

On SQED₃ and SQCD₃: phase transitions and integrability

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We study supersymmetric Yang-Mills theories on the three-sphere, with massive matter and Fayet-Iliopoulos parameter, showing second order phase transitions for the non-Abelian theory, extending a previous result for the Abelian theory. We study both partition functions and Wilson loops and also discuss the case of different R -charges. Two interpretations of the partition function as eigenfunctions of the A_1 and free A_{N-1} hyperbolic Calogero-Moser integrable model are given as well.

The study of supersymmetric gauge theories in curved space-times has been pushed forward considerably in the last decade due to the extension of the localization method of path integrals [1, 2]. By using localization, a much simpler integral representation of the observables of the gauge theories is achieved. In turn, these seemingly simple representations, in general of the matrix model type, contain a wealth of information of different type. First, they are very useful for asymptotic analysis and, in suitable large N double scaling limits, have predicted phase transitions in the theory [3–6]. Secondly, in many cases, especially for three dimensional theories, they are amenable to exact analytical solutions, even for finite N [4, 7]. Such exact evaluation, or the procedure leading to it, oftentimes may point towards a connection between the gauge theory and, for example, integrable systems [8].

All these aspects of the localization integral formulas will be exposed in what follows, as we will not only study finite and large N properties, together with phase transitions in double scaling limits, but also give an integrable systems view of the gauge theory, by showing a connection with the hyperbolic Calogero-Moser system.

In what follows, we will consider $\mathcal{N} = 4$ theory on the $3d$ sphere \mathbb{S}^3 , with gauge group $U(n)$ and an even number $N_f = 2N$ of massive chiral multiplets in the fundamental, N of them with mass m and N with mass $-m$, arranged into N hypermultiplets. We also insert a Fayet-Iliopoulos (FI) term. Localization [2, 9, 10] gives the integral representation of the partition function:

$$\mathcal{Z}_N^{U(n)} = \int_{\mathbb{R}^n} d^n x \prod_{1 \leq j < k \leq n} \left(2 \sinh \frac{x_j - x_k}{2} \right)^2 \times \prod_{j=1}^n \frac{e^{i\eta x_j}}{2^N [\cosh(x_j) + \cosh(m)]^N}, \quad (1)$$

where we set the radius of \mathbb{S}^3 to $1/2\pi$ and η is the FI parameter. We will eventually be interested in the limit in which the number of flavours $N_f = 2N$ is large, while the number of colours n is kept finite. Therefore, we consider $N_f = 2N \geq 2n$, so that the integral (1) is convergent, besides the theory is “good” (or “ugly”, if $N = n$) according to the classification [11].

The Abelian case $n = 1$ was studied in detail in [5]. In what follows, we will extend the results of [5], including $1/N$ corrections and the analysis of Wilson loops, as well as carrying over the study to non-Abelian theories, $n > 1$. In the simplest non-Abelian case $n = 2$ we will also compute $1/N$ corrections to the large N limit.

Abelian theory at finite N . The partition function of the Abelian theory reads:

$$\mathcal{Z}_N^{U(1)} = 2^{-N} \int_{-\infty}^{+\infty} dx e^{i\eta x} [\cosh(x) + z]^{-N}, \quad (2)$$

where $z \equiv \cosh(m)$. The expression is significantly simpler than any non-Abelian case, since the one-loop determinant of the vector multiplet is trivial for $n = 1$. The partition function (2) can be computed exactly in terms of a hypergeometric function [5], as

$$\mathcal{Z}_N^{U(1)} = \frac{\sqrt{2\pi}}{2^N (1+z)^{N-\frac{1}{2}}} \frac{\Gamma(N+i\eta)\Gamma(N-i\eta)}{\Gamma(N)\Gamma(N+\frac{1}{2})} \times {}_2F_1\left(\frac{1}{2}-i\eta, \frac{1}{2}+i\eta, N+\frac{1}{2}, \frac{1-z}{2}\right). \quad (3)$$

Using an Euler transformation for the hypergeometric [12, Ch.2], we can rewrite (3) when $\eta \geq 1, m \geq 1$ as:

$$\mathcal{Z}_N^{U(1)} = \frac{e^{i\eta m}}{2^N (\sinh(m))^N} \frac{\Gamma(N-i\eta)\Gamma(i\eta)}{\Gamma(N)} \times {}_2F_1(1-N, N, 1-i\eta, -(e^{2m}-1)^{-1}) + (\text{replace } i\eta \longleftrightarrow -i\eta). \quad (4)$$

This latter form is illustrative: since the first coefficient, $a = 1 - N$, is a nonpositive integer, the hypergeometric series terminates and gives a polynomial of degree $N - 1$ in the variable $y \equiv -(e^{2m} - 1)^{-1}$. Moreover, in our case the second coefficient $b = N = 1 - a$, thus the hypergeometric function is actually an associated Legendre function of imaginary order [13]:

$${}_2F_1(1-N, N, 1-i\eta, y) = \Gamma(1-i\eta) \left(\frac{y}{1-y}\right)^{\frac{i\eta}{2}} P_{N-1}^{i\eta}(1-2y).$$

The partition function reads:

$$\mathcal{Z}_N^{U(1)} = \frac{\pi e^{-\frac{\pi\eta}{2}} \Gamma(N - i\eta)}{2^N i \sinh(\pi\eta) \sinh(m)^N \Gamma(N)} P_{N-1}^{i\eta}(\coth(m)) \\ + (\text{replace } i\eta \longleftrightarrow -i\eta),$$

where we used the property $\Gamma(1 - i\eta)\Gamma(i\eta) = \pi / \sin(i\pi\eta)$.

We can represent the function (3) in yet another form, in terms of a conical function [5, 14]:

$$\mathcal{Z}_N^{U(1)} = \frac{\sqrt{2\pi}}{2^N (\sinh(m))^{N-\frac{1}{2}}} \frac{\Gamma(N + i\eta)\Gamma(N - i\eta)}{\Gamma(N)} P_{-\frac{1}{2}+i\eta}^{\frac{1}{2}-N}(z),$$

where $P_{-\frac{1}{2}+i\eta}^{\frac{1}{2}-N}(z)$ is an associated Legendre function of negative order and complex degree. This latter form is the most suitable to study the asymptotics for large mass. Indeed, when $m \rightarrow \infty$, $z = \cosh(m) \rightarrow \infty$ as well and we can use the approximation of [15]:

$$P_{-\frac{1}{2}+i\eta}^{\frac{1}{2}-N}(z) \approx \sqrt{\frac{2\pi}{z}} \frac{\sin(\eta \log(2z) + \theta_1 + \theta_2)}{\sinh(\pi\eta) |\Gamma(1 + i\eta)\Gamma(N + i\eta)|} \\ = \sqrt{\frac{2}{\pi z}} \frac{\sin(\eta \log(2z) + \theta_1 + \theta_2)}{\prod_{k=0}^{N-1} \sqrt{k^2 + \eta^2}},$$

where $\theta_1 = \arg \Gamma(1 + i\eta)$ and $\theta_2 = \arg \Gamma(N - i\eta)$, and in the second line we used elementary identities for the Γ function. Altogether, and approximating the hyperbolic functions for $m \rightarrow \infty$, we have:

$$\mathcal{Z}_N^{U(1)} \approx \frac{e^{-mN} \pi \prod_{k=1}^{N-1} \sqrt{k^2 + \eta^2}}{2^{N-1} \Gamma(N) \sinh(\pi\eta)} \sin(\eta m + \theta_1 + \theta_2). \quad (5)$$

This approximation is in agreement with the large mass approximation found in [5] (Eq. (8) therein) applying a different Euler transformation to (3), which led to:

$$\mathcal{Z}_N^{U(1)} \approx \frac{2\pi e^{-mN}}{\Gamma(N) \sinh(\pi\eta)} \mathfrak{S} \left(e^{im\eta} \prod_{k=1}^{N-1} (k - i\eta) \right). \quad (6)$$

See figure 1 for the match of expressions (5) and (6).

The exact evaluation (3) of the partition function, or its equivalent representation as a conical function, relies on the hypothesis $\cosh(m) \geq 1$, thus on reality of the mass. However, the dependence of $\mathcal{Z}_N^{U(1)}$ on m should be holomorphic [9, 16]. For arbitrary complex masses the integral (2) can be evaluated by residue theorem [17], and we checked for many values of N that the result coincide with the prolongation of (3) to complex masses.

Integrability. The partition function satisfies the second-order differential equation [5]

$$\frac{d^2 \mathcal{Z}_N}{dm^2} + 2N \coth(m) \frac{d\mathcal{Z}_N}{dm} + (\eta^2 + N^2) \mathcal{Z}_N = 0, \quad (7)$$

which becomes the Schrödinger equation with a hyperbolic Pöschl-Teller potential, for the function $Z(m) =$

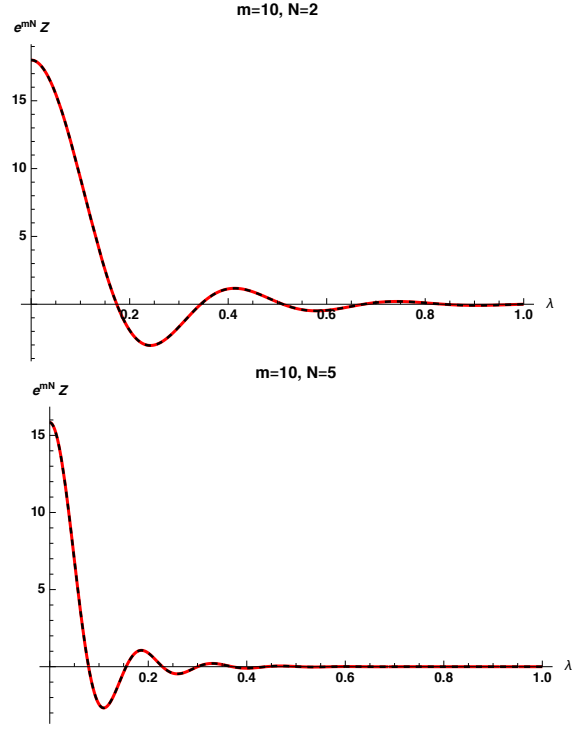


FIG. 1. Approximation of $e^{mN} \mathcal{Z}_N^{U(1)}$ at large $m = 10$ as a function of $\lambda = \eta/N$, using (5) (red) and (6) (black, dashed), for $N = 2$ (above) and $N = 5$ (below).

$\sinh(m)^N \mathcal{Z}_N$ [5]. This quantum mechanical model has a discrete energy spectrum [18], and $Z(m)$ represents the wave function of a state with positive energy proportional to η^2 . Furthermore, the fact that the potential appears with integer coefficient N implies that the wave function propagates without reflection.

The appearance of the quantum mechanical interpretation with a solvable Pöschl-Teller potential immediately suggests a possible role of the hyperbolic Calogero-Moser model, the celebrated integrable system, which can be seen as the many-body generalization of the quantum mechanical problem above. The Hamiltonian of the $A_{\widehat{N}-1}$ hyperbolic Calogero-Moser model is [19, 20]

$$H = \sum_{1 \leq j < k \leq \widehat{N}} \left[-\hbar^2 \partial_{x_j} \partial_{x_k} + \frac{g(g - \hbar) \mu^2}{4 \sinh^2(\mu(x_j - x_k)/2)} \right], \quad (8)$$

and there exists $\widehat{N} - 1$ additional independent partial differential operators H_l of order l , such that the PDOs form a commutative family. The simplest is the momentum operator

$$H_1 = -i\hbar \sum_{j=1}^{\widehat{N}} \partial_{x_j} \quad (9)$$

whereas the others are made of correspondingly higher derivatives (and lower order terms as well). Here, $\widehat{N} =$

$n + 1$. Consider the two-particle case, the family is then the Hamiltonian and the momentum operator, (8) and (9).

The result in what follows appears to have some similarities with the work [21] (further extended in [22, 23]) where conformal blocks of scalar 4-point functions in d -dimensional conformal field theory are mapped to eigenfunctions of the two particle hyperbolic Calogero-Moser system. The relevant model there corresponds to the BC_2 case rather than the A_1 or $A_{\widehat{N}-1}$ here (see below), due to the orthogonal symmetry there.

Using recent work on the construction, by a recursive method, of the joint eigenfunctions of this integrable system [20], we show now that the Abelian theory above can be identified with this two-particle A_1 hyperbolic Calogero-Moser, where the coupling constant g in (8) will be identified with the half-number of flavours N . In particular, this two-particle interpretation follows from considering the function

$$\Psi_2(g; x, y) \equiv e^{iy_2(x_1+x_2)} \int_{-\infty}^{\infty} e^{i(y_1-y_2)z} K_2(g; x, z) dz,$$

where the kernel, with $g > 0$, $x, y \in \mathbb{R}^2$, is

$$K_2(g; x, z) = \frac{[4 \sinh^2(x_1 - x_2)]^{g/2}}{\prod_{j=1}^{\widehat{N}} [2 \cosh(x_j - z)]^g},$$

and is central in the recursion, taking the $\widehat{N} - 1$ eigenfunction to the \widehat{N} eigenfunction. The connection with the function $Z(m)$ defined above follows immediately from the identifications $g = N$, $x_1 = m/2 = -x_2$ and $(y_1 - y_2)/2 = \eta$. It is shown in [20] that

$$\begin{aligned} H_1 \Psi_2(x, y) &= (y_1 + y_2) \Psi_2(x, y), \\ H \Psi_2(x, y) &= (y_1^2 + y_2^2) \Psi_2(x, y). \end{aligned}$$

A different type of connection also exists relating the non-Abelian theory, with $\widehat{N} = N$, with the free case of the integrable system, given by $g = \hbar$ in (8). Using the customary adimensional coupling $\widehat{\lambda} \equiv g/\hbar = 1$, (8) is then the free N -body Hamiltonian. Thus, there is no identification here between g and number of flavours and is a very different relationship compared to the two-particle one. The integral representation given for $\Psi_N(\widehat{\lambda}; x, y)$ [20] is then evaluated exactly for $\widehat{\lambda} = 1$ and the explicit expression [20, Theorem 3.1.] is the one for the partition function of the $T[SU(N)]$ linear quiver [17, 24, 25].

The relationship between the integral expressions in [20] and the well-known Heckman-Opdam hypergeometric functions [26], which are also relevant in [21, 22], is explained in [20]. By factorizing Ψ_N in two pieces, one describing the centre of mass, it is shown in [20] that

the remaining piece is the A_{N-1} Heckman-Opdam hypergeometric function. In terms of two sets of N variables $(m_j, \zeta_j)_{j=1}^N$, this hypergeometric satisfies the condition $\sum_j m_j = 0 = \sum_j \zeta_j$, with $\zeta_j \in \mathbb{R}$ and complex m_j such that $|\Im(m_j - m_k)| < \pi$, cfr. [20, Theorem 7.1]. On the gauge theory side, those are exactly the constraints on the $T[SU(N)]$ theory [17], the first being the $SU(N)$ flavour symmetry and the latter arising from the redundancy of the N number of ζ_j variables, defined from the original $N - 1$ FI parameters as $\zeta_j = \eta_j - \eta_{j+1}$ [27]. We underline that the partition function of the $T[SU(N)]$ quiver is evaluated for real masses and FI parameters, but can, by holomorphicity, hold on the stripes $|\Im(m_j - m_k)| < \pi$, hence the identification is exact.

Abelian theory at large N . Sending $N \rightarrow \infty$ in the double scaling limit with $\lambda \equiv \eta/N$ fixed, the leading contribution to the partition function (2) comes from the saddle points of the action

$$S_1(x) = -i\lambda x + \frac{\sinh(x)}{\cosh(x) + z}, \quad (10)$$

which are given by the set $\mathcal{S} = \{x_s^\pm + i2\pi k, k \in \mathbb{Z}\}$, with

$$x_s^\pm = \log \left(\frac{-\lambda z \pm i\Delta}{i + \lambda} \right), \quad (11)$$

where $\Delta \equiv \sqrt{1 - \lambda^2 \sinh(m)^2}$ and we recall that $z \equiv \cosh(m)$. The curve $\lambda \sinh(m) = 1$ determines a critical line in parameter space, along which the free energy $\mathcal{F} = -\frac{1}{N} \log \mathcal{Z}$ has a discontinuity in its second derivative. In the *sub-critical phase* $\lambda \sinh(m) < 1$, the leading contribution comes from x_s^+ and $k = 0$, while in the *super-critical phase* $\lambda \sinh(m) > 1$ both x_s^\pm contribute, being complex conjugate and $S_1(x_s^-) = S_1(x_s^+)^*$.

Close to the saddle points $\bar{x} \in \mathcal{S}$, we can change variables $x = \bar{x} + t/\sqrt{N}$ and expand

$$S_1(x) = S_1(\bar{x}) + \frac{t^2 S_1''(\bar{x})}{2N} + \frac{t^3 S_1'''(\bar{x})}{6N^{\frac{3}{2}}} + \frac{t^4 S_1^{(iv)}(\bar{x})}{24N^2} + \dots$$

We now plug this expansion into (2) and keep the Gaussian part in t exponentiated, while expanding the rest of the exponential function. Elementary integration provides:

$$\begin{aligned} \mathcal{Z}^{U(1)} &= 2^{-N} \sqrt{\frac{2\pi}{N}} \sum_{\bar{x} \in \mathcal{S}} \frac{e^{-NS_1(\bar{x})}}{\sqrt{S_1''(\bar{x})}} \left[1 + \right. \\ &\quad \left. \frac{1}{24N} \left(\frac{5S_1'''(\bar{x})}{(S_1''(\bar{x}))^3} - \frac{3S_1^{(iv)}(\bar{x})}{(S_1''(\bar{x}))^2} \right) + \mathcal{O}(N^{-2}) \right]. \end{aligned}$$

The relevant expressions for the derivatives of the action S_1 are reported in the Appendix A. When $\lambda \sinh(m) < 1$,

only x_s^+ contributes, and we get:

$$\mathcal{Z}_{\text{sub.}}^{U(1)} = 2^{-N} \sqrt{\frac{2\pi}{N}} \frac{e^{-NS_1(x_s^+)}}{\sqrt{S_1''(x_s^+)}} \left[1 + \frac{1}{24N} \left(\frac{5S_1'''(x_s^+)}{(S_1''(x_s^+))^3} - \frac{3S_1^{(iv)}(x_s^+)}{(S_1''(x_s^+))^2} \right) \right] + \mathcal{O}(N^{-2}),$$

while in the supercritical phase $\lambda \sinh(m) > 1$ both x_s^\pm must be taken into account, leading to:

$$\mathcal{Z}_{\text{super.}}^{U(1)} = 2\Re \left(\mathcal{Z}_{\text{sub.}}^{U(1)} \right) + \mathcal{O}(N^{-2}).$$

Dropping sub-leading corrections, one can evaluate \mathcal{F} in both phases:

$$\mathcal{F}_{\text{sub.}}^{U(1)} = S_1(x_s^+), \quad \mathcal{F}_{\text{super.}}^{U(1)} = \Re \left(S_1(x_s^+) \right), \quad (12)$$

with discontinuous second derivative:

$$\frac{\partial^2 \mathcal{F}_{\text{sub.}}^{U(1)}}{\partial \lambda^2} - \frac{\partial^2 \mathcal{F}_{\text{super.}}^{U(1)}}{\partial \lambda^2} = \frac{z}{(1 + \lambda^2)\Delta}.$$

Therefore, not only the susceptibility $\frac{\partial^2 \mathcal{F}}{\partial \lambda^2}$ is discontinuous, but it is divergent as $(\lambda - \lambda_c)^{-\gamma_c}$, and we identify the critical exponent $\gamma_c = \frac{1}{2}$. The free energy yields analogous discontinuity with respect to the mass:

$$\frac{\partial^2 \mathcal{F}_{\text{sub.}}^{U(1)}}{\partial m^2} - \frac{\partial^2 \mathcal{F}_{\text{super.}}^{U(1)}}{\partial m^2} = \frac{z\Delta}{\sinh(m)^2} - \frac{\lambda z}{\Delta},$$

hence the critical exponent for the mass is again $\delta_c = \frac{1}{2}$.

In figure 2 we present the convergence of the exact solution (3) and the large N expression (12) as N is increased.

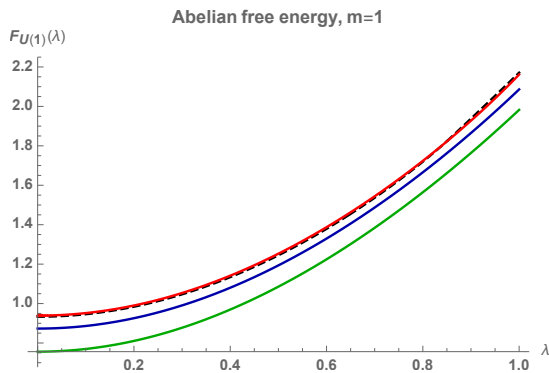


FIG. 2. Exact solution of $\mathcal{F}^{U(1)}$ as a function of $\lambda = \eta/N$ at $m = 1$, for $N = 4, 7, 20$ (in green, blue, red, respectively) and large N expression (black, dashed).

Wilson loops. Irreducible complex representations of $U(1)$ are labelled by $r \in \mathbb{Z}$, thus Wilson loops can be

written as $W_r = \text{Tr}_r e^x = e^{rx}$ (recall that the radius of the three-sphere is $1/2\pi$), and their expectation value is:

$$\begin{aligned} \langle W_r \rangle &= \frac{1}{2^N \mathcal{Z}_N^{U(1)}} \int_{-\infty}^{+\infty} dx \frac{e^{(i\eta+r)x}}{[\cosh(x) + z]^N} \\ &= \frac{\Gamma(N+r+i\eta)\Gamma(N-r-i\eta)}{\Gamma(N+i\eta)\Gamma(N-i\eta)} \\ &\times \frac{{}_2F_1\left(\frac{1}{2}-r-i\eta, \frac{1}{2}+r+i\eta, N+\frac{1}{2}, \frac{1-z}{2}\right)}{{}_2F_1\left(\frac{1}{2}-i\eta, \frac{1}{2}+i\eta, N+\frac{1}{2}, \frac{1-z}{2}\right)}, \quad (13) \end{aligned}$$

where we stress that the insertion of a Wilson loop is analogous to the complexification of the FI coupling. The integral representation (13) is well-defined as $\eta \rightarrow 0$ only for representations of size $|r| < N$: this is reflected in the poles of the Γ function at negative integers.

The quantum mechanical interpretation carries over for the Wilson loop without FI term, $\eta = 0$. In this case, $w_r \equiv [\sinh(m)^N \mathcal{Z}_N \langle W_r \rangle]_{\eta=0}$ satisfies the Schrödinger equation with Pöschl-Teller potential:

$$\left[\frac{d^2}{dm^2} - \frac{N(N-1)}{\sinh(m)^2} \right] w_r = r^2 w_r.$$

The latter equation describes the wave function of a bound state with energy proportional to r^2 , for integer $|r| < N$, which is indeed the case at hand [18].

For $\eta \neq 0$, however, the resulting potential acquires an imaginary part, seemingly spoiling unitarity of the evolution operator and producing a dissipation-like term in the probability conservation.

At large N with the size r of the representation fixed, the Wilson loop can be approximated by the value of the integrand in (13) at the saddle points. Nevertheless, we can also consider the case of large representations, in which r scales with N , *i.e.* $f \equiv r/N$ is kept fixed as $N \rightarrow \infty$. Let us turn off the FI term for simplicity, $\eta = 0$, the saddle points of the action are given by:

$$\bar{x} = \log \left(\frac{f \cosh(m) \pm \sqrt{1 + f^2 \sinh(m)^2}}{1 - f} \right) + i2\pi k,$$

with $k \in \mathbb{Z}$, that are real for every $-1 < f < 1$ [28]. Therefore, the Wilson loops without FI term do not experience phase transition. The limit with both η and r scaling with N is commented in Appendix B.

J_3 correlators. We can also consider other families of operators, besides Wilson loops. Higgs branch operators in $3d \mathcal{N} = 4$ can be analyzed through localization techniques [29], and therefore represent a suitable choice for the present setting. In particular, we focus our attention on the gauge invariant, quadratic operator

$$J_3 = \frac{1}{N} \left[\tilde{Q}_{+,j} Q_+^j - \tilde{Q}_{-,j} Q_-^j \right],$$

where $Q_{\pm,j}$, $j = 1, \dots, N$, are the hypermultiplets of mass $\pm m$. The expectation value of this operator is [5]

$$\langle J_3 \rangle = \frac{1}{2N \mathcal{Z}_N} \frac{d\mathcal{Z}_N}{dm},$$

and correlation functions of J_3 are generated by higher derivatives.

The differential equation (7) satisfied by \mathcal{Z}_N can be translated into a recursion relation for correlators of J_3 :

$$\langle J_3 J_3 \rangle = -\coth(m) \langle J_3 \rangle - \frac{1}{4N} \left(1 + \frac{\eta^2}{N^2} \right).$$

Taking the first derivative of Eq. (7) gives $\frac{d^3 \mathcal{Z}_N}{dm^3}$ as a function of the first and second derivative of \mathcal{Z}_N , but the second order term can be eliminated using (7). Hence, we immediately obtain:

$$\begin{aligned} \langle J_3 J_3 J_3 \rangle &= \langle J_3 \rangle \left[\frac{2N \cosh(m)^2 + 1}{2N \sinh(m)^2} - \frac{1}{4} \left(1 + \frac{\eta^2}{N^2} \right) \right] \\ &+ \frac{1}{4} \left(1 + \frac{\eta^2}{N^2} \right). \end{aligned}$$

One can take further derivatives and systematically plug (7) in the resulting expression. This allows to recursively compute k -point correlation functions of J_3 : exploiting Eq. (7), the final result will be an expression only in terms of $\langle J_3 \rangle$, hyperbolic functions of m and polynomials in $(1 + \eta^2/N^2)$.

Non-Abelian theory: $SU(2)$. The simplest non-Abelian theory corresponds to the gauge group $SU(2)$. The partition function is again a single integral, but now the one-loop determinant of the vector multiplet contributes. Also, the $SU(2)$ vector multiplet cannot be coupled to an FI background, therefore $\eta = 0$. The partition function is:

$$\mathcal{Z}_N^{SU(2)} = \int_{-\infty}^{+\infty} dx \frac{\sinh(x)^2}{2^N [\cosh(x) + z]^N}.$$

Writing $\sinh(x)$ in terms of exponentials, we can see the $SU(2)$ partition function as a combination of expectation values of Wilson loops in the Abelian theory:

$$\mathcal{Z}_N^{SU(2)} = \left[\frac{\mathcal{Z}_N^{U(1)}}{2} (\langle W_2 \rangle - 2 + \langle W_{-2} \rangle) \right]_{\eta=0},$$

with the expectation value $\langle W_r \rangle$ given in Eq. (13).

Due to the absence of FI term, the unique saddle point is $x_s = 0$, and the phase structure at large N is trivial.

Non-Abelian theory: $U(2)$. We now apply the same procedure to the $U(2)$ theory, *i.e.* two colours. Specialization of (1) for $n = 2$ gives:

$$\mathcal{Z}_N^{U(2)} = \int_{\mathbb{R}^2} \frac{e^{i\eta(x_1+x_2)} (2 \sinh \frac{x_1-x_2}{2})^2 dx_1 dx_2}{2^{2N} [(\cosh(x_1) + z)(\cosh(x_2) + z)]^N}, \quad (14)$$

where, as above, $z \equiv \cosh(m)$. Through the equivalent representation of (14) as a determinant, one could write an exact solution

$$\mathcal{Z}_N^{U(2)} = 2! \det_{1 \leq j, k \leq 2} [Z_{jk}],$$

with Z_{jk} entries of a 2×2 matrix formally given by (3) up to a shift in the FI coupling $i\eta \mapsto i\eta + j + k - 2$, $j, k \in \{1, 2\}$. This equals the determinant of a matrix whose entry (j, k) is the expectation value, in the Abelian matrix model, of a Wilson loop in the irreducible representation labelled by $j + k - 2$:

$$\mathcal{Z}_N^{U(2)} = 2(\mathcal{Z}_N^{U(2)})^2 (\langle W_2 \rangle - \langle W_1 \rangle^2).$$

To study (14) in the limit in which the number of flavours N is large, we notice that the interaction between eigenvalues is sub-leading in $1/N$, thus the saddle points of the $U(2)$ theory are those of the action $S_1(x_1) + S_1(x_2)$:

$$\mathcal{S}^2 = \{ (x_s^\pm + 2\pi k_1, x_s^\pm + 2\pi k_2), k_{1,2} \in \mathbb{Z} \}.$$

We proceed as in the Abelian case: we change variables $x_{1,2} = \bar{x}_{1,2} + t_{1,2}/\sqrt{N}$ and expand both the action and the hyperbolic interaction around the saddle point (\bar{x}_1, \bar{x}_2) . Expanding up to $\mathcal{O}(N^{-1})$ and integrating we obtain, for the *sub-critical phase*:

$$\begin{aligned} \mathcal{Z}_{\text{sub.}}^{U(2)} &= \frac{\pi}{2^{2(N-1)} N^2} \frac{e^{-2NS_1(x_s^+)}}{(S_1''(x_s^+))^2} \left[1 + \right. \\ &+ \left. \frac{1}{2N} \left(\frac{1}{S_1''(x_s^+)} + \frac{17(S_1'''(x_s^+))^2}{6(S_1''(x_s^+))^3} - \frac{3S_1^{(iv)}(x_s^+)}{2(S_1''(x_s^+))^2} \right) \right], \end{aligned}$$

while the expression in the *super-critical phase* $\lambda \sinh(m) > 1$ is a sum of four pieces, and is reported in Appendix C.

Dropping $1/N$ corrections, the free energy is simply $\mathcal{F}^{U(2)} = 2\mathcal{F}^{U(1)}$, in particular the phase transition is second order with the same critical exponent $\gamma_c = \frac{1}{2}$. In figure 3 we show how the exact solution approaches the large N expression as N is increased.

We study the most general non-Abelian case in Appendix D, and only report here the main result. The free energy at large N of the $U(n)$ theory is n times the free energy of the Abelian theory:

$$\mathcal{F}^{U(n)} = n\mathcal{F}^{U(1)}.$$

Other R -charges. To conclude, we show how the features of the $\mathcal{N} = 4$ theory with $2N$ chiral multiplets with R -charge $q = \frac{1}{2}$ can be extended to the $\mathcal{N} = 2$ theory with $2N$ chiral multiplets with more general assignment of R -charge q . The expressions for the partition function and the saddle point equation for arbitrary q are reported in Appendix E. Here we comment on how the theory at half-integer $q \in \frac{1}{2}\mathbb{Z}$ can be obtained by simple modification of the results in [5].

$q = 1$. In this case the action is pure imaginary, already at finite N , and admits no saddle point.

$q \in \frac{1}{2} + \mathbb{Z}$. The saddle point equation reduces to:

$$\frac{\sinh(x)}{\cosh(x) + z} = \frac{i\lambda}{2(1-q)},$$

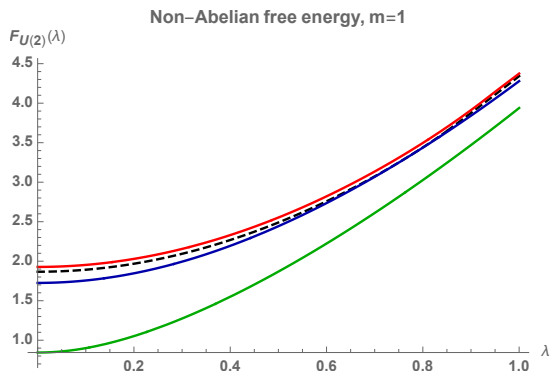


FIG. 3. Exact solution from determinants of $\mathcal{F}^{U(2)}$ as a function of $\lambda = \eta/N$ at $m = 1$, for $N = 4, 7, 100$ (in green, blue, red, respectively) and large N expression (black, dashed).

and the large N behaviour is identical to the case $q = \frac{1}{2}$ upon scaling $\lambda \mapsto \frac{\lambda}{2(1-q)}$.

$q \in \mathbb{Z} \setminus \{1\}$. For integer non-unit q the saddle point equation simplifies into:

$$\frac{\sinh(x)}{\cosh(x) - z} = \frac{i\lambda}{2(1-q)},$$

and the phase structure at large N is identical to the case $q = \frac{1}{2}$, up to scaling $\lambda \mapsto \frac{\lambda}{2(1-q)}$ and replace in the formulae $z \mapsto -z$. The critical line is $\lambda \sinh(m) = 2|1-q|$.

As a future direction, it would be interesting to study the large N free energy for more general R -charges and determine the R -symmetry in the IR by \mathcal{F} -extremization [9, 16]. A crucial question then would be whether there exists more than one solution q_{IR} , and analyze the corresponding theories as a function of λ , along the lines of [30, 31].

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Appendix A. Derivatives of the action S_1

Here we present the full expressions for the derivatives of the action in the Abelian theory, evaluated at the saddle point $\bar{x} = x_s^+$. In what follows, we denote $z \equiv \cosh(m)$, $\xi \equiv \lambda \sinh(m)$ and $\Delta \equiv \sqrt{1 - \lambda^2 \sinh(m)^2}$.

$$\begin{aligned} S_1(x_s^+) &= \log \left(\frac{\Delta z + i\lambda \sinh(m)^2 + 1}{(i + \lambda)(\lambda z - i\Delta)} \right) - i\lambda \log \left(\frac{-\lambda z + i\Delta}{i + \lambda} \right), \\ S_1''(x_s^+) &= \lambda^2 \left[1 + \frac{z\Delta - 1}{\xi^2} \right], \\ S_1'''(x_s^+) &= \frac{\lambda(1 - i\lambda)}{4 \sinh(m)^2 (\lambda \sinh(m)^2 - iz\Delta - i)^2} [2\xi - 6\xi z\Delta + 2(4\xi^2 - 3)\xi \cosh(2m) + 8i\xi^2 \sinh(m)^3 \\ &\quad + 2\xi \cosh(3m)\Delta + 7i \sinh(m) - 8i\xi^2 \sinh(2m)\Delta + 2i \sinh(2m)\Delta - i \sinh(3m)], \\ S_1^{(iv)}(x_s^+) &= -6\lambda^4 + \frac{23\lambda^2}{2 \sinh^2 m} - \frac{\Delta(48\lambda^2 \sinh(m)^2 - 23)z + 2(7\lambda^2 \sinh(m)^2 - 4)\cosh(2m) + \Delta \cosh(3m) - 16}{4 \sinh^4 m}. \end{aligned}$$

The values of the derivatives of S_1 when evaluated at x_s^- are immediately obtained through the relations:

$$S_1(x_s^-) = (S_1(x_s^+))^*, \quad S_1''(x_s^-) = (S_1''(x_s^+))^*, \quad S_1'''(x_s^-) = -(S_1'''(x_s^+))^*, \quad S_1^{(iv)}(x_s^-) = (S_1^{(iv)}(x_s^+))^*$$

Appendix B. Multiple scaling limit of Wilson loops

The large N limit of Eq. (13) with $\lambda \equiv \eta/N$ and $f \equiv r/N$ fixed, with $0 \leq f < 1$, is obtained from the contributions of the saddle points:

$$\bar{x} = \log \left(\frac{-(\lambda z + L \sin \frac{\theta}{2}) + i(fz + L \cos \frac{\theta}{2})}{\lambda + i(1 - f)} \right) + i2\pi k, \quad k \in \mathbb{Z},$$

where we defined L and θ as:

$$L \equiv L(\lambda, f) = \sqrt{1 + (\lambda^2 + f^2)^2 \sinh(m)^4 - 2(\lambda^2 - f^2) \sinh(m)^2}, \quad \sin \theta = \frac{2\lambda f \sinh(m)^2}{L}, \quad \cos \theta = \frac{1 - (\lambda^2 - f^2) \sinh(m)^2}{L}.$$

Those saddle points are in general complex, and there is no critical surface in parameter space signalling a phase transition. The sub-critical phase of the case $r = 0$ now corresponds to the system living in the surface in the (λ, f, m) space determined by the equation:

$$\left(\lambda z + L \sin \frac{\theta}{2} \right)^2 + \left(fz + L \cos \frac{\theta}{2} \right)^2 = \lambda^2 + (1 - f)^2,$$

while the rest of $3d$ parameter space is qualitatively analogous to the super-critical phase of the partition function.

Appendix C. Partition function in the *super-critical phase* for $n = 2$

The non-Abelian theory with $n = 2$ has four relevant saddle points, obtained from the combinations $(\bar{x}_1, \bar{x}_2) = (x_s^\pm, x_s^\pm)$. In the *sub-critical phase*, only (x_s^+, x_s^+) contributes, but in the *super-critical phase* all four saddle points are to be taken into account, and the partition function is therefore the sum of four pieces:

$$\mathcal{Z}_{\text{super.}}^{U(2)} = \mathcal{Z}(x_s^+, x_s^+) + \mathcal{Z}(x_s^+, x_s^-) + \mathcal{Z}(x_s^-, x_s^+) + \mathcal{Z}(x_s^-, x_s^-).$$

Taking advantage of the relations of Appendix A, one immediately finds:

$$\mathcal{Z}(x_s^+, x_s^+) + \mathcal{Z}(x_s^-, x_s^-) = \mathcal{Z}_{\text{sub.}}^{U(2)} + c.c.,$$

at order $\mathcal{O}(N^{-1})$. The sum of the other two contributions is:

$$\begin{aligned} \mathcal{Z}(x_s^+, x_s^-) + \mathcal{Z}(x_s^-, x_s^+) &= \frac{\pi e^{-2N\Re S_1(x_s^+)}}{2^{2(N-1)}N^2} \left\{ \frac{2\Re S_1''(x_s^+)}{|S_1''(x_s^+)|^3} \right. \\ &+ \frac{1}{N} \left[\frac{(\Re S_1''(x_s^+))^2}{|S_1''(x_s^+)|^5} - \frac{\Re \left((S_1''(x_s^+))^2 (S_1^{(iv)}(x_s^+))^* (5S_1''(x_s^+) + (S_1''(x_s^+))^*) \right)}{4|S_1''(x_s^+)|^7} \right] + \\ &\left. + \frac{5\Re \left((S_1''(x_s^+))^3 ((S_1'''(x_s^+))^*)^2 (7S_1''(x_s^+) + (S_1''(x_s^+))^*) \right) - 6|S_1''(x_s^+)|^4 |S_1'''(x_s^+)|^2}{12|S_1''(x_s^+)|^9} \right\}. \end{aligned}$$

Appendix D. Non-Abelian theory: the general case

The same procedure applied in the text for the case of $U(2)$ yields in principle for any $U(n)$ theory, *i.e.* arbitrary number of colours, as long as n is kept fixed in the large N limit. At finite N , one has the determinantal representation:

$$\mathcal{Z}_N^{U(n)} = N! \det_{1 \leq j, k \leq N} [Z_{jk}] = N! \det_{1 \leq j, k \leq N} \langle W_{j+k-2} \rangle.$$

Here we compute the large N limit of the partition function (1) of the $U(n)$ theory, and the $1/N$ corrections might be obtained in the same fashion as for the $U(2)$ case. The key observation is that, for every n , the interaction among eigenvalues is sub-leading as $N \rightarrow \infty$, and therefore the set of saddle points of the $U(n)$ theory is given by n copies of the set \mathcal{S} of the Abelian theory. Another simplification arises from the observation that, at leading order in $1/N$, the determinant is linearized:

$$\prod_{1 \leq j < k \leq n} \left(2 \sinh \frac{x_j - x_k}{2} \right)^2 = \prod_{1 \leq j < k \leq n} \frac{(t_j - t_k)^2}{N} + \mathcal{O}(N^{-2}).$$

Consequently, at large N the partition function $\mathcal{Z}_N^{U(n)}$ converges to:

$$\mathcal{Z}_{\text{sub.}}^{U(n)} = \frac{e^{-nNS_1(x_s^+)}}{2^{nN}N^{\frac{n^2}{2}}} \mathcal{Z}_{\text{GUE}}(S_1''(x_s^+)) = \frac{(2\pi)^{\frac{n}{2}} e^{-nNS_1(x_s^+)}}{2^{nN}N^{\frac{n^2}{2}}(S_1''(x_s^+))^{\frac{n^2}{2}}} G(n+2),$$

when $\lambda \sinh(m) < 1$, where $\mathcal{Z}_{\text{GUE}}(g)$ denotes the partition function of a Gaussian ensemble with coefficient g in the exponent, and $G(n+2) = \prod_{k=0}^n (k!)$ is the Barnes G -function. In the *super-critical phase*, $\mathcal{Z}_{\text{super.}}^{U(n)}$ is a sum over all possible combinations $(\bar{x}_1, \dots, \bar{x}_n) = (x_s^\pm, \dots, x_s^\pm)$. It is formally given by:

$$\mathcal{Z}_{\text{super.}}^{U(n)} = \frac{(2\pi)^{\frac{n}{2}}}{2^{nN}N^{\frac{n^2}{2}}} \sum_{(\bar{x}_1, \dots, \bar{x}_n) \in \mathcal{S}^n} \prod_{j=1}^n \frac{e^{-NS_1(\bar{x}_j)}}{(S_1''(\bar{x}_j))^{n-\frac{1}{2}}} P_n(S_1''(\bar{x}_1), \dots, S_1''(\bar{x}_n)),$$

with $P_n(s_1, \dots, s_n)$ a symmetric polynomial of degree $n(n-1)/2$ in n variables, subject to the additional constraint:

$$P_n(s, \dots, s) = G(n+2) s^{n(n-1)/2}.$$

For example, in the $U(3)$ theory it is:

$$P_3(s_1, s_2, s_3) = 3 (s_1^2 s_2 + s_1^2 s_3 + s_1 s_2^2 + s_1 s_3^2 + s_2^2 s_3 + s_2 s_3^2 - 2s_1 s_2 s_3),$$

and for $U(4)$ it is:

$$\begin{aligned} P_4(s_1, s_2, s_3, s_4) &= 9 \left\{ 5s_2 s_3 s_4 [s_2(s_3 - s_4)^2 + s_2^2(s_3 + s_4) + s_3 s_4(s_3 + s_4)] \right. \\ &+ s_1^3 [5s_2^2(s_3 + s_4) + 5s_3 s_4(s_3 + s_4) + s_2(5s_3^2 - 18s_3 s_4 + 5s_4^2)] \\ &+ s_1^2 [5s_3(s_3 - s_4)^2 s_4 + 5s_2^3(s_3 + s_4) - 2s_2^2(5s_3^2 - 2s_3 s_4 + 5s_4^2) + s_2(s_3 + s_4)(5s_3^2 - s_3 s_4 + 5s_4^2)] \\ &\left. + s_1 [5s_3^2 s_4^2(s_3 + s_4) + s_2^3(5s_3^2 - 18s_3 s_4 + 5s_4^2) + s_2^2(s_3 + s_4)(5s_3^2 - s_3 s_4 + 5s_4^2) - 2s_2 s_3 s_4(9s_3^2 - 2s_3 s_4 + 9s_4^2)] \right\}. \end{aligned}$$

The expression may be further simplified, using the fact that every combination $(\bar{x}_1, \dots, \bar{x}_n)$ with a fixed number l of entries equal to x_s^+ , and the remaining $n-l$ equal to x_s^- , give the same contribution, independently on the position the x_s^\pm appear. We obtain:

$$\mathcal{Z}_{\text{super.}}^{U(n)} = \frac{(2\pi)^{\frac{n}{2}}}{2^n N N^{\frac{n-2}{2}}} \sum_{l=0}^n \frac{e^{-NlS_1(x_s^+) - N(n-l)S_1(x_s^-)}}{(S_1''(x_s^+))^{(n-\frac{1}{2})l} (S_1''(x_s^-))^{(n-\frac{1}{2})(n-l)}} \binom{n}{l} P_n(\underbrace{s, \dots, s}_l, \underbrace{s^*, \dots, s^*}_{n-l}),$$

where for shortness we denoted $s \equiv S_1''(x_s^+)$ and used $S_1''(x_s^-) = S_1''(x_s^+)^* \equiv s^*$ from Appendix A.

To find the free energy, we reason as in [5] for the Abelian case. We write:

$$\begin{aligned} \mathcal{Z}_{\text{super.}}^{U(n)} &\propto \sum_{l=0}^n \exp[-NlS_1(x_s^+) - N(n-l)S_1(x_s^-) + \dots] \\ &= \exp\left[-nN\Re(S_1(x_s^+)) + \log\left(1 + \sum_{l=0}^n \cos(lN\Im(S_1(x_s^+)))\right) + \dots\right], \end{aligned}$$

where the dots contain sub-leading terms at large N , and arrive to a closed formula for the free energy in the arbitrary $U(n)$ case:

$$\mathcal{F}^{U(n)} = n\mathcal{F}^{U(1)}.$$

Appendix E. General R -charges

The partition function of the $U(1)$ $\mathcal{N} = 2$ theory with N chiral multiplets of mass m and N chiral multiplets of mass $-m$ with arbitrary R -charge q , and coupled to a FI background, is [9, 10]:

$$\begin{aligned} \mathcal{Z}_{Z,q}^{U(1)} &= \int_{-\infty}^{\infty} dx \exp\left\{i\eta x + N\left[\ell\left(1 - q + \frac{i(x+m)}{2\pi}\right) + \ell\left(1 - q - \frac{i(x+m)}{2\pi}\right)\right]\right. \\ &\quad \left.+ N\left[\ell\left(1 - q + \frac{i(x-m)}{2\pi}\right) + \ell\left(1 - q - \frac{i(x-m)}{2\pi}\right)\right]\right\}, \end{aligned}$$

where we recall that the theory is put on a three-sphere of radius $1/2\pi$. Here, $e^{\ell(z)}$ is the double sine function, defined as [9]:

$$\ell(u) = -u \log(1 - e^{i2\pi u}) + \frac{i\pi}{2} u^2 + \frac{i}{2\pi} \text{Li}_2(e^{i2\pi u}) - \frac{i\pi}{12}, \quad u \in \mathbb{C}.$$

This function has logarithmic singularities when $q \in \mathbb{Z}$, or, more in general, when $\Im(\tilde{m}) + 2\pi(1-q) \in 2\pi\mathbb{Z}$, where \tilde{m} denotes a complexified mass parameter. Nevertheless, the partition function does not develop singularities, and in fact is holomorphic in \tilde{m} , as the divergences cancel. This can be seen, for instance, from the identity

$$\ell\left(1 - q - \frac{i(x-m)}{2\pi}\right) \stackrel{\text{reg.}}{=} -\ell\left(1 - q + \frac{i(x-\tilde{m})}{2\pi}\right), \quad \tilde{m} = m - i4\pi(1-q),$$

where the equality is exact for the infinite product representation of the one-loop determinants and extends to the function ℓ through regularization by ζ -function.

The derivative of the double sine function satisfies the simple property:

$$\frac{d\ell}{du} = -\pi u \cot(\pi u).$$

Therefore, in the double scaling large N limit, we arrive to the saddle point equation:

$$\frac{\frac{(x+m)}{2\pi} \sin(2\pi(1-q)) - i(1-q) \sinh(x+m)}{\cosh(x+m) - \cos(2\pi(1-q))} + \frac{\frac{(x-m)}{2\pi} \sin(2\pi(1-q)) - i(1-q) \sinh(x-m)}{\cosh(x-m) - \cos(2\pi(1-q))} = \lambda. \quad (15)$$

It is a simple exercise to see that, setting $q = \frac{1}{2}$, one recovers the saddle points of the $\mathcal{N} = 4$ theory [5]. When q is half-integer, the trigonometric functions take simple values and we can solve the saddle point equation exactly, as showed in the main text.

We found out that, for $q = 1$, the action admits no saddle point. Here, we study what happens close to that point, for real $q = 1 - \varepsilon$. We assume ε small and approximate the expression at $\mathcal{O}(\varepsilon)$. From (15) we get:

$$\frac{\sinh(x)}{\cosh(x) - \cosh(m)} + i \frac{x(\cosh(x)\cosh(m) - 1) + m\sinh(x)\sinh(m)}{(\cosh(x) - \cosh(m))^2} = \frac{i\lambda}{2\varepsilon}.$$

The equation is still transcendental, but we can find an approximate solution in the large mass limit:

$$\sinh(x) \approx \frac{\lambda e^m}{2m\varepsilon} \implies x \approx \log \left[\frac{\lambda e^m}{2m\varepsilon} \left(1 + \sqrt{1 + \frac{4m^2\varepsilon^2 e^{-2m}}{\lambda^2}} \right) \right] \approx m + \log \frac{\lambda}{m\varepsilon}. \quad (16)$$

In figure 4 we compare this expression at large m with a numerical solution to the saddle point equation.

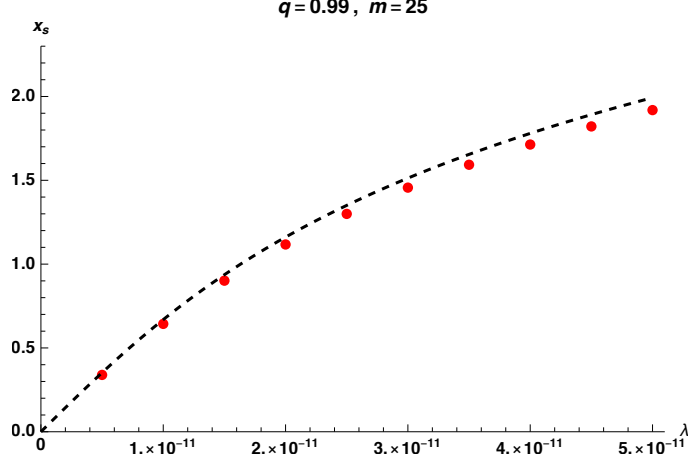


FIG. 4. Comparison of the numerical solution of the saddle point equation (15) (red dots) and expression (16) (black, dashed line), for $q = 0.99$ and $m = 25$.

Comment on squashed geometry

If the supersymmetric $\mathcal{N} = 2$ theory is put on a squashed three-sphere instead than a round one, the partition function is obtained replacing the double sine functions by their squashed version [32]

$$\ell \left(1 - q + \frac{i\sigma}{2\pi} \right) \ell \left(1 - q - \frac{i\sigma}{2\pi} \right) \mapsto \ell_b \left(\frac{1}{2} \left(b + \frac{1}{b} \right) (1 - q) + \frac{i\sigma}{2\pi} \right) \ell_b \left(\frac{1}{2} \left(b + \frac{1}{b} \right) (1 - q) - \frac{i\sigma}{2\pi} \right), \quad \sigma = x \pm m,$$

where $b = \sqrt{r_1/r_2}$ is the squashing parameter, and the average radius is $\sqrt{r_1 r_2} = 1/2\pi$. We now take advantage of the remarkable property of the double sine function:

$$\exp \left\{ \ell_b \left(\frac{b}{2} + \frac{i\sigma}{2\pi} \right) \ell_b \left(\frac{b}{2} - \frac{i\sigma}{2\pi} \right) \right\} = \frac{1}{2 \cosh \left(\frac{b\sigma}{2} \right)},$$

which holds for every real non-negative b , and for the round case $b = 1$ provides the partition function (2). Therefore, for hypermultiplets with R -charge $0 < q < 1$, we may tune the geometry of the manifold so that $(b + b^{-1})(1 - q) = b$, that is we may squash the sphere as

$$b = \sqrt{\frac{1 - q}{q}},$$

and the partition function reads:

$$\mathcal{Z}_{N, \text{squash}}^{U(1)} = \int_{-\infty}^{\infty} dx \frac{e^{i\eta x}}{2^N [\cosh(bx) + \cosh(bm)]^N}.$$

Notice that this procedure would also affect the one-loop determinant of the vector multiplet, but this is irrelevant in the Abelian theory, being such determinant trivial. Also, we see that the symmetry $q \leftrightarrow 1 - q$ at the matrix model level is translated into a symmetry $b \leftrightarrow b^{-1}$ in the geometry. We therefore obtain a simple relation between the partition function of the $\mathcal{N} = 2$ theory with arbitrary R -charge $0 < q < 1$ posed on a suitably squashed sphere and the $\mathcal{N} = 4$ theory with R -charge $q = \frac{1}{2}$ on the round \mathbb{S}^3 :

$$\mathcal{Z}_{N,\text{squash}}^{U(1)}(m, \eta, q) = \frac{1}{b} \mathcal{Z}_{N,\text{round}}^{U(1)}\left(bm, \frac{\eta}{b}, q = \frac{1}{2}\right), \quad b = \sqrt{\frac{1-q}{q}}.$$

As a byproduct, this equivalence holds for the $U(2)$ theory in the large N approximation. In fact, the squashing would modify:

$$\left(\sinh \frac{x_1 - x_2}{2}\right)^2 \mapsto \left(\sinh \frac{b(x_1 - x_2)}{2}\right) \left(\sinh \frac{x_1 - x_2}{2b}\right),$$

and, as we have seen, the determinant is linearized at first order in $1/N$, producing cancellation of the b -dependence.