,\alpha_{\mathfrak{n}}=1}^{n} \frac{t^{\alpha_1}\dots t^{\alpha_{\mathfrak{n}}}}{\mathfrak{n}!}\int_{\overline{\mathcal{M}}_{0,\mathfrak{n}}} \Omega_{0,\mathfrak{n}}\left(\Delta_{\alpha_1}\otimes\cdots\otimes\Delta_{\alpha_{\mathfrak{n}}}\right). \tag{6.3}$$

The power series $\mathcal{F}_0 \in k[\![\boldsymbol{t}]\!]$ is a solution of WDVV equations, and it defines a formal Frobenius manifold (over k) on (H,η,e), see [KM94, Man99]. The CohFT will be said to be *semisimple* if the corresponding formal Frobenius manifold is semisimple.

If $E = \sum_\alpha (w_\alpha t^\alpha + y_\alpha)\,\partial_\alpha$ is a Killing-conformal vector field on H, i.e. $\mathfrak{L}_E\eta = (2-d)\eta$ for some $d \in k$, we have a natural action of E on the CohFT $(\Omega_{g,\mathfrak{n}})_{g,\mathfrak{n}}$. Denote by $\deg\colon H^\bullet(\overline{\mathcal{M}}_{g,\mathfrak{n}},k) \to H^\bullet(\overline{\mathcal{M}}_{g,\mathfrak{n}},k)$ the operator which acts on H^{2k} by multiplication by k. Then we set

$$(E\Omega)_{g,\mathfrak{n}}\left(\bigotimes_{j=1}^{\mathfrak{n}}\Delta_{\alpha_j}\right) := \left(\deg + \sum_{\ell=1}^{\mathfrak{n}} w_\ell\right)\Omega_{g,\mathfrak{n}}\left(\bigotimes_{j=1}^{\mathfrak{n}}\Delta_{\alpha_j}\right) + \pi_*\Omega_{g,\mathfrak{n}+1}\left(\bigotimes_{j=1}^{\mathfrak{n}}\Delta_{\alpha_j}\otimes\sum_{\ell=1}^{\mathfrak{n}} y_\ell\Delta_\ell\right).$$

A CohFT is called *homogeneous in genus g* if $(E\Omega)_{g,\mathfrak{n}} = [(g-1)d+n]\Omega_{g,\mathfrak{n}}$ for all $\mathfrak{n} > 2-2g$. When a CohFT is homogeneous in genus zero, E is an Euler vector field for the underlying formal Frobenius manifold.

Remark 6.1. Teleman Reconstruction Theorem [Tel12, Th. 1] asserts that a CohFT, semisimple and homogeneous in all genera, can be uniquely reconstructed from the underlying formal Frobenius manifold. The reconstruction is performed via the Givental group action [Giv01].

The following result immediately follows from Theorem 5.1.

Theorem 6.2. *For any semisimple and homogeneous (at least in genus 0) CohFT over $k = \mathbb{C}$, the potential $\mathcal{F}_0(\boldsymbol{t})$ is convergent. In particular, there exist real positive constants $m, \rho_1, \dots, \rho_n$ such that*

$$\left|\int_{\overline{\mathcal{M}}_{0,|\boldsymbol{\alpha}|}} \Omega_{0,|\boldsymbol{\alpha}|}\left(\Delta_1^{\otimes\alpha_1}\otimes\cdots\otimes\Delta_n^{\otimes\alpha_n}\right)\right| \leqslant m\prod_{i=1}^{n}\rho_i^{\alpha_i}, \quad \boldsymbol{\alpha}\in\mathbb{N}^n,$$

where we set $|\boldsymbol{\alpha}| := \sum_k \alpha_k$. □

6.2. **Gromov-Witten theory.** Let X be a smooth complex projective variety with vanishing odd cohomology $H^{\mathrm{odd}}(X;\mathbb{C})=0$. Let $(\Delta_1,\dots,\Delta_n)$ be a homogeneous basis of $H^\bullet(X;\mathbb{C})$, with $\Delta_1=1$ and $(\Delta_2,\dots,\Delta_{r+1})$ a NEF basis of $H^2(X;\mathbb{Z})$. Denote by η the Poincaré metric $\eta(\alpha,\beta):=\int_X\alpha\cup\beta$. Introduce indeterminates $\mathbf{Q}:=(Q_1,\dots,Q_r)$, and define the *Novikov ring* $\Lambda:=\mathbb{Q}[\![\mathbf{Q}]\!]$.

Gromov-Witten theory naturally provides a CohFT over the Λ-module $H^\bullet(X;\Lambda)$ with Λ-bilinearly extended Poincaré metric η. The maps $\Omega_{g,\mathfrak{n}}$ are given by the counting of curves on X,

$$\Omega_{g,\mathfrak{n}}\left(\bigotimes_{i=1}^{\mathfrak{n}}\Delta_{\alpha_i}\right):=\sum_\beta\phi_*\left([\overline{\mathcal{M}}_{g,\mathfrak{n}}(X,\beta)]^{\mathrm{vir}}\cap\prod_{i=1}^{\mathfrak{n}}\mathrm{ev}_i^*\Delta_{\alpha_i}\right)\mathbf{Q}^\beta\in H^\bullet(\overline{\mathcal{M}}_{0,\mathfrak{n}};\Lambda),\tag{6.4}$$

where $\mathbf{Q}^\beta:=\prod_{i=1}^r Q_i^{\int_\beta\Delta_{i+1}}$, $\overline{\mathcal{M}}_{g,\mathfrak{n}}(X,\beta)$ is the Deligne-Mumford moduli space of $\mathfrak{n}$-pointed stable maps with target X, genus g and degree β, $\mathrm{ev}_i\colon\overline{\mathcal{M}}_{g,\mathfrak{n}}(X,\beta)\to X$ are the evaluation morphisms and $\phi\colon\overline{\mathcal{M}}_{g,\mathfrak{n}}(X,\beta)\to\overline{\mathcal{M}}_{g,\mathfrak{n}}$ is the morphism forgetting the map.

Equation (6.3) defines then a formal power series $F_0^X\in\Lambda[\![\boldsymbol{t}]\!]$, called the genus 0 Gromov-Witten potential of X. The corresponding formal Frobenius manifold over $k=\Lambda$ is the *quantum cohomology* of X. In order to work with formal Frobenius manifold over $\mathbb{C}$ we make the following assumption.

Assumption A: There exist a point $\boldsymbol{q}\in\mathbb{C}^r$ such that the series $\int_{\overline{\mathcal{M}}_{0,\mathfrak{n}}}\Omega_{0,\mathfrak{n}}\left(\bigotimes_{i=1}^{\mathfrak{n}}\Delta_{\alpha_i}\right)|_{\mathbf{Q}=\boldsymbol{q}}$ are convergent for any $\mathfrak{n}\geqslant 3$.

If Assumption A holds true, then the specialization $F_0^X|_{\mathbf{Q}=\boldsymbol{q}}$ is a formal power series in $\mathbb{C}[\![\boldsymbol{t}]\!]$. We call *big quantum cohomology of X (at $\mathbf{Q}=\boldsymbol{q}$)* the corresponding formal Frobenius manifold over $\mathbb{C}$. We call *small quantum cohomology of X (at $\mathbf{Q}=\boldsymbol{q}$)* the Frobenius $\mathbb{C}$-algebra structure defined on $H^\bullet(X;\mathbb{C})$ with structure constants $c_{\alpha\beta}^\gamma:=\eta^{\gamma\mu}\int_{\overline{\mathcal{M}}_{0,3}}\Omega_{0,3}\left(\Delta_\alpha\Delta_\beta\Delta_\mu\right)|_{\mathbf{Q}=\boldsymbol{q}}$.

Remark 6.3. Assumption A holds true for all Fano varieties. This is because any sum $\sum_\beta$ in (6.4) reduces to a finite number of terms, so that $\Omega_{0,n}\left(\bigotimes_{i=1}^{\mathfrak{n}}\Delta_{\alpha_i}\right)\in\mathbb{Q}[\mathbf{Q}]$. See e.g. [CK99, Prop. 8.1.3].

Remark 6.4. By the Divisor axiom of Gromov-Witten invariants, it follows that the potential F_0^X can be seen as a formal power series in $\mathbb{Q}[\![t^1,Q_1e^{t^2},\dots,Q_re^{t^{r+1}},t^{r+2},\dots,t^n]\!]$, see [CK99, Man99]. If Assumption A holds true, without loss of generalities we can assume that $\boldsymbol{q}=(1,1,1,\dots,1)$: this correspond to a shift of coordinates $t^{i+1}\mapsto t^{i+1}-\log q_i$ for $i=1,\dots,r$.

Remark 6.5. If X has generically semisimple quantum cohomology (as a formal Frobenius manifold over Λ), then X is of Hodge-Tate type, i.e. the Hodge numbers $h_{p,q}(X):=\dim_{\mathbb{C}}H^q(X,\Omega^p)$ vanish for $p\neq q$, see [HMT09].

Theorem 5.1 implies then the following result.

Theorem 6.6. *Assume that Assumption A holds true. Then, if the small quantum cohomology of X at $\boldsymbol{q}$ is semisimple, then the function $F_0^X(\boldsymbol{t})|_{\mathbf{Q}=\boldsymbol{q}}$ has a non-empty domain of convergence $M_{\boldsymbol{q}}\subseteq H^\bullet(X;\mathbb{C})$, which is equipped of a Dubrovin-Frobenius manifold structure.* $\square$

Theorem 6.6 should be compared with other results in literature, differing in techniques. In [Iri07], H. Iritani proved convergence of the big quantum cohomology of X under a different assumption, namely that $H^\bullet(X;\mathbb{C})$ is generated by $H^2(X;\mathbb{C})$, see [Iri07, Corollary 5.9]. Subsequently, in [CI15] T. Coates and H. Iritani proved the convergence (suitably defined) of all potentials $\mathcal{F}_g^X$ given by (6.1), by assuming both convergence of F_0^X and semisimplicity.

Whenever the three-point Gromov-Witten correlators $\int_{\overline{\mathcal{M}}_{0,3}}\Omega_{0,3}\left(\Delta_\alpha\Delta_\beta\Delta_\mu\right)$ of X are explicitly known, and thus generators and relations for the small quantum cohomology ring are given, it is a problem purely in computational commutative algebra to check generic semisimplicity of the small quantum cohomology. Here, we limit ourselves to the following claim[6], which easily follows from [BM04, BM19, CMP10, Cio04, Cio05, Iri07, Per14].

Corollary 6.7. *We have $F_0^X\in\mathbb{Q}\{\mathbf{Q},\boldsymbol{t}\}$ in the following cases (not mutually excluding):*

(1) *$X=G/P$ is a (co)minuscule homogeneous variety;*
(2) *X is a del Pezzo surface;*
(3) *X is a Fano toric variety;*
(4) *X is one of the following Fano threefolds:*
   - $\mathbb{P}^3$*, a quadric* Q_3, V_5, V_{22}*,*
   - M_k^2 *with* $21\leqslant k\leqslant 36$ *and* $k\neq 23,25,28$*,*
   - M_k^3 *with* $k=10,12,15,17,18,20,24,25,27,28,30,31$*,*
   - $\mathbb{P}^1\times\mathbb{P}_k^2$ *where* $\mathbb{P}_k^2$ *is the blow-up of* $\mathbb{P}^2$ *at* k *points* $(1\leqslant k\leqslant 8)$*;*
(5) *X is a Fano general hyperplane section with index $i(X)>\frac{1}{2}\dim_{\mathbb{C}}X$ of a homogeneous space in the following list:*

$$\begin{gathered}\mathbb{P}^n,\quad \text{the } n\text{-dimensional quadric } Q_n,\quad LG(3,6),\quad F_4/P_1\\ Gr(2,2n+1),\quad OG(5,10),\quad OG(2,2n+1),\quad G_2/P_1;\end{gathered}$$

(6) *X is the Cayley Grassmannian parametrizing four dimensional subalgebras of the complex octonions.* □

Remark 6.8. It is known that there exist homogeneous spaces with non-semisimple small quantum cohomology, [CMP10, CP11]. Isotropic Grassmannians $IG(2,2n)$ furnish an example. It is also known, however, that their big quantum cohomology is generically semisimple [GMS15, Per14, CMMPS19]. For these varieties, the results of the current paper do not allow to infer the convergence of the genus zero Gromov–Witten potential, a working assumption in [CMMPS19, Th. B].

There is an intriguing conjecture due to B. Dubrovin [Dub98, Conj. 4.2.2] stating the equivalence of the semisimplicity of the (big) quantum cohomology of a variety X (originally assumed to be Fano) and the existence of full exceptional collections in the derived category of coherent sheaves $\mathcal{D}^b(X)$. In its most updated formulation, under the assumption of convergence of the genus zero Gromov-Witten potential F_0^X, Dubrovin's conjecture also predicts the monodromy data of the system (5.9) (in the terminology of the current paper, the admissible data $\mathfrak{M}$) in terms of characteristic classes of the objects of these exceptional collections, see [GGI16, CDG18, Cot20]. In [Dub98, §4.2, Problem 1] Dubrovin also briefly addressed the problem of convergence of the genus zero Gromov-Witten potential F_0^X. In

[6]Surely enough, such a list does not cover all the known cases of semisimple small quantum cohomologies available in literature.

this regard, Dubrovin adds: «*Hopefully, in the semisimple case the convergence can be proved on the basis of the differential equations of n.3*». Theorems 6.6 fulfills Dubrovin's hope.

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