PRINCIPAL SCHOTTKY BUNDLES OVER RIEMANN SURFACES

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Abstract. We introduce and study (strict) Schottky G-bundles over a compact Riemann surface X, where G is a connected reductive algebraic group. Strict Schottky representations are shown to be related to branes in the moduli space of G-Higgs bundles over X, and we prove that all Schottky G-bundles have trivial topological type. Generalizing the Schottky moduli map introduced in [Flo01] to the setting of principal bundles, we prove its local surjectivity at the good and unitary locus. Finally, we prove that the Schottky map is surjective onto the space of flat bundles for two special classes: when G is an abelian group over an arbitrary X, and the case of a general G-bundle over an elliptic curve.

1. Introduction and Main Results

1.1. **Schottky uniformizations.** The classical Fuchsian uniformization theorem provides an explicit parameterization of all Riemann surfaces X of genus $g \geq 2$: every such X can be obtained as $\mathbb{H}/\Gamma$, a quotient of the upper half-plane $\mathbb{H}$ by a Fuchsian group $\Gamma \subset PSL_2\mathbb{R}$, isomorphic to the fundamental group of X, $\pi_1(X)$. A less well-known result, the so-called "retrosection theorem", or Schottky uniformization, asserts that we can also write $X \cong \Omega/\Sigma$, for a certain *free group* of Möbius transformations $\Sigma \subset PSL_2\mathbb{C}$ of rank g (called, in this context, a *Schottky group*) and region of discontinuity (for the Σ-action) $\Omega \subset \mathbb{CP}^1$ (see [Ber75, For51]).

These are two very different parametrizations: the Fuchsian one is essentially unique, and provides an identification between Teichmüller space and one component of the (real) *character variety* $\mathrm{Hom}(\pi_1(X), PSL_2\mathbb{R})/PSL_2\mathbb{R}$ (the quotient of representations $\pi_1(X) \to PSL_2\mathbb{R}$ by conjugation); by contrast, the Schottky one is defined on a less explicit subset of $\mathrm{Hom}(\Sigma, PSL_2\mathbb{C})/PSL_2\mathbb{C}$, having the advantage of providing manifestly holomorphic coordinates.

Passing from surfaces to holomorphic bundles over a *fixed* Riemann surface X, it is natural to consider analogous explicit parametrizations. In their famous papers [NS65, NS64], Narasimhan and Seshadri proved that every polystable vector bundle over X, of degree zero, can be obtained from a (unique up to conjugation) unitary representation. Ramanathan generalised Narasimhan-Seshadri's results to principal G-bundles, where G is any reductive algebraic group over $\mathbb{C}$ (see [Ram75, Ram96]).

More precisely, a representation $\rho : \pi_1(X) \to K \subset G$ into K, a maximal compact subgroup of G, defines a holomorphic G-bundle over $X = \mathbb{H}/\pi_1(X)$, equipped with a natural flat connection:

$$E_\rho := (\mathbb{H} \times G)/_\rho\, \pi_1(X), \tag{1.1}$$

using the diagonal action of $\pi_1(X)$, via ρ, on the trivial G-bundle $G \times \mathbb{H} \to \mathbb{H}$. A special case of the results of Narasimhan, Seshadri and Ramanathan is that a holomorphic G-bundle over

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X, which admits a flat connection, is polystable if and only if it can be written in the above form, for some $\rho : \pi_1(X) \to K$, unique up to conjugation. Their result can thus be seen as a bundle version of classical Fuchsian uniformization, and identifies the moduli space of flat semistable G-bundles with the "real character variety"

$$\mathrm{Hom}(\pi_1(X), K)/K.$$

The question of whether some sort of Schottky uniformization can be obtained for a large class of holomorphic G-bundles is still an open problem, as far as we know.[1] Florentino studied the case of *vector* bundles and obtained some partial results ([Flo01]), showing that all flat line bundles, and all flat vector bundles over an elliptic curve are Schottky bundles: these can be defined as in (1.1) for certain representations ρ of a *free group of rank g* into the general linear group $GL_n\mathbb{C}$. Moreover, an open subset of the moduli space of degree zero semistable vector bundles consists of Schottky vector bundles. This study was motivated by an attempt to develop an analytic theory of non-abelian theta functions and their relation to the spaces of conformal blocks in conformal field theory (see [Bea95, FMN03, Tyu03]).

Schottky (*principal*) G-bundles were defined by Florentino and Ludsteck, for a general complex reductive algebraic group G ([FL14]). They showed that there exists a natural equivalence between the categories of unipotent representations of a Schottky group of rank g and unipotent holomorphic vector bundles over Riemann surface of genus g.

In this paper, we generalize the results of the article [Flo01] in two different ways: we replace $GL_n\mathbb{C}$ by an arbitrary connected complex reductive group G, and we consider a more general definition of Shottky representations, allowing all marked generators to be represented in the center of G.

1.2. **Main results.** We now summarize our main results, emphasizing the novelties in the principal bundle case, while describing the contents of each section. Consider the usual presentation

$$\pi_1(X) = \left\langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g \mid \textstyle\prod_{i=1}^{g} \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1} = 1 \right\rangle, \tag{1.2}$$

of the fundamental group of a fixed Riemann surface X, of genus $g \geq 1$ (we are implicitly choosing a base point $x_0 \in X$, but this is irrelevant when considering isomorphism classes of representations). A representation $\rho : \pi_1(X) \to G$ is said to be *Schottky* (with respect to our choice of generators above) if $\rho(\alpha_i)$ is in the *center* $Z = Z_G$ of G for all $i = 1, \cdots, g$. These include what we call *strict Schottky representations*, which verify $\rho(\alpha_i) = e$ for all $i = 1, \cdots, g$, with e the identity of G. Although the definitions require a choice of generators for $\pi_1(X)$, our results are independent of such choices. Thus, from an algebro-geometric perspective, Schottky representations (up to conjugation) are naturally parametrized by the affine *geometric invariant theory* (GIT) quotient

$$\mathbb{S} := \mathrm{Hom}\,(F_g,\, Z \times G) /\!\!/ G,$$

where F_g denotes a fixed free group of rank g (see Proposition 2.4). Besides these definitions and first properties, in Section 2 we describe the irreducible components of the Schottky space $\mathbb{S}$ and prove the existence of good and unitary Schottky representations for $g \geq 2$.

Strict Schottky representations have the following natural topological interpretation. Suppose that M is a 3-manifold whose boundary is X, and the natural morphism $i_* : \pi_1(X) \to \pi_1(M)$ induced by the inclusion $i : X \hookrightarrow M$, has all the α_i in its kernel and the β_i are free,

[1] Interestingly, the consideration of the Schottky uniformization problem for vector bundles over Mumford curves, in the framework of p-adic analysis, has furnished stronger results. (see [Fal83]).

$i = 1, \cdots, g$. Then it is easy to see that *strict* Schottky representations are the representations of $\pi_1(X)$ which "extend to M", meaning that they factor through i_* (note that $\pi_1(M)$ is indeed a free group of rank g). In addition to its relation to the uniformization problems for holomorphic G-bundles, Schottky representations also appear in a different context, related to non-abelian Hodge theory: recently, Baraglia and Schaposnik considered G-Higgs bundles over a Riemann surface equipped with an anti-holomorphic involution and showed that, inside the moduli space of G-Higgs bundles, the locus of those which are fixed by an associated involution define what is called an (A, B, A)-brane ([BS14]). In Section 3, we identify all strict Schottky representations as elements of this brane (see [BS14, Proposition 43] and Proposition 3.2). The study of branes is of great interest in connection with mirror symmetry and the geometric Langlands correspondence (see [KW07]).

Section 4 provides the definition of Schottky G-bundles and their relation to Schottky vector bundles in terms of associated bundles. A Schottky (principal) G-bundle over X is defined to be a holomorphic bundle which is isomorphic to a bundle of the form (1.1), for some Schottky representation ρ (so that its conjugation class $[\rho]$ belongs to $\mathbb{S}$). Similarly, we define strict Schottky bundles. Note that all Schottky bundles, being defined by representations of $\pi_1(X)$, necessarily admit a flat holomorphic connection.

The association of a Schottky G-bundle to a Schottky representation defines what we call the *Schottky uniformization map*:

$$\mathbf{W} : \mathbb{S} \to M_G,$$

where M_G stands for the set of isomorphism classes of G-bundles over X admitting a flat connection. Two important properties of $\mathbf{W}$ are in clear contrast with the Narasimhan-Seshadri-Ramanathan uniformization (see Remark 7.6(1)):

(1) A (strict) Schottky bundle is not necessarily semistable (contrary to those coming from unitary representations $\rho : \pi_1(X) \to K$);

(2) If $E = E_\rho$ is a Schottky bundle, then $[\rho] \in \mathbb{S}$ is not unique in general, and the preimage $\mathbf{W}^{-1}([E])$ is typically infinite.

By results of Ramanathan [Ram75], further developed in [Li93], the topological invariants of flat G-bundles are labeled by elements in $\pi_1(DG)$, where DG is the derived group of G. Moreover, there exist flat G-bundles with all possible topological types. In Section 5, we prove that the Schottky case is particularly simple (Theorem 5.3):

Theorem. (A) *Every Schottky G-bundle is topologically trivial.*

In Section 6 we define and study the notion of *analytic equivalence* of representations and consider the period map, for later use in computing the derivative of the Schottky map. In general, Schottky representations and strict ones are distinct. Analytic equivalence allows to prove that, for Schottky bundles, the distinction between the strict and the general case is not relevant when G has a *connected* center (Proposition 6.4).

In Section 7, we consider the tangent spaces to Schottky space, describe them in terms of the first cohomology group of F_g in certain F_g-modules, and compute the dimension of the Schottky space $\mathbb{S} := \mathcal{S}/\!\!/G$. We characterize the kernel of the derivative of the Schottky moduli map at a good Schottky representation. We also prove that the good locus of strict Schottky space is a Lagrangian submanifold of the complex manifold of the smooth points of $\mathrm{Hom}(\pi_1(X), G)/\!\!/G$.

Let $\mathcal{M}_G$ denote the moduli space of semistable G-bundles over X and consider the restricted map called the *Schottky moduli map*

$$\mathbf{V} : \mathbb{S}^* \to \mathcal{M}_G,$$

where $\mathbb{S}^* := \mathbf{W}^{-1}(\mathcal{M}_G \cap M_G)$ is a dense subset of $\mathbb{S}$. With their natural complex structures, this gives now a holomorphic map between the smooth locus of the corresponding spaces. In Section 8 we compute the derivative of the Schottky moduli map at a good and unitary representation (assuming also that $[E_\rho]$ is a smooth point of $\mathcal{M}_G$), proving that it is an isomorphism when G is semisimple (Corollary 8.8). In the more general case of reductive G, the Schottky moduli map will be a submersion (Theorem 8.6).

Theorem. (B) *Let $\rho : \pi_1(X) \to G$ be a good and unitary Schottky representation, such that $[E_\rho]$ is a smooth point in $\mathcal{M}_G$. Then, the derivative of the Schottky moduli map at $[\rho] \in \mathbb{S}^*$ has maximal rank. In particular, locally around $[\rho]$, the Schottky moduli map $\mathbf{V} : \mathbb{S}^* \to \mathcal{M}_G$ is a submersion, and* $\dim \mathbf{V}^{-1}([E_\rho]) = g \dim Z$.

Finally, in Section 9, we consider two special classes of Schottky principal bundles: the first case are G-bundles where $G = (\mathbb{C}^*)^m$, for some $m \in \mathbb{N}$, over a general surface X. In this case, since our definition is more general than the one in [Flo01], the strict Schottky condition turns out to be equivalent to flatness (Proposition 9.1). The second special class consists of Schottky G-bundles over a compact Riemann surface of genus $g = 1$, which needs a distinct treatment than the case $g \geq 2$ (Theorem 9.7). Again, in this case, the Schottky condition is equivalent to flatness.

Theorem. (C) *Let X be an elliptic curve and E a G-bundle over X. Then E is Schottky if and only if it admits a flat connection.*

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2. Schottky Representations

Given a compact Riemann surface X of genus $g \geq 2$, the classical Schottky uniformization theorem (see [For51, Ber75]) states that X is isomorphic to a quotient Ω_Σ / Σ, where $\Sigma \subset PSL_2\mathbb{C}$ is a Schottky group and $\Omega_\Sigma \subset \mathbb{CP}^1$ is the corresponding region of discontinuity in the Riemann sphere. Schottky groups are finitely generated free purely loxodromic subgroups of the Möbius group $PSL_2\mathbb{C}$ (see also [Mas67]), and so, Σ is the image of a free group F_g, of g generators, under a homomorphism $\rho : F_g \to PSL_2\mathbb{C}$. Naturally, conjugate homomorphisms define isomorphic surfaces.

In this section, we consider the space of isomorphism classes of representations of F_g into a general complex reductive algebraic group G, and prove some properties of the corresponding algebraic variety. This is an extension of the notion of Schottky representations studied in [Flo01], which were associated to representations of F_g into $GL_n\mathbb{C}$.

We begin by fixing some notation. Denote by $\pi_1 = \pi_1(X)$ the fundamental group of X, and fix generators α_i, β_i, $i = 1, \cdots, g$, of π_1 giving the usual presentation

$$\pi_1 = \left\langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g \mid \textstyle\prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1 \right\rangle. \tag{2.1}$$

Let G be a complex connected reductive algebraic group and denote by F_g a fixed free group of rank g, with g fixed generators $\gamma_1, \cdots, \gamma_g$. Since G is algebraic, and π_1 and F_g are finitely presented, both $\mathrm{Hom}(\pi_1, G)$ and $\mathrm{Hom}(F_g, G)$ are affine algebraic varieties.

The reductive group G acts by conjugation on $\mathrm{Hom}(\pi_1, G)$ and hence, one can define a geometric invariant theory (GIT) quotient, called the G-character variety of π_1 (also called the *Betti space* in the context of the non-abelian Hodge theory, see [Sim94]), as

$$\mathbb{B} := \mathrm{Hom}(\pi_1, G)/\!\!/G. \tag{2.2}$$

This is a categorical quotient which, as an affine algebraic variety, is the maximal spectrum of the $\mathbb{C}$-algebra of G-invariant regular functions in $\mathbb{C}[\mathrm{Hom}(\pi_1, G)]$ (see, for example [New78, Theorem 3.5]).

2.1. **Schottky representations.** Denote by $e \in G$, the unit element of G, and by $Z = Z_G$ the center of G.

Definition 2.1. A representation $\rho : \pi_1 \to G$ is called:

(1) a *Schottky representation* if $\rho(\alpha_i) \in Z$ for all $i \in \{1, \cdots, g\}$
(2) a *strict Schottky representation* if $\rho(\alpha_i) = e$ for all $i \in \{1, \cdots, g\}$

The set of Schottky representations is denoted by $\mathcal{S} \subset \mathrm{Hom}(\pi_1, G)$ and the strict ones by $\mathcal{S}_s$. Of course, $\mathcal{S}_s \subset \mathcal{S}$ and they coincide when $Z = \{e\}$ (i.e., for adjoint groups).

For a useful alternative characterization of Schottky representations, consider the natural short exact sequence of groups

$$1 \to \ker \varphi \hookrightarrow \pi_1 \stackrel{\varphi}{\to} F_g \to 1$$

where φ is the natural epimorphism given, in terms of the generators, by

$$\varphi(\alpha_i) = e, \quad \text{and } \varphi(\beta_i) = \gamma_i, \quad \forall i = 1, \cdots, g,$$

so that $\ker \varphi$ is the normal subgroup of π_1 generated by all α_i. Schottky representations can also be defined with respect to the map φ as in the following lemma, whose proof is straightforward (see also [FL14]).

Lemma 2.2. *Let $\rho \in \mathrm{Hom}(\pi_1, G)$ and let $\varphi : \pi_1 \to F_g$ be as above. Then*

(1) *ρ is a Schottky representation if and only if $\rho(\ker \varphi) \subset Z$;*
(2) *ρ is a strict Schottky representation if and only if $\rho(\ker \varphi) = \{e\}$.*

Using our fixed generators, we can see $\mathcal{S}$ as an algebraic subvariety of $\mathrm{Hom}(\pi_1, G)$, isomorphic to $(Z \times G)^g$ (and $\mathcal{S}_s$ as a smooth subvariety of $\mathcal{S}$, isomorphic to G^g). A Schottky representation $\rho \in \mathcal{S} \subset \mathrm{Hom}(\pi_1, G)$ may also be viewed as a representation

$$\rho_F = (\rho_1, \rho_2) : F_g \to Z \times G.$$

Indeed, given $\rho \in \mathcal{S}$, define $\rho_1 : F_g \to Z$, and $\rho_2 : F_g \to G$ by

$$\rho_F(\gamma_i) = (\rho_1(\gamma_i), \rho_2(\gamma_i)) := (\rho(\alpha_i), \rho(\beta_i)) \in Z \times G, \qquad i = 1, \cdots, g. \tag{2.3}$$

Conversely, given $\rho_F = (\rho_1, \rho_2) : F_g \to Z \times G$, we obtain a Schottky representation $\rho \in \mathcal{S} \subset \mathrm{Hom}(\pi_1, G)$ defined by setting $\rho(\alpha_i) := \rho_1(\gamma_i)$ and $\rho(\beta_i) := \rho_2(\gamma_i)$, $i = 1, \cdots, g$. It is clear that this defines an inclusion of algebraic varieties

$$\psi : \mathrm{Hom}(F_g, Z \times G) \hookrightarrow \mathrm{Hom}(\pi_1, G), \tag{2.4}$$

identifying $\mathrm{Hom}(F_g, Z \times G)$ with its image, which is precisely $\mathcal{S}$. The strict Schottky locus $\mathcal{S}_s$ is then identified with $\mathrm{Hom}(F_g, \{e\} \times G) \simeq \mathrm{Hom}(F_g, G) \simeq G^g$, where the last isomorphism is the evaluation map: $(\sigma : F_g \to G) \mapsto (\sigma(\gamma_1), \cdots, \sigma(\gamma_g))$.

Remark 2.3. Our identifications depend on the choice of generators for π_1 and F_g, but the algebraic structure is independent of those choices (different choices provide isomorphic varieties), as can be easily seen.

It is clear that the conjugation action of the reductive group G on $\mathrm{Hom}(\pi_1, G)$ restricts to an action on $\mathcal{S}$ and on $\mathcal{S}_s$. In terms of the identification $\mathcal{S} \cong \mathrm{Hom}(F_g, Z \times G)$, each element $g \in G$ acts as follows:

$$(g \cdot \rho_F)(\gamma) = \big(\rho_1(\gamma),\ g\,\rho_2(\gamma)\,g^{-1}\big) \quad \text{for all } \gamma \in F_g, \tag{2.5}$$

where $\rho_F = (\rho_1, \rho_2)$ as above. As before, there exist a GIT quotient

$$\mathbb{S} := \mathcal{S}/\!\!/G \cong \mathrm{Hom}\,(F_g, Z \times G)\,/\!\!/G,$$

which we call the *Schottky space* (in particular, it is a character variety of F_g). Moreover, since $\psi : \mathrm{Hom}\,(F_g, Z \times G) \hookrightarrow \mathrm{Hom}(\pi_1, G)$ in (2.4) is clearly a G-equivariant inclusion of affine algebraic varieties, in view of Equation (2.3) and (2.4), we have shown the following.

Proposition 2.4. *There are the following morphisms between algebraic G-varieties:*

$$\mathcal{S}_s \cong \mathrm{Hom}(F_g, G) \cong G^g \quad \subset \quad \mathcal{S} \cong \mathrm{Hom}(F_g, Z \times G) \cong (Z \times G)^g \quad \subset \quad \mathrm{Hom}(\pi_1, G).$$

In particular, $\mathcal{S}$ and $\mathcal{S}_s$ are smooth. In turn, these induce morphisms of affine GIT quotients:

$$\mathbb{S}_s = \mathcal{S}_s/\!\!/G \cong G^g/\!\!/G \quad \subset \quad \mathbb{S} = \mathcal{S}/\!\!/G \cong (Z \times G)^g/\!\!/G \quad \subset \quad \mathbb{B} = \mathrm{Hom}(\pi_1, G)/\!\!/G.$$

Note that, because the conjugation action is trivial on Z, we can also write

$$\mathbb{S} \cong (Z \times G)^g/\!\!/G = Z^g \times (G^g/\!\!/G) = Z^g \times \mathbb{S}_s. \tag{2.6}$$

The GIT quotient under G of an irreducible variety is irreducible. Thus, $\mathbb{S}_s \cong G^g/\!\!/G$ is irreducible. However, $\mathbb{S}$ can have several irreducible components, in bijection with the components of Z^g. It is well known that the connected component of the identity of Z is an algebraic torus Z°, and the quotient $Z_f := Z/\,Z^\circ$ is finite.

Proposition 2.5. *All irreducible components of $\mathbb{S}$ are isomorphic to*

$$\mathrm{Hom}\,(F_g, Z^\circ \times G)\,/\!\!/G \cong (Z^\circ)^g \times (G^g/\!\!/G) \cong (Z^\circ)^g \times \mathbb{S}_s,$$

and the number of irreducible components of $\mathbb{S}$ is given by $|Z_f|^g$.

Proof. As a variety, we can write Z as a cartesian product of the above subgroups, $Z = Z_f \times Z^\circ$. So, we get the following isomorphism of varieties, from Equation (2.6)

$$\mathbb{S} \cong Z^g \times \mathbb{S}_s \cong (Z_f)^g \times (Z^\circ)^g \times \mathbb{S}_s \cong (Z_f)^g \times \mathrm{Hom}\,(F_g, Z^\circ \times G)\,/\!\!/G$$

which immediately proves the proposition. □

Remark 2.6. (1) Clearly, $\mathbb{S} = \mathbb{S}_s$, hence irreducible, when the center of G is trivial.
(2) Replacing F_g by other finitely generated groups can give very different results on components. For example, when $G = PSL_2\mathbb{C}$ it is known that $\mathrm{Hom}(\pi_1, G)/\!\!/G$ has several irreducible components, and only two of them correspond to representations that uniformize a Riemann surface (Kleinian representations). On the other hand $\mathrm{Hom}(\pi_1, SL_2\mathbb{C})/\!\!/SL_2\mathbb{C}$ is irreducible (see [Gol88]).

2.2. **Good and unitary representations.** Although $\mathcal{S}$ and $\mathcal{S}_s$ are smooth, the algebraic varieties $\mathbb{S}$ and $\mathbb{S}_s$ are singular in general. The notion of a good representation allows us to consider smooth points of the GIT quotient, as we will see. Let Γ be a finitely generated group, for example the fundamental group of a compact manifold. Given a representation $\rho : \Gamma \to G$ we denote by

$$Z(\rho) = \{h \in G : \ \rho(\gamma)h = h\rho(\gamma)\ \forall \gamma \in \Gamma\}$$

its stabilizer in G, and denote by $G \cdot \rho$ its G-orbit in the algebraic variety $\mathrm{Hom}(\Gamma, G)$. Recall the following standard definitions.

Definition 2.7. Let $\rho : \Gamma \to G$ be a representation. We say that ρ is:

(a) *polystable* if $G \cdot \rho$ is (Zariski)-closed,

(b) *reducible* if $\rho(\Gamma)$ is contained in a proper parabolic subgroup of G,

(c) *irreducible* if it is not reducible,

(d) *good* if ρ is irreducible and $Z(\rho) = Z$.

Remark 2.8. Note that ρ is polystable if and only if $Z(\rho)$ is a reductive group itself, and it is irreducible if and only if $Z(\rho)$ is reductive and a finite extension of Z (see [Sik12]). Moreover, ρ is irreducible if and only if it is *stable* in the appropriate affine GIT sense (see [FC12]).

Now we apply these notions to the case of Schottky representations.

Definition 2.9. A representation $\rho \in \mathcal{S} \subset \operatorname{Hom}(\pi_1, G)$ is said to be *polystable* (resp. *irreducible*, *good*) if ρ is polystable (resp. irreducible, good) as an element of $\operatorname{Hom}(\pi_1, G)$.

Denote the set of all good (resp. good Schottky) representations by $\operatorname{Hom}^{\mathsf{gd}}(\pi_1, G)$ (resp. $\mathcal{S}^{\mathsf{gd}}$). Since these notions are well defined under conjugation, we can define the corresponding quotient spaces:

$$\mathbb{B}^{\mathsf{gd}} := \operatorname{Hom}^{\mathsf{gd}}(\pi_1, G) /\!\!/ G \quad \text{and} \quad \mathbb{S}^{\mathsf{gd}} := \mathcal{S}^{\mathsf{gd}} /\!\!/ G,$$

and, from Proposition 2.4, we have the inclusion $\mathbb{S}^{\mathsf{gd}} \subset \mathbb{B}^{\mathsf{gd}}$.

The sets of good, polystable and irreducible representations are Zariski open in $\mathcal{S}$ (see for example [Sik12]). By [Mar00, Lemma 4.6] there exists a good representation in $\operatorname{Hom}(\pi_1, G)$, that is, $\operatorname{Hom}^{\mathsf{gd}}(\pi_1, G) \neq \varnothing$, if X has genus $g \geq 2$. Note that the case $g = 1$ is slightly different (see Section 9).

To show that $\mathcal{S}^{\mathsf{gd}}$ is nonempty, we start by relating the relevant properties of $\rho \in \mathcal{S}$ with the corresponding properties of $\rho_2 : F_g \to G$.

Proposition 2.10. *Let $\rho \in \mathcal{S} \subset \operatorname{Hom}(\pi_1, G)$ be given by $\rho_F = (\rho_1, \rho_2) : F_g \to Z \times G$ as in (2.3). Then:*

(a) $Z(\rho) = Z(\rho_2) \subset G$,

(b) ρ is irreducible if and only if ρ_2 is irreducible,

(c) ρ is a good Schottky representation if and only if ρ_2 is a good representation of F_g.

Proof. (a) Denote by $C(h)$ the centralizer of an element $h \in G$, $C(h) := \{g \in G : \ hg = gh\}$. Since ρ is completely defined by the image of the generators of π_1, the stabilizer of ρ is the intersection of the centralizers of the images of the generators α_i, β_i of π_1 and γ_i of F_g:

$$Z(\rho) = \textstyle\bigcap_{i=1}^{g} C(\rho(\alpha_i)) \bigcap_{i=1}^{g} C(\rho(\beta_i)) = \bigcap_{i=1}^{g} C(\rho_2(\gamma_i)) = Z(\rho_2),$$

because $\rho(\alpha_i) = \rho_1(\gamma_i) \in Z$, which implies $C(\rho(\alpha_i)) = G$.

(b) Let us suppose that $\rho : \pi_1 \to G$ is reducible. By definition, $\rho(\pi_1) \subset P$ for some proper parabolic subgroup $P \subset G$. This means that $\rho(\alpha_i), \rho(\beta_i) \in P$, $\forall i = 1, \cdots, g$. So,

$$\rho(\beta_i) = \rho_2(\gamma_i) \in P, \forall i \quad \Leftrightarrow \quad \rho_2(F_g) \subset P,$$

proving that ρ_2 is reducible. The proof of the converse is analogous, using again $\rho(\alpha_i) = \rho_1(\gamma_i) \in Z$, and also the fact that any parabolic subgroup contains the center of G.

(c) This follows immediately from (a) and (b). □

Recall that, for a connected reductive algebraic group G over $\mathbb{C}$, there exists a maximal compact connected *real* Lie group K whose complexification coincides with G. Ramanathan showed that the moduli space of semistable G-bundles over X, which admit a flat connection, is homeomorphic to $\operatorname{Hom}(\pi_1, K)/K$ ([Ram75]).

We now show that good Schottky representations exist, and these can be taken to be unitary, as well.

Lemma 2.11. *Let K be a maximal compact subgroup of G. If H is a subgroup of K which is dense in the manifold topology of K, then $Z_G(H) = Z_G(K) = Z$.*

Proof. Being the intersection of centralizers of single elements, the centralizer of any subgroup of G is an algebraic subgroup of G, hence Zariski closed. In particular, $Z_G(K)$ centralizes the Zariski closure of K, which is well known to be G. So $Z_G(K) = Z_G(G) = Z$. Moreover, since H is dense in K, their centralizers are equal, $Z_G(H) = Z_G(K)$. □

Now recall that any connected compact Lie group can be generated by two elements.

Theorem 2.12. [Aue34] *Let K be a connected compact Lie group. Then there are two elements $c, d \in K$ such that the closure of the subgroup they generate, $\overline{\langle c, d\rangle}$, equals K. Moreover, the set of such pairs $\{(c, d)\}$ is dense in $K \times K$.*

Proposition 2.13. *Let $g \geq 2$. Then, there is always a good strict Schottky representation $\rho : \pi_1 \to G$. Moreover, such a representation can be defined to take values in K.*

Proof. Let $c, d \in K$ be two elements of K, such that $\overline{\langle c, d\rangle} = K$, as in Theorem 2.12. Then we explicitly define a unitary representation $\rho : \pi_1 \to K$ by:

$$\rho(\alpha_i) = e, \quad \forall i = 1, \cdots, g, \qquad \text{and} \qquad \begin{cases} \rho(\beta_1) = c \\ \rho(\beta_2) = d \\ \rho(\beta_i) = e, \quad \forall i = 3, \cdots, g, \end{cases} \tag{2.7}$$

Since the subgroup $H := \langle c, d\rangle$ is dense in K, the subgroup $\rho(\pi_1) \subset K$ is also dense in K. So $Z_G(\rho) = Z$, by Lemma 2.11, which proves that ρ is a good strict Schottky representation. □

Theorem 2.14. *Let $g \geq 2$. The subsets of good representations* $\mathrm{Hom}^{\mathsf{gd}}(\pi_1, G)$ *and* $\mathcal{S}^{\mathsf{gd}}$ *are Zariski open in* $\mathrm{Hom}(\pi_1, G)$ *and $\mathcal{S}$, respectively. A good representation defines a smooth point in the corresponding geometric quotient. Thus, the geometric quotients $\mathbb{B}^{\mathsf{gd}}$ and $\mathbb{S}^{\mathsf{gd}}$ are complex manifolds, and $\mathbb{S}^{\mathsf{gd}}$ is a complex submanifold of $\mathbb{B}^{\mathsf{gd}}$.*

Proof. In Proposition 2.13 we constructed a good Schottky representation, for $g \geq 2$. By [Sik12, Proposition 33], the subspaces of good representations in $\mathrm{Hom}(\pi_1, G)$ and $\mathcal{S}$ are Zariski open. Thus, $\mathrm{Hom}^{\mathsf{gd}}(\pi_1, G)$ and $\mathcal{S}^{\mathsf{gd}}$ are open. Since we are considering either surface groups or free groups, [Sik12, Corollary 50] shows that if $\rho \in \mathrm{Hom}^{\mathsf{gd}}(\pi_1, G)$, respectively $\rho \in \mathcal{S}^{\mathsf{gd}}$, then its class $[\rho]$ is a smooth point of $\mathbb{B}$, respectively $\mathbb{S}$. □

3. Higgs bundles and Schottky representations

In this section, we relate Schottky representations to certain Lagrangian subspaces of the moduli space of Higgs G-bundles. It is a fundamental result in the theory of Higgs bundles, the so-called non-abelian Hodge theorem, that by considering the Hitchin equations for G-Higgs fields, one obtains a homeomorphism between the Betti space $\mathbb{B} = \mathrm{Hom}(\pi_1, G)/\!\!/G$ and the moduli space of semistable G-Higgs bundles over X, denoted by $\mathcal{H}$.

It is a recent observation in [BS14] that, when considering G-Higgs bundles over Riemann surfaces with a real structure, one is naturally lead to representations into G of the fundamental group of a 3-manifold with boundary X. These are naturally related to Schottky representations, as we present below. Our approach via Schottky representations has one advantage: by showing the vanishing of the complex symplectic form on the strict Schottky locus (see Proposition 7.3), we get a simple argument for the fact that (at least a natural component of) the Baraglia-Schaposnik brane is indeed non-empty and Lagrangian with respect to the natural complex structure of $\mathbb{B}$ (coming from the complex structure of G).

3.1. **Schottky representations and flat connections on a three manifold.** Suppose that our Riemann surface X, of genus g, is the boundary ∂M, of a compact 3-manifold M. Choose a basepoint in this boundary, $x_0 \in X \subset M$. From the inclusion of pointed spaces $(X, x_0) \hookrightarrow (M, x_0)$ one gets an induced homomorphism:

$$\varphi : \pi_1 = \pi_1(X, x_0) \to \pi_1(M, x_0), \tag{3.1}$$

between their fundamental groups.

One particularly interesting case is when X bounds a 3-dimensional handlebody M, so that $\pi_1(M, x_0)$ is free of rank g. In this case, by carefully choosing the generators of each fundamental group, we can arrange so that φ coincides with the map defining Schottky representations (see Lemma 2.2).

Proposition 3.1. *Let M be a compact 3-dimensional handlebody of genus g whose boundary is a compact surface X. Then, the moduli space $\mathbb{S}_s$ of strict Schottky representations with respect to φ coincides with the moduli space $\mathbb{F}_M(G)$ of flat G-connections over M.*

Proof. By hypothesis $\pi_1(M, x_0)$ is a free group of rank g, and π_1 has a "symplectic presentation" in terms of generators α_i and β_i, $i = 1, \cdots, g$, as in Equation (2.1), so that

$$\varphi(\alpha_i) = 1, \qquad \varphi(\beta_i) = \gamma_i, \qquad i = 1, \cdots, g,$$

where $\gamma_1, \cdots, \gamma_g$ form a free basis of $\pi_1(M, x_0)$. Thus, a strict Schottky representation $\rho : \pi_1 \to G$ with respect to φ factors through a representation of $\pi_1(M, x_0) \cong F_g$ via φ. By standard differential geometry arguments, this is precisely the same as saying that the corresponding flat connection ∇_ρ on X extends, as a flat connection, to the 3-manifold M. Conversely, a flat G-connection on M induces a representation $\rho : \pi_1 \to G$ satisfying $\rho(\ker \varphi) = \{e\}$, and thus it is a strict Schottky representation of π_1 (with respect to φ), by Lemma 2.2. This correspondence is well defined up to conjugation by G, and so, we have a natural identification:

$$\mathbb{S}_s = \mathrm{Hom}(F_g, G)/\!\!/G \cong \mathbb{F}_M(G),$$

as wanted. □

3.2. **Schottky representations and (A, B, A)-branes.** Suppose now that we have an anti-holomorphic involution $f : X \to X$, defining a real structure on X. This induces, as in [BS14, §3], an anti-holomorphic involution

$$f^* : \mathcal{H} \to \mathcal{H}, \tag{3.2}$$

where $\mathcal{H}$ is the moduli space of G-Higgs bundles over X. Following [BS14, §3], denote the set of fixed points of f^* in $\mathcal{H}$ by $\mathcal{L}_G$, and call it the *Baraglia-Schaposnik brane* inside $\mathcal{H}$.

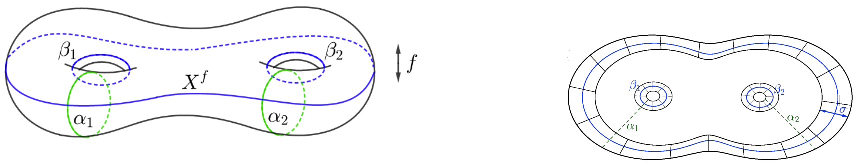
Consider the 3-manifold with boundary $\hat{X} := X \times [-1, 1]$. The anti-holomorphic involution $f : X \to X$ defines now an orientation preserving involution $\sigma : \hat{X} \to \hat{X}$ given by

$$\sigma(x, t) = (f(x), -t).$$

Note that the boundary of $\hat{X}$ consists of two copies of X, but the boundary of the compact 3-manifold $M := \hat{X}/\sigma$ is homeomorphic to X.

Proposition 3.2. *Let $f : X \to X$ be an anti-holomorphic involution such that M is a handlebody of genus g, and let $x_0 \in X \subset M$ be fixed by f. Then, the moduli space $\mathbb{S}_s$ of strict Schottky representations with respect to the map φ in* (3.1) *is included in the Baraglia-Schaposnik brane $\mathcal{L}_G$.*

Proof. In [BS14, Prop. 43], Baraglia and Schaposnik show that any flat G-connection on M defines, under the non-abelian Hodge theorem sending $\mathbb{B}$ to $\mathcal{H}$, a G-Higgs bundle which



(A) Involution f on X, and its fixed curves X^f (B) Involution σ restricted to $X^f \times I$

FIGURE 3.1. Involutions

is fixed by the involution f^*. Thus, they have produced a map, which they prove to be an inclusion:

$$\mathbb{F}_M(G) \to \mathcal{L}_G \subset \mathcal{H}.$$

Since, by Proposition 3.1, $\mathbb{S}_s$ can be identified with $\mathbb{F}_M(G)$ the proposition follows. □

Remark 3.3. The assumption of the previous proposition is verified when the anti-holomorphic involution f has as fixed point locus, X^f, the union of $g+1$ disjoint loops and disconnected orientation double cover (see [BS14, Proposition 3] and Figure 3.1). In this case, [BS14, Proposition 10] says that the set of smooth points of $\mathcal{L}_G$ is a non-empty Lagrangian submanifold of $\mathcal{H}$. In a future work, we plan to further address this construction.

4. Schottky G-bundles

Let again X be a compact Riemann surface, with fundamental group π_1 and $\rho : \pi_1 \to G$ be a representation into a reductive group. The associated bundle construction, from a universal cover $p : Y \to X$, defines a G-bundle over X associated to ρ. We write this G-bundle as $E_\rho := (Y \times G)/_\rho \pi_1$, with equivalence classes given by[2]

$$(y,\ g) \sim (\gamma \cdot y,\ \rho(\gamma) \cdot g)\,, \qquad \forall \gamma \in \pi_1,\ (y, g) \in Y \times G. \tag{4.1}$$

4.1. **Schottky principal bundles and the uniformization map.** Thus, the space of representations parametrizes holomorphic G-bundles, and we can view this construction as providing a natural map, that we call the *uniformization map*:

$$\begin{array}{rccc} \mathbf{E}: & \mathbb{B} & \to & M_G \\ & [\rho] & \mapsto & [E_\rho] \end{array} \tag{4.2}$$

Here, M_G represents the set of isomorphism classes of G-bundles that admit a holomorphic flat connection. To simplify terminology, we say that a bundle *is flat* if it admits a holomorphic flat connection. Note that $\mathbf{E}$ is well defined on conjugacy classes, since if ρ and σ are conjugate representations, then $E_\rho \cong E_\sigma$. Moreover, by considering the holonomy representation of a given flat G-bundle, the map $\mathbf{E}$ is easily seen to be surjective.

Definition 4.1. A G-bundle E over the Riemann surface X is called:

(1) a *Schottky G-bundle* if E is isomorphic to E_ρ for some Schottky representation $\rho : \pi_1 \to G$, that is, $\rho(\alpha_i) \in Z$ for all $i = 1, \cdots, g$.
(2) a *strict Schottky G-bundle* if E is isomorphic to E_ρ for some strict Schottky representation $\rho : \pi_1 \to G$, that is, $\rho(\alpha_i) = e$ for all $i = 1, \cdots, g$.

Remark 4.2. (1) *Schottky vector bundles* were defined by [Flo01] as vector bundles isomorphic to $V_\rho := (Y \times \mathbb{C}^n)/_\rho \pi_1$ for a representation $\rho : \pi_1 \to GL_n\mathbb{C}$ with $\rho(\alpha_i) = e$ for all $i = 1, \cdots, g$.

[2]We are using a left action both on Y and on G; this was chosen (other options would be equivalent) for a standard use of Fox calculus in section 8.

Then, the associated *frame bundle* is, by definition the $GL_n\mathbb{C}$-bundle defined by the same representation: $E_\rho = (Y \times GL_n\mathbb{C})/\!\!/_\rho\, \pi_1$. So, if V is a Schottky vector bundle then the associated frame bundle is a *strict* Schottky $GL_n\mathbb{C}$-bundle. In other words, according to our definition, Schottky vector bundles are the same as strict Schottky (principal) $GL_n\mathbb{C}$-bundles. See, however, Proposition 6.4 and Example 6.5.
(2) In terms of the uniformization map in Equation (4.2) we simply say that E is Schottky (resp. strict Schottky) if and only if $\mathbf{E}^{-1}([E]) \subset \mathbb{S}$ (resp. $\mathbf{E}^{-1}([E]) \subset \mathbb{S}_s$).

4.2. **Associated Schottky bundles.** In the following, we describe how the Schottky property is transferred to associated bundles. Throughout this section, G and H denote connected reductive algebraic groups, Z_G and Z_H the corresponding centers.

Suppose we have a G-bundle E over X. Then, the H-bundle over X, obtained from the trivial bundle $E \times H \to E$ by letting G act on H through a homomorphism $\phi : G \to H$ is denoted by $E(H) := (E \times H)/\!\!/_\phi\, G$, and we say that $E(H)$ is obtained from E by *extension of structure group.* This is, conceptually, the same as the construction of the bundle E_ρ starting from universal cover of X, the π_1 bundle $Y \to X$, and the homomorphism $\rho : \pi_1 \to G$, as in (4.1).

Proposition 4.3. *Let $\phi : G \to H$ be a group homomorphism and E be a Schottky G-bundle. Then:*

(1) *If E is a* strict *Schottky G-bundle, then $E(H)$ is a strict Schottky H-bundle.*
(2) *If $\phi(Z_G) \subset Z_H$, then $E(H)$ is a Schottky H-bundle.*

Proof. First note that if $E = E_\rho$, for some $\rho : \pi_1 \to G$, then $E(H) = E_{\phi\circ\rho}$. Then, assuming ρ is a strict Schottky representation, $\rho(\ker\varphi)$ is the identity of G (as in Lemma 2.2). This implies that $(\phi\circ\rho)(\ker\varphi) = \phi(e) = e_H$, the identity of H, so $E_{\phi\circ\rho}$ is a strict Schottky bundle, as wanted. The second case is similar, using the hypothesis $\phi(Z_G) \subset Z_H$. □

A $G \times H$-bundle E can be seen as an ordered pair (E_G, E_H), with E_G and E_H a G-bundle and a H bundle respectively. Indeed, from E we can define $E_G := E(G)$ and $E_H := E(H)$, where there are considered the projections $\pi_G : G\times H \to G$ and $\pi_H : G\times H \to H$, respectively. So, the following proposition is an easy consequence of the previous one.

Proposition 4.4. *A $(G \times H)$-bundle E is (strict) Schottky if and only if the E_G and E_H are (strict) Schottky principal bundles.*

Proof. Assume that $E = E_\rho$ for a certain Schottky representation $\rho = (\rho_G, \rho_H) : \pi_1 \to G\times H$. Then, both ρ_G and ρ_H are Schottky representations because $(\rho_G(\alpha_i), \rho_H(\alpha_i)) = \rho(\alpha_i) \in Z_{G\times H} = Z_G \times Z_H$ for $i = 1, \cdots, g$. By Proposition 4.3, $E_{\pi_G\circ\rho}$ is Schottky. On the other hand, it is easy to see that $E_G = E_{\rho_G} = E_{\pi_G\circ\rho}$, so E_G is Schottky. The same argument applies to E_H. The converse statement and the strict case are treated in a similar fashion. □

Let now $\mathfrak{g} = \mathsf{Lie}(G)$ be the Lie algebra of G. Given a G-bundle E, the $GL(\mathfrak{g})$-bundle associated to the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ corresponds, via the frame bundle construction, to the vector bundle (with $\mathfrak{g}$ as fiber):

$$\mathrm{Ad}(E) := E \times_{\mathrm{Ad}} \mathfrak{g}. \tag{4.3}$$

called the *adjoint bundle.*

Proposition 4.5. *If E is a Schottky G-bundle then the adjoint bundle* $\mathrm{Ad}(E)$ *is a Schottky vector bundle.*

Proof. Since E is a Schottky G-bundle, there is $\rho \in \mathrm{Hom}(\pi_1, G)$ with $\rho(\alpha_i) \in Z$ for all $i = 1, \cdots, g$, such that $E \cong E_\rho = (Y \times G)/_\rho \pi_1$. By construction, the vector bundle associated to E by the adjoint representation can be seen as

$$\mathrm{Ad}(E) = E \times_{\mathrm{Ad}} \mathfrak{g} \cong (Y \times \mathfrak{g})/_{\mathrm{Ad}_\rho} \pi_1$$

where $\mathrm{Ad}_\rho : \pi_1 \to G \to GL(\mathfrak{g})$ is the composition of the representations Ad and ρ. Because $\rho(\alpha_i) \in Z$ and since $\ker(\mathrm{Ad}) = Z$, we see that $\mathrm{Ad}_\rho(\alpha_i)$ is the identity map, for all $i = 1, \cdots, g$. Thus, we obtain a strict Schottky representation $\mathrm{Ad}_\rho : \pi_1 \to GL(\mathfrak{g})$. So, $\mathrm{Ad}(E) \cong Y \times_{\mathrm{Ad}\,\rho} \mathfrak{g}$ is a strict Schottky $GL(\mathfrak{g})$-bundle. □

The following simple example shows that the converse of Proposition 4.5 is not valid.

Example. Consider the $\mathbb{C}^*$-bundle $E \to X$ defined as the frame bundle of a line bundle L with non-zero first Chern class. Then, $\mathrm{Ad}(E)$ is the trivial line bundle, as conjugation is trivial in this case, so that $\mathrm{Ad}(E)$ is trivially a Schottky vector bundle. But E is not Schottky, as it does not admit a flat holomorphic connection (Weil's theorem [Wei38]).

However, by only requiring that E admits a flat holomorphic connection, we obtain a necessary and sufficient condition.

Proposition 4.6. *Suppose that the G-bundle E admits a flat holomorphic connection. Then, E is a Schottky G-bundle if and only if* $\mathrm{Ad}(E)$ *is a Schottky vector bundle.*

Proof. If E is Schottky, Proposition 4.5 implies that $\mathrm{Ad}(E)$ is Schottky. Conversely, suppose that E admits a flat G-connection. Then it is of the form $E \cong E_\rho$, for some $\rho : \pi_1 \to G$. Note that $\mathrm{Ad}(E_\rho) \cong E_{\mathrm{Ad}_\rho}$. Since by hypothesis $\mathrm{Ad}(E)$ is a Schottky vector bundle, this means that $\mathrm{Ad}_\rho(\alpha_i)$ is the identity morphism, $\forall i = 1, \cdots, g$. As $\ker(\mathrm{Ad}) = Z$ (because G is reductive), we may conclude that $\rho(\alpha_i) \in Z$ for all $i = 1, \cdots, g$, that is, $E \cong E_\rho$ where ρ is a Schottky representation. □

Moreover, when G is a connected semisimple algebraic group, we can drop the flatness condition above.

Theorem 4.7. *Let G be a connected semisimple algebraic group. Then E is a Schottky G-bundle if and only if the adjoint bundle* $\mathrm{Ad}(E)$ *is a Schottky vector bundle.*

Proof. By Proposition 4.5, if E is Schottky, $\mathrm{Ad}(E)$ is Schottky too. Conversely, assume that $\mathrm{Ad}(E)$ is a Schottky vector bundle. Then, $\mathrm{Ad}(E)$ admits a flat connection and [AB03, Proposition 2.2] proved that, because G is semisimple, E admits a flat connection too. So, the conditions of Proposition 4.6 are fulfilled, and E is a Schottky G-bundle. □

5. Topological type

The moduli space of G-bundles over a compact Riemann surface is a disjoint union of connected components indexed by $\pi_1(G)$, the fundamental group of G (see [Ram75], [GPO17]). In this section, we show that all Schottky G-bundles over a compact Riemann surface X have trivial topological type, corresponding to the identity element $0 \in \pi_1(G)$. Therefore, any Schottky G-bundle E is globally trivial in the smooth category, although it is generally non-trivial as a flat, or as an algebraic principal bundle.

5.1. **Topological types of G-bundles.** In this subsection, G is just a *connected topological group* which admits a universal cover (this is the case provided G is locally path connected and semilocally simply connected). To characterize G-bundles topologically, consider the short exact sequence of group homomorphisms

$$1 \to \ker p \to \widetilde{G} \xrightarrow{p} G \to 1, \tag{5.1}$$

where $p : \widetilde{G} \to G$ is a universal cover. It is known that $\ker p \cong \pi_1(G)$ is a discrete subgroup of the center of $\widetilde{G}$, so that (5.1) defines $\widetilde{G}$ as a central extension of G (cf. also Lemma 5.1 below). The exact sequence (5.1) induces a short exact sequence of sheaves

$$1 \to \pi_1(G) \to \underline{\widetilde{G}} \xrightarrow{p} \underline{G} \to 1,$$

where the underline denotes the sheaf of continuous functions defined on open subsets of the base X into the corresponding group. In turn, we get an exact sequence in (non-abelian) sheaf cohomology, with an associated coboundary map:

$$H^1(X, \underline{G}) \xrightarrow{\delta} H^2(X, \pi_1(G)) \cong \pi_1(G),$$

whose right isomorphism comes from using the orientation on X (see, for example [Gol88]). The map δ serves to define the *topological type* of a G-bundle. Namely, interpreting an isomorphism class of a G-bundle E as an element of $H^1(X, \underline{G})$ we define its topological type as (see also [Ram75, Remark 5.2])

$$\delta(E) := \delta([E]) \in \pi_1(G).$$

The topological type is functorial in the sense that, if a H-bundle E_H is obtained from a G-bundle E_G by extension of the structure group $\phi : G \to H$, then:

$$\delta(E_H) = \phi_*(\delta(E_G)), \tag{5.2}$$

using the induced morphism $\phi_* : \pi_1(G) \to \pi_1(H)$ (see [Ram75, Remark 5.1]). The following simple lemma should be well known, but we include a proof for convenience of the reader.

Lemma 5.1. *Let G be a connected, locally path connected and semilocally simply connected topological group, and $p : \widetilde{G} \to G$ be a universal cover of G. Then $Z_{\widetilde{G}} = p^{-1}(Z_G)$ and $p(Z_{\widetilde{G}}) = Z_G$.*

Proof. Let $\tilde{z} \in Z_{\widetilde{G}}$. Since p is surjective, for all $h \in G$, there is $\tilde{h} \in p^{-1}(h) \subset \widetilde{G}$, and we obtain

$$hp(\tilde{z}) = p(\tilde{h})p(\tilde{z}) = p(\tilde{h}\tilde{z}) = p(\tilde{z})p(\tilde{h}) = p(\tilde{z})h,$$

showing that $p(\tilde{z}) \in Z_G$. We conclude that $Z_{\widetilde{G}} \subset p^{-1}(Z_G)$.

Conversely, let $z \in Z_G$ and fix $\tilde{z} \in p^{-1}(z) \subset \widetilde{G}$. We want to show that $\tilde{z} \in Z_{\widetilde{G}}$. Since $\widetilde{G}$ is path connected, given $\tilde{h} \in \widetilde{G}$ there is a continuous path $\lambda : [0,1] \to \widetilde{G}$ with $\lambda(0) = \tilde{e}$ and $\lambda(1) = \tilde{h}$, where $\tilde{e}$ is the identity element of $\widetilde{G}$. Since $p(\tilde{z}\lambda(t)\tilde{z}^{-1}\lambda(t)^{-1}) = zp(\lambda(t))z^{-1}p(\lambda(t))^{-1} = e$, the following map is well defined and continuous:

$$\begin{array}{rcc} \Psi : [0,1] & \to & \ker p \\ t & \mapsto & \tilde{z}\lambda(t)\tilde{z}^{-1}\lambda(t)^{-1}. \end{array}$$

Noting that $\ker p \cong \pi_1(G) \subset Z_{\widetilde{G}}$ is a discrete subgroup of $\widetilde{G}$, the image of Ψ is constant and so $\Psi([0,1]) = \{\tilde{e}\}$. Thus:

$$\tilde{e} = \Psi(0) = \Psi(1) = \tilde{z}\tilde{h}\tilde{z}^{-1}\tilde{h}^{-1},$$

showing that $\tilde{z} \in Z_{\widetilde{G}}$. Finally $p(Z_{\widetilde{G}}) = Z_G$ is a simple consequence of $Z_{\widetilde{G}} = p^{-1}(Z_G)$. □

5.2. **Topological triviality of Schottky G-bundles.** Now, we return to the case where G is a *connected complex reductive group*, and suppose that E is a flat G-bundle, a bundle isomorphic to E_ρ for some $\rho : \pi_1 \to G$. Then, the value $\delta(E)$ lies, in fact, in the subgroup

$\pi_1(DG) \subset \pi_1(G)$ coming from the natural inclusion $DG \hookrightarrow G$, where DG is the derived group of G. Moreover, in [Ram75], Ramanathan defined a natural map from connected components of $\mathrm{Hom}(\pi_1, G)$ to $\pi_1(DG)$. More precisely, the following statement was recently shown in [LR15, Appendix] (following [Li93] and [Ram75]).

Theorem 5.2. *For any complex reductive group G, there is a natural bijection*

$$\pi_0(\mathrm{Hom}(\pi_1, G)) \cong \pi_1(DG).$$

In particular, for groups whose derived group is not simply connected, there exist *flat bundles E* which are *not topologically trivial.* By contrast, all Schottky bundles are topologically trivial, as we now show.

Theorem 5.3. *Let G be a connected complex reductive group, and let E be a Schottky G-bundle. Then E has trivial topological type.*

Proof. If $E \cong E_\rho$ is Schottky then it is defined by a representation $\rho : F_g \to Z_G \times G$. Since F_g is free, we can lift ρ to a representation $\widetilde{\rho} : F_g \to \widetilde{G} \times \widetilde{G}$, verifying $\rho = (p \times p) \circ \widetilde{\rho}$, where as above, $p : \widetilde{G} \to G$ is the universal cover ($\widetilde{G}$ is a complex Lie group, not necessarily algebraic). By Lemma 5.1, we see that $\widetilde{\rho}$ has image in $Z_{\widetilde{G}} \times \widetilde{G} \subset \widetilde{G} \times \widetilde{G}$, so it defines a Schottky representation inside $\mathrm{Hom}(\pi_1, \widetilde{G})$. The corresponding $\widetilde{G}$-bundle $E_{\widetilde{\rho}}$ is topological trivial, since $\pi_1(\widetilde{G})$ is trivial. Finally, as $E_\rho = E_{(p\times p)\circ\widetilde{\rho}}$ is obtained by extension of structure group, it is also topological trivial, by Equation (5.2) with $\phi = p \times p$. □

By [Ram75, Theorem 5.9], the components of the moduli space of *semistable* G-bundles $\mathcal{M}_G$ over a Riemann surface X are normal projective varieties indexed by the topological types of G-bundles. Thus, we can write the moduli space $\mathcal{M}_G$ as a disjoint union

$$\mathcal{M}_G = \bigsqcup_{\delta \in \pi_1(G)} \mathcal{M}_G^\delta.$$

where $\mathcal{M}_G^\delta$ denotes the moduli space of semistable G-bundles with topological type $\delta \in \pi_1(G)$. In [GPO17], the theory of G-Higgs bundles is used to prove the non-emptiness of the moduli spaces $\mathcal{M}_G^\delta$, for each topological type $\delta \in \pi_1(G)$ (see also [Ram96, Proposition 7.7]). In particular, $\mathcal{M}_G^0 \subset \mathcal{M}_G$ is connected.

Corollary 5.4. *The isomorphism class of a semistable Schottky G-bundle E lies in the connected component $\mathcal{M}_G^0$.*

6. The Uniformization Map

The association of a G-bundle to a representation of π_1 was called the uniformization map in section 4. In this section, we introduce the notion of analytic equivalence (see [Flo01]), consider the tangent space of Schottky space at good representations, and define the period map.

6.1. **Analytic equivalence.** Recall that the uniformization map (4.2)

$$\begin{array}{rccl} \mathbf{E}: & \mathbb{B} := \mathrm{Hom}\,(\pi_1, G) /\!\!/ G & \to & M_G \\ & [\rho] & \mapsto & [E_\rho] \end{array} \tag{6.1}$$

is surjective but, in general, not injective. This leads us to consider what we call *analytic equivalence.*

Definition 6.1. Two representations $\rho, \sigma \in \mathrm{Hom}\,(\pi_1, G)$ are called *analytically equivalent* if their associated G-bundles are isomorphic, so that $E_\rho \cong E_\sigma$, or equivalently $\mathbf{E}[\rho] = \mathbf{E}[\sigma]$.

The next result provides two useful criteria for analytic equivalence, generalizing Lemma 2 of ([Flo01]) (see also [Gun67]), one of them in terms of holomorphic sections of Ω^1_X, the canonical line bundle of X. Let $p : Y \to X$ be a universal covering map of X.

Theorem 6.2. *Let $\rho, \sigma \in \mathrm{Hom}(\pi_1, G)$ and $y_0 \in Y$. Then the following conditions are equivalent:*

(1) *$E_\sigma \cong E_\rho$, that is σ and ρ are analytically equivalent;*
(2) *There exists a holomorphic function $h : Y \to G$ such that*

$$h(\gamma \cdot y) = \rho(\gamma) h(y) \sigma(\gamma)^{-1}, \quad \forall \gamma \in \pi_1, y \in Y;$$

(3) *There exists $\omega \in H^0(X, \mathrm{Ad}(E_\sigma) \otimes \Omega^1_X)$ such that*

$$\sigma(\gamma) = h_\omega(\gamma \cdot y) \rho(\gamma) h_\omega(y)^{-1}, \quad \forall \gamma \in \pi_1, y \in Y$$

where h_ω is the unique solution of the differential equation $h^{-1}dh = \omega$ with the initial condition $h(y_0) = e \in G$.

Proof. (1) $\Leftrightarrow$ (2) Since the pullback $p^*(E_\sigma) \to Y$ using $p : Y \to X$ is a holomorphically trivial G-bundle on Y, its sections s_σ can be viewed as holomorphic maps $s_\sigma : Y \to G$ satisfying $s_\sigma(\gamma \cdot y) = \sigma(\gamma) s_\sigma(y)$ for all $\gamma \in \pi_1$, $y \in Y$ (and similarly for E_ρ). Analogously, an isomorphism $\psi : E_\sigma \to E_\rho$ is given by an isomorphism between the pullback bundles $\tilde{\psi} : p^*(E_\sigma) \to p^*(E_\rho)$ satisfying $\tilde{\psi}(y, g) = (y, h(y)\, g)$, $\forall (y, g) \in Y \times G$, for some holomorphic $h : Y \to G$. Since $\tilde{\psi}$ sends a section of $p^*(E_\sigma)$ to a section of $p^*(E_\rho)$ we have $h(y) s_\sigma(y) = s_\rho(y)$, for all $y \in Y$, which implies, $h(\gamma \cdot y) \sigma(\gamma) s_\sigma(y) = \rho(\gamma) h(y) s_\sigma(y)$, as wanted.

(2) $\Leftrightarrow$ (3) Writing $h^\gamma(y) := h(\gamma \cdot y)$ we have, from (2), the equation $\rho(\gamma) = h^\gamma \sigma(\gamma) h^{-1}$, for all $\gamma \in \pi_1$, as holomorphic functions on Y. Taking the exterior derivative we get

$$0 = d(h^\gamma)\gamma' \sigma(\gamma) h^{-1} - h^\gamma \sigma(\gamma) h^{-1} dh\, h^{-1},$$

(γ' is the derivative of $\gamma : y \mapsto \gamma y$) which is equivalent to $(h^\gamma)^{-1} dh^y\, \gamma' = \sigma(\gamma) h^{-1} dh \sigma(\gamma)^{-1}$, for all $\gamma \in \pi_1$. Setting $\omega := h^{-1}dh$, this equation can be rewritten as $\mathrm{Ad}_\sigma(\gamma) \cdot \omega = \omega^\gamma \gamma'$, which precisely means that the 1-form ω is the pullback to Y of a holomorphic section (still denoted the same) $\omega \in H^0(X, \mathrm{Ad}(E_\sigma) \otimes \Omega^1_X)$. Conversely, since the solution of the differential equation $\omega = h^{-1}dh$ with the condition $h(y_0) = e$ over the simply connected space Y is unique and satisfies the equality (3) then, obviously it satisfies (2). □

It is clear that Schottky space is different from the strict Schottky space, when the center Z is nontrivial. However, there is no need to distinguish the strict and non-strict cases when considering their associated bundles, in the case that Z is itself connected, as we now see.

Proposition 6.3. *Let G be a complex connected reductive group with center Z, and let $\rho : \pi_1 \to G$ and $\sigma, \nu : \pi_1 \to Z$ be representations. If there is an isomorphism $E_\sigma \cong E_\nu$ of Z-bundles, then the representations $\sigma\rho$, $\nu\rho \in \mathrm{Hom}(\pi_1, G)$, give isomorphic G-bundles $E_{\sigma\rho} \cong E_{\nu\rho}$.*

Proof. By Theorem 6.2, there exists a holomorphic function $h : Y \to Z$ such that $\nu(\gamma) h(y) = h(\gamma \cdot y) \sigma(\gamma)$, for every $\gamma \in \pi_1$, $y \in Y$. Considering this equation in G, and since ν, σ are in the center of G, we can multiply by $\rho(\gamma)$, obtaining:

$$\nu(\gamma) \rho(\gamma) h(y) = h(\gamma \cdot y) \sigma(\gamma) \rho(\gamma), \qquad \forall \gamma \in \pi_1,\ y \in Y.$$

Thus, $\nu\rho : \pi_1 \to G$ is analytically equivalent to the Schottky representation $\sigma\rho : \pi_1 \to G$. So again by Theorem 6.2, $E_{\sigma\rho} \cong E_{\nu\rho}$. □

Proposition 6.4. *Suppose that Z is connected. Then E is a G-Schottky bundle if and only if it is a strict G-Schottky bundle.*

Proof. A strict Schottky bundle is trivially a Schottky bundle. So, let $E = E_\rho$ be a Schottky G-bundle, with $\rho : \pi_1 \to G$ a Schottky representation and, using Theorem 6.2, we look for a strict Schottky representation analytically equivalent to ρ.

Let DG be the derived group of G. In terms of the well-known decomposition $G = Z \cdot DG$, and our usual generators, we can write $\rho(\alpha_i) = \nu(\alpha_i)\tilde{\rho}(\alpha_i)$ and $\rho(\beta_i) = \nu(\beta_i)\tilde{\rho}(\beta_i)$ for every $i = 1, \cdots, g$, for some $\nu(\alpha_i), \nu(\beta_i) \in Z$, with $\tilde{\rho}(\beta_i) \in DG$ and $\tilde{\rho}(\alpha_i) = e$. This assignment defines representations $\nu : \pi_1 \to Z$ and $\tilde{\rho} : \pi_1 \to DG$ satisfying $\rho(\gamma) = \nu(\gamma)\tilde{\rho}(\gamma)$ for all $\gamma \in \pi_1$.

The representation ν defines a Schottky Z-bundle, E_ν. As Z is connected, by Proposition 9.1 there is an isomorphism of Z-bundles $E_\nu \cong E_\sigma$, where E_σ is the Z-bundle associated to a strict Schottky representation $\sigma : \pi_1 \to Z$, (so that $\sigma(\alpha_i) = e$). By Proposition 6.3, $E_\rho = E_{\nu\tilde{\rho}} \cong E_{\sigma\tilde{\rho}}$. Since $\sigma\tilde{\rho} : \pi_1 \to G$ is a strict Schottky representation, we are done. □

Example 6.5. Since $\mathbb{C}^*$, the center of $GL_n\mathbb{C}$, is connected, every Schottky $GL_n\mathbb{C}$-bundle is strict Schottky. On the other hand, for vector bundles with trivial determinant, corresponding to $G = SL_n\mathbb{C}$, because $Z = \mathbb{Z}_n$, our definition of Schottky bundles is more general than the one used in [Flo01].

For later use, we now provide another description of the fiber of the uniformization map.

Definition 6.6. Given a representation $\rho \in \mathrm{Hom}\,(\pi_1,\, G)$, we define the following map, called the *orbit map*

$$\begin{array}{rcl} Q_\rho : H^0(X, \mathrm{Ad}\, E_\rho \otimes \Omega^1_X) & \to & \mathbb{B} \\ \omega & \mapsto & Q_\rho\,(\omega) := [\sigma]\,, \end{array}$$

with $\sigma \in \mathrm{Hom}\,(\pi_1,\, G)$ the representation given by

$$\sigma\,(\gamma) := h_\omega\,(\gamma \cdot y)\,\rho\,(\gamma)\,h_\omega\,(y)^{-1}\,, \quad \gamma \in \pi_1,\ y \in Y.$$

Here, h_ω is defined in Theorem 6.2 (3), whose proof readily shows the following.

Lemma 6.7. *The fibre $\mathbf{E}^{-1}([E_\rho])$ coincides with $Q_\rho\left(H^0\left(X, \mathrm{Ad}\, E_\rho \otimes \Omega^1_X\right)\right)$, the image of the orbit map. In other words, $E_\rho \cong E_\sigma$ if and only if $[\sigma] \in Im\,(Q_\rho)$.*

6.2. **Tangent spaces and group cohomology.** We now describe the tangent space of $\mathbb{B} = \mathrm{Hom}\,(\pi_1,\, G) /\!\!/ G$, at a good representation, in terms of the group cohomology of π_1.

More generally, let Γ denote a finitely generated group and fix $\rho \in \mathrm{Hom}\,(\Gamma, G)$. The adjoint representation on the Lie algebra of G, $\mathfrak{g} = \mathsf{Lie}(G)$, composed with ρ, that is

$$\mathrm{Ad}_\rho : \Gamma \to G \to GL\,(\mathfrak{g})\,, \tag{6.2}$$

induces on $\mathfrak{g}$ a Γ-module structure, which we denote by $\mathfrak{g}_{\mathrm{Ad}_\rho}$. The cohomology groups of Γ with coefficients in $\mathfrak{g}_{\mathrm{Ad}_\rho}$, are explicitly given by:

$$H^0\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) := Z^0\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = \left(\mathfrak{g}_{\mathrm{Ad}_\rho}\right)^\Gamma \qquad (\Gamma \text{ invariants in } \mathfrak{g}_{\mathrm{Ad}_\rho}),$$
$$H^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) := Z^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)/B^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$$

where (see, e.g., [Bro94])

$$\begin{array}{rcl} Z^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) & := & \{\phi : \Gamma \to \mathfrak{g} \,|\, \phi(\gamma_0\gamma_1) = \phi(\gamma_0) + \mathrm{Ad}_\rho(\gamma_0)\cdot\phi(\gamma_1)\,,\ \forall \gamma_0, \gamma_1 \in \Gamma\}\,, \\ B^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) & := & \{\phi : \Gamma \to \mathfrak{g} \,|\, \exists a \in \mathfrak{g},\ \ \phi(\gamma_0) = \mathrm{Ad}_\rho(\gamma_0)\cdot a - a\,,\ \forall \gamma_0 \in \Gamma\}\,. \end{array}$$

Let us recall the isomorphism between the Zariski tangent space of the character variety at a good representation ρ, and the first cohomology group $H^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$. The following result was proved by Goldman [Gol84], Martin [Mar00] (generalizing the case of $G = GL_n\mathbb{C}$ proved by Weil [Wei38]) and Lubotzky and Magid [LM85], see also [Sik12].

Theorem 6.8. *For a good representation $\rho \in \mathrm{Hom}\,(\Gamma, G)$ we have,*

$$T_{[\rho]}\left(\mathrm{Hom}\,(\Gamma, G)/\!\!/G\right) \cong H^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right).$$

The identification between tangent spaces to character varieties and group cohomology spaces is very useful in many situations. In particular, we can use it to compute the dimension of the complex manifolds $\mathbb{B}^{\mathsf{gd}} = \mathrm{Hom}\,(\pi_1,\ G)^{\mathsf{gd}}/\!\!/G$ and $\mathbb{S}^{\mathsf{gd}} \subset \mathbb{B}^{\mathsf{gd}}$, consisting of classes of good representations, when Γ is the fundamental group π_1 of a surface of genus g. In fact, by [Mar00, Lemma 6.2], we have, for $\rho \in \mathbb{B}^{\mathsf{gd}}$:

$$\dim Z^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = (2g-1)\dim G + \dim Z,$$
$$\dim B^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = \dim G - \dim Z,$$

and also the following.

Proposition 6.9. [Mar00] *If $[\rho] \in \mathbb{B}^{\mathsf{gd}}$, then*

$$T_{[\rho]}\mathbb{B} \cong H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$$

and $\dim T_{[\rho]}\mathbb{B} = (2g-2)\dim G + 2\dim Z$.

6.3. **The period map.** As in Theorem 6.2, seing holomorphic sections $\omega \in H^0(X, \mathrm{Ad}\,E_\rho \otimes \Omega^1_X)$ as 1-forms on the universal cover Y, we can integrate them along paths to obtain elements in group cohomology of π_1. This defines the period map.

Fix $y \in Y$ and $\omega \in H^0(X, \mathrm{Ad}\,E_\rho \otimes \Omega^1_X)$. Let us denote by ϕ^ω_y the map:

$$\begin{array}{cccc} \phi^\omega_y: & \pi_1 & \to & \mathfrak{g} \\ & \gamma & \mapsto & \phi^\omega_y(\gamma) := \int_y^{\gamma\cdot y}\omega, \end{array}$$

where we denote also by ω its pullback to Y. In fact, ϕ^ω_y is a cocycle in $Z^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho})$, and its cohomology class only depends on ω (and not on the basepoint $y \in Y$).

Proposition 6.10. *Fix a representation $\rho : \pi_1 \to G$, and $y \in Y$. Then, for every ω, $\phi^\omega_y \in Z^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho})$. Moreover, the assignement*

$$\begin{array}{rcl} P_{\mathrm{Ad}_\rho} : H^0\left(X, \mathrm{Ad}\,E_\rho \otimes \Omega^1_X\right) & \to & H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \\ \omega & \mapsto & [\phi^\omega_y], \end{array}$$

is a well defined linear map between finite dimensional $\mathbb{C}$-vector spaces, and is independent of $y \in Y$.

Definition 6.11. We call $P_{\mathrm{Ad}\,\rho}$, as defined above, the *period map* associated with ρ.

Proof. Using the action on 1-forms $\gamma\cdot\omega = \mathrm{Ad}_\rho(\gamma)\cdot\omega = (\omega\circ\gamma)\gamma'$, $\gamma \in \pi_1$, as in Theorem 6.2, we compute, by linearity and change of variable:

$$\begin{array}{rcl} \phi^\omega_y\left(\gamma_1\gamma_2\right) & = & \int_y^{\gamma_1\cdot y}\omega + \int_{\gamma_1\cdot y}^{\gamma_1\cdot(\gamma_2\cdot y)}\omega \\ & = & \phi^\omega_y(\gamma_1) + \int_y^{\gamma_2\cdot y}(\omega\circ\gamma_1)\,\gamma_1' \\ & = & \phi^\omega_y(\gamma_1) + \gamma_1\cdot\phi^\omega_y(\gamma_2), \end{array}$$

which shows that ϕ^ω_y is a cocycle in $Z^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$. The proof that $P_{\mathrm{Ad}_\rho}(\omega)$ is independent of the base point follows a similar computation to conclude that $\phi^\omega_y - \phi^\omega_{y'}$ is 1-coboundary, for another $y' \in Y$. □

Recall that the orbit map

$$Q_\rho : \ H^0(X, \operatorname{Ad} E_\rho \otimes \Omega^1_X) \to \mathbb{B},$$

(see Definition 6.6) verifies $Q_\rho(0) = [\rho]$, and its derivative at the identity, for a good representation ρ is a map:

$$d_0 Q_\rho : \ H^0(X, \operatorname{Ad} E_\rho \otimes \Omega^1_X) \to T_{[\rho]}\mathbb{B} \cong H^1(\pi_1, \mathfrak{g}_{\operatorname{Ad}_\rho}).$$

Lemma 6.12. *For a representation $\rho \in \operatorname{Hom}(\pi_1, G)$, the image of $d_0 Q_\rho$ coincides with the kernel of the map $d_{[\rho]}\mathbf{E}$, the derivative of $\mathbf{E}$ at $[\rho]$.*

Proof. Using Theorem 6.2, and Lemma 6.7, the proof is analogous to the proof of [Flo01, Lemma 4(a)]. □

For a good representation $\rho \in \operatorname{Hom}(\pi_1, G)$, such that $[\rho] \in \mathbf{E}^{-1}(\mathcal{M}_G)$, we can form the diagram

$$\begin{array}{ccccc} H^0\left(X, \operatorname{Ad} E_\rho \otimes \Omega^1_X\right) & \xrightarrow{d_0 Q_\rho} & T_{[\rho]}\mathbb{B} & \xrightarrow{d_{[\rho]}\mathbf{E}} & T_{[E_\rho]}\mathcal{M}_G \\ & \searrow^{P_{\operatorname{Ad}_\rho}} & \downarrow \cong & & \\ & & H^1\left(\pi_1, \mathfrak{g}_{\operatorname{Ad}_\rho}\right). & & \end{array} \tag{6.3}$$

The next result shows that, in fact, the triangle above is commutative.

Proposition 6.13. *For each good representation ρ in $\operatorname{Hom}(\pi_1, G)$, the maps $d_0 Q_\rho$ and $P_{\operatorname{Ad}_\rho}$, coincide under the vertical isomorphism of diagram (6.3).*

Proof. Since G is a connected reductive group over complex numbers, there is a faithful representation $\phi : G \to GL_n\mathbb{C}$. By associating a representation $\rho \in \operatorname{Hom}(\pi_1, G)$ to the composition $\phi \circ \rho = \bar{\rho} \in \operatorname{Hom}(\pi_1, GL_n\mathbb{C})$, ϕ induces an injective morphism of algebraic varieties $\bar{\phi} : \mathbb{B} \to \mathbb{G}_n$, where $\mathbb{G}_n = \operatorname{Hom}(\pi_1, GL_n\mathbb{C}) /\!\!/ GL_n\mathbb{C}$. The Lie algebra $\mathfrak{g}$, can be seen as a subalgebra of the Lie algebra $\mathfrak{gl} = M_n\mathbb{C}$ of $GL_n\mathbb{C}$, and we obtain an inclusion of π_1-modules $\mathfrak{g}_{\operatorname{Ad}_\rho} \subset \mathfrak{gl}_{\operatorname{Ad}_{\bar{\rho}}}$. On the other hand, Florentino proved in [Flo01, Lemma 4(b)] this result for $G = GL_n\mathbb{C}$. So, we obtain the following diagram, where $E_{\bar{\rho}}$ is the associated vector bundle of E_ρ.

$$\begin{array}{ccccccc} H^1\left(\pi_1, \mathfrak{g}_{\operatorname{Ad}_\rho}\right) & & & \longrightarrow & & & H^1\left(\pi_1, \mathfrak{gl}_{\operatorname{Ad}_{\bar{\rho}}}\right) \\ \uparrow & \nwarrow^{\cong} & & & & \nearrow^{\cong} & \uparrow \\ P_{\operatorname{Ad}_\rho} & & T_{[\rho]}\mathbb{B} & \xrightarrow{d_{[\rho]}\bar{\phi}} & T_{[\bar{\rho}]}\mathbb{G}_n & & P_{\operatorname{Ad}_{\bar{\rho}}} \\ \mid & \nearrow_{d_0 Q_\rho} & & & & \nwarrow_{d_0 Q_{\bar{\rho}}} & \mid \\ H^0\left(X, \operatorname{Ad}(E_\rho) \otimes \Omega^1_X\right) & & & \longrightarrow & & & H^0\left(X, \operatorname{End}(E_{\bar{\rho}}) \otimes \Omega^1_X\right) \end{array}$$

Above, the horizontal arrows are inclusions of vector spaces, because H^0 and H^1 behave functorially. Finally, since the triangle on the right is commutative, the same holds for the left triangle, as wanted. □

7. Schottky space

In this section we compute the dimension of Schottky space and prove that the strict Schottky space is a Lagrangian subspace of the Betti space. We also define the Schottky uniformization and moduli maps, by restricting the uniformization map to Schottky representations, and to those representations whose flat bundles are semistable.

7.1. **Dimension of Schottky space.** We now compute the dimensions of $\mathbb{S}$ and $\mathbb{S}_s$, using the techniques of group cohomology. By the density result (Theorem 2.14), the computations can be carried out at good representations. Using formula (2.6) we can write the inclusion $\mathbb{S}^{\mathsf{gd}} \subset \mathbb{B}^{\mathsf{gd}}$ as

$$\begin{array}{rcl} \mathrm{Hom}(F_g, Z) \times \mathrm{Hom}(F_g, G)^{\mathsf{gd}} /\!\!/ G \cong Z^g \times \mathbb{S}_s^{\mathsf{gd}} & \hookrightarrow & \mathbb{B}^{\mathsf{gd}} \cong \mathrm{Hom}(\pi_1, G)^{\mathsf{gd}} /\!\!/ G \\ (\rho_1, [\rho_2]) & \mapsto & [\rho]. \end{array}$$

Above, the notation should be clear according to Section 2. Correspondingly, from Theorem 6.8, we obtain the inclusion of tangent spaces:

$$T_{[\rho]}\mathbb{S} = T_{\rho_1}(Z^g) \oplus T_{[\rho_2]}\mathbb{S}_s \cong \mathfrak{z}^g \oplus H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_2}}\right) \hookrightarrow H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = T_{[\rho]}\mathbb{B} \tag{7.1}$$

for $[\rho] \in \mathbb{S}^{\mathsf{gd}}$. Recall that $\mathfrak{g}_{\mathrm{Ad}_{\rho_2}}$ denotes the F_g-module $\mathsf{Lie}(G) = \mathfrak{g}$, with the F_g-action given by the composition $F_g \stackrel{\rho_2}{\to} G \stackrel{\mathrm{Ad}}{\to} GL(\mathfrak{g})$.

Proposition 7.1. *Let $g \geq 2$. We have $\dim \mathbb{S}_s = (g-1)\dim G + \dim Z$.*

Proof. Since good representations are dense in $\mathbb{S}_s$, it is enough to compute the dimension at a strict good representation, $[\rho] \in \mathbb{S}_s^{\mathsf{gd}}$, $\rho : F_g \to G$. By Theorem 6.8, we know

$$\dim \mathbb{S}_s = \dim T_{[\rho]}\mathbb{S}_s = \dim H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right).$$

Since F_g is a free group, there is no cocycle condition, so any 1-cocycle is completely defined by the image of its generators; this means that $Z^1(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}) \cong \mathfrak{g}^g$. In order to compute the dimension of the space of 1-coboundaries, $B^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$, we consider the linear map between vector spaces

$$\begin{array}{rccl} \psi_\rho : & \mathfrak{g} & \to & \mathfrak{g}^g \\ & v & \mapsto & \left(\rho(\gamma_1) v \rho(\gamma_1)^{-1} - v, \cdots, \rho(\gamma_g) v \rho(\gamma_g)^{-1} - v\right), \end{array}$$

and note that $B^1(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}) = \psi_\rho(\mathfrak{g})$. Thus:

$$\dim B^1(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}) = \dim \psi_\rho(\mathfrak{g}) = \dim \mathfrak{g} - \dim \ker \psi_\rho = \dim \mathfrak{g} - \dim \mathfrak{z}(\rho)$$

where $\mathfrak{z}(\rho) := \{v \in \mathfrak{g} \mid v\rho(\gamma_i) = \rho(\gamma_i)v, \forall i = 1, \cdots, g\}$ is the Lie algebra of the stabilizer of ρ, $Z(\rho)$. Finally,

$$\dim H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = \dim Z^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) - \dim B^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = g \dim G - \dim G + \dim Z(\rho).$$

Since ρ is good, by definition $Z(\rho) = Z$, and the proof is finished. □

Corollary 7.2. *For $g \geq 2$, the Schottky space $\mathbb{S}$ is equidimensional (all irreducible components have the same dimension). Moreover,*

$$\dim \mathbb{S} = (g-1)\dim G + (g+1)\dim Z.$$

Proof. This follows immediately from the previous result and from Proposition 2.5, as $\dim Z^\circ = \dim Z$. □

7.2. **Lagrangian subspaces of $\mathbb{S}_s$.** Recall that a Lagrangian submanifold $L \subset M$ of a symplectic manifold M is a half dimensional submanifold such that the symplectic form vanishes on any tangent vectors to L.

It is well known that character varieties of surface group representations have a natural symplectic structure ([Gol84]), which can be constructed as follows. Consider an Ad-invariant bilinear form $\langle\, , \rangle$ on $\mathfrak{g}$. Then, using the cup product on group cohomology

$$\cup : H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \otimes H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \to H^2\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right), \tag{7.2}$$

and composing it with the contraction with $\langle\,,\,\rangle$ and with the evaluation on the fundamental 2-cycle, we obtain a non-degenerate bilinear pairing:

$$H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \otimes H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \xrightarrow{\cup} H^2\left(\pi_1,\, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \xrightarrow{\langle\,,\,\rangle} H^2\left(\pi_1,\, \mathbb{C}\right) \cong \mathbb{C} \tag{7.3}$$

Under the identification of $H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ with the tangent space at a good representation $\rho \in \mathbb{B}^{\mathsf{gd}}$, this pairing defines a complex sympletic form on the complex manifold $\mathbb{B}^{\mathsf{gd}}$. This symplectic form is complex analytic with respect to the complex structure on $\mathbb{B}^{\mathsf{gd}}$ coming from the complex structure on G, and $\mathbb{S}_s^{\mathsf{gd}} \subset \mathbb{B}^{\mathsf{gd}}$ is Lagrangian.[3]

Proposition 7.3. *The good locus of the strict Schottky space $\mathbb{S}_s^{\mathsf{gd}}$ is a Lagrangian submanifold of $\mathbb{B}^{\mathsf{gd}}$.*

Proof. The restriction of the map (7.2) to $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ is a vanishing map:

$$\cup : H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \otimes H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \to H^2\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = 0,$$

because free groups have vanishing higher cohomology groups (see [Bro94]). Since the tangent space, at a good point, to the strict Schottky locus $\mathbb{S}_s$ is identified with $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ (see Theorem 6.8), this means that the symplectic form, defined above on $\mathbb{B}^{\mathsf{gd}}$, vanishes on any two tangent vectors to $\mathbb{S}_s^{\mathsf{gd}}$. Since the dimension of $\mathbb{B}^{\mathsf{gd}}$ is twice the dimension of $\mathbb{S}_s^{\mathsf{gd}}$ (see Proposition 6.9 and Proposition 7.1), we conclude the result. □

Remark 7.4. (1) The proof that $\mathcal{L}_G$ is Lagrangian is only done for complex semisimple groups in [BS14]. Thus, Proposition 3.2 generalizes that statement for reductive complex algebraic groups. Moreover, since there are good strict Schottky representations for every $g \geq 2$, the current approach furnishes a proof that the Baraglia-Schaposnik branes are non-empty, at least in the conditions of Remark 3.3.
(2) Proposition 3.2 shows that we have an inclusion $\mathbb{S}_s \subset \mathcal{L}_G$ in the (A, B, A)-branes of [BS14] and in the case G is an adjoint group, $\mathbb{S} = \mathbb{S}_s \subset \mathcal{L}_G$. In a future work, we plan to study the conditions under which this inclusion is actually a bijection.

7.3. **The Schottky uniformization and moduli maps.**

Definition 7.5. The *Schottky uniformization map*

$$\mathbf{W} : \mathbb{S} \to M_G \tag{7.4}$$

is defined by $\mathbf{W}[\rho] := [E_\rho]$, the isomorphism class of the Schottky G-bundle E_ρ. From (4.2), $\mathbf{W} = \mathbf{E} \circ i$ where $i : \mathbb{S} \to \mathbb{B}$ is the inclusion from Proposition 2.4.

Remark 7.6. (1) As mentioned above, $\mathbf{W}[\rho]$ is not necessarily semistable. In fact, maximally unstable rank vector 2 bundles with trivial determinant are Schottky (see [Flo01]). Also, $\mathbf{W}$ is not injective in general: this happens already for the line bundle case (see [Flo01]).
(2) Recall that, from Theorem 5.3, $\mathbf{W}[\rho]$ has trivial topological type.

As defined, the target of the Schottky uniformization map M_G is a set, and it can be given the structure of a stack. However, since we want to consider the relation between Schottky space $\mathbb{S}$ and the moduli space of G-bundles, we need to further restrict $\mathbf{W}$ to be a morphism of algebraic varieties.

Let $\mathcal{M}_G^F = \mathcal{M}_G \cap M_G$ be the moduli space of semistable G-bundles on X that admit a flat connection. It is a, generally singular, projective complex algebraic variety. In order to characterize the derivative of the Schottky map $\mathbf{W}$, we will consider the subsets

$$\mathbb{B}^* := \mathbf{E}^{-1}\left(\mathcal{M}_G^F\right), \qquad \mathbb{S}^* := \mathbf{W}^{-1}\left(\mathcal{M}_G^F\right),$$

[3]For a general real Lie group, the analogous pairing defines a smooth (C^∞) symplectic structure, see [Gol84].

consisting of representations (resp. Schottky representations) $[\rho]$ whose associated bundles E_ρ are semistable.

Proposition 7.7. *For $g \geq 2$, the subset $\mathbb{S}^* \subset \mathbb{S}$ contains the unitary Schottky representations. Moreover, $\mathbb{S}^* \cap \mathbb{S}^{\mathsf{gd}}$ is open in $\mathbb{S}$.*

Proof. By Proposition 2.13 and Theorem 2.14 we know that $\mathbb{S}^{\mathsf{gd}}$ contains unitary representations and it is smooth and open in $\mathbb{S}$, since $g \geq 2$. If $\rho \in \mathcal{S}$ is a unitary representation, then E_ρ is semistable by Ramanathan's theorem. So, $[E_\rho] \in \mathcal{M}_G^F$ and $[\rho] \in \mathbf{W}^{-1}(E_\rho) \subset \mathbb{S}^*$. Thus $\mathbb{S}^* \cap \mathbb{S}^{\mathsf{gd}}$ is non-empty, so it is open in $\mathbb{S}$, by the coarse moduli property. □

Definition 7.8. The *Schottky moduli map*

$$\mathbf{V} : \mathbb{S}^* \to \mathcal{M}_G \tag{7.5}$$

is defined to be the restriction of the Schottky uniformization map $\mathbf{W}$ to the subset $\mathbb{S}^* = \mathbf{W}^{-1}(\mathcal{M}_G^F) \subset \mathbb{S}$ of representations defining semistable G-bundles.

Theorem 7.9. *Let ρ be a good Schottky representation, then*

$$\ker d_{[\rho]}\mathbf{V} \cong T_{[\rho]}\mathbb{S} \bigcap \operatorname{Im} d_0 Q_\rho. \tag{7.6}$$

Proof. It is immediate from Lemma 6.12. □

8. Surjectivity of the Schottky moduli map

In this section, we consider the image of the Schottky moduli map inside the moduli space of semistable G-bundles. The main result is the proof that this map is a local submersion at a good and unitary Schottky representation (see Theorem 8.6).

8.1. **Bilinear relations.** Let again K denote a maximal compact subgroup of the complex connected reductive algebraic group G. We fix an hermitian structure on the complex Lie algebra $\mathfrak{g}$ of G, denoted by $\langle\, ,\, \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ ($\mathbb{C}$-linear on the first entry) which is invariant under the adjoint action of K on $\mathfrak{g}$. For example, if $G = GL_n\mathbb{C}$, we can take $\langle A,\, B\rangle := \operatorname{tr}(AB^*)$, $\forall A,\, B \in \mathfrak{g}$,where $*$ means conjugate transpose and tr the matrix trace.

We now define an hermitian inner product on $H^0\left(X, \operatorname{Ad}(E_\rho) \otimes \Omega_X^1\right)$, when $\rho : \pi_1 \to K \subset G$ is a unitary representation. As before, Y is a universal cover of the compact Riemann surface X of genus $g \geq 2$, and we let $D \subset Y$ denote a fundamental domain for the quotient $X = Y/\pi_1$.

Definition 8.1. Let $\omega_1, \omega_2 \in H^0\left(X, \operatorname{Ad} E_\rho \otimes \Omega_X^1\right)$, with $\rho : \pi_1 \to K \subset G$. Define the following hermitian inner product

$$(\omega_1,\, \omega_2) := i \int_X \langle \omega_1, \omega_2 \rangle := i \int_D \langle h_1(z),\, h_2(z)\rangle\, dz \wedge d\bar{z} \tag{8.1}$$

where $\omega_i = h_i(z)dz$ for $z \in Y$.

Remark. The above integral depends on the choice of the hermitian inner product on $\mathfrak{g}$. However, by unitarity of ρ, it is independent of the choice of the fundamental domain D.

To prove the unitarity of the period map $\omega \mapsto P_{\mathrm{Ad}_\rho}(\omega)$, at unitary representations, generalizing [Flo01, Proposition 5], we need also a pairing on $H^1\left(\pi_1,\, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$. We use the so-called Fox calculus, and extend 1-cocycles $\phi : \pi_1 \to \mathfrak{g}_{\mathrm{Ad}_\rho}$ by $\mathbb{Z}$-linearity to the group ring $\mathbb{Z}[\pi_1]$ (see [Flo01, Gol84]). The boundary ∂D can be considered as a $4g$ polygon, with a vertex $z_0 \in Y$, and the other vertices ordered as:

$$\left\{z_0,\, \alpha_1 z_0,\, \alpha_1\beta_1 z_0,\, \alpha_1\beta_1\alpha_1^{-1} z_0,\, R_1 z_0,\, R_1\alpha_2 z_0, \cdots,\, R_g z_0 = z_0\right\}$$

where $R_k = \prod_{i=1}^{k} \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$, and define $R := R_g$. The Fox derivatives of R give:

$$\frac{\partial R}{\partial \alpha_i} := R_{i-1} - R_i\beta_i, \qquad \frac{\partial R}{\partial \beta_i} := R_{i-1}\alpha_i - R_i \ .$$

Introduce also a $\mathbb{Z}$-linear involution $\sharp$ on $\mathbb{Z}[\pi_1]$ defined by $\sharp\left(\sum n_i\gamma_i\right) := \sum n_i\gamma_i^{-1}$, $n_i \in \mathbb{Z}$. In particular:

$$\sharp\frac{\partial R}{\partial \alpha_i} = R_{i-1}^{-1} - \beta_i^{-1}R_i^{-1} \quad \text{and} \quad \sharp\frac{\partial R}{\partial \beta_i} = \alpha_i^{-1}R_{i-1}^{-1} - R_i^{-1}. \tag{8.2}$$

Definition 8.2. Define a pairing on $H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ by

$$\langle\!\langle \phi_1, \phi_2 \rangle\!\rangle := i\sum_{j=1}^{g} \left\langle \phi_1\left(\sharp\tfrac{\partial R}{\partial \beta_j}\right), \phi_2\left(\beta_j\right)\right\rangle - \left\langle \phi_1\left(\sharp\tfrac{\partial R}{\partial \alpha_j}\right), \phi_2\left(\alpha_j\right)\right\rangle,$$

for $\phi_1, \phi_2 \in Z^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$.

Remark 8.3. It can be shown that this pairing is well defined on cohomology classes, and is hermitian (being a complex analogue of the pairing in [Gol84]). Moreover, it coincides (up to the factor i) with the cup product pairing in (7.2), when using our hermitian structure on $\mathfrak{g}$.

Theorem 8.4. *Let $\rho : \pi_1 \to K \subset G$ be a unitary and good representation. Then, for all $\omega_1, \omega_2 \in H^0\left(X, \mathrm{Ad}\left(E_\rho\right) \otimes \Omega_X^1\right)$, we have:*

$$\left(\omega_1, \omega_2\right) = \left\langle\!\left\langle P_{\mathrm{Ad}_\rho}\left(\omega_1\right), P_{\mathrm{Ad}_\rho}\left(\omega_2\right)\right\rangle\!\right\rangle.$$

In other words, at a good and unitary representation, the period map is unitary.

Proof. Fix a base point $y = z_0 \in Y$, let $\phi_1(\gamma) := \int_y^{\gamma\cdot y}\omega_1$ be a 1-cocycle representing $P_{\mathrm{Ad}_\rho}(\omega_1)$, and write $\omega_2 = h(z)\,dz$. Define also $f : Y \to \mathfrak{g}$ by $f(z) := \int_y^z \omega_1$, so that we have $\omega_1 = df$ (with a slight abuse of notation we identify forms in $D \subset X$ with their pullbacks to Y). Computing as in Proposition 6.10, this function verifies

$$f(\gamma z) = \phi_1(\gamma) + \int_{\gamma\cdot y}^{\gamma\cdot z}\omega_1 = \phi_1(\gamma) + \int_y^z \gamma\cdot\omega = \phi_1(\gamma) + \gamma\cdot f(z), \tag{8.3}$$

where we write $\gamma \cdot f$ for the Ad_ρ-action of π_1 on functions on Y. Note that, for 1-forms on Y, we have $\gamma \cdot h\,dz = h(\gamma z)\gamma'(z)\,dz$.

Using $\langle\omega_1, \omega_2\rangle = d\langle f, hdz\rangle = d\left(\langle f, h\rangle\,\overline{dz}\right)$, applying Stokes' theorem to (8.1), and decomposing the boundary ∂D as the $4g$ polygon described above, we get:

$$\begin{aligned} (\omega_1, \omega_2) &= i\int_{\partial D}\langle f(z), h(z)\rangle\,\overline{dz} = \\ &= i\int_y^{\alpha_1 y}\langle f(z), h(z)\rangle\ \overline{dz} + \cdots + i\int_{R_{g-1}\alpha_g\beta_g\alpha_g^{-1}y}^{R_g y}\langle f(z), h(z)\rangle\ \overline{dz} \end{aligned} \tag{8.4}$$

For each $j = 1, \cdots, g$, we reduce the pair of integrals:

$$\int_{R_{j-1}y}^{R_{j-1}\alpha_j y}\langle f, h\rangle\,\overline{dz} + \int_{R_{j-1}\alpha_j\beta_j y}^{R_{j-1}\alpha_j\beta_j\alpha_j^{-1}y}\langle f, h\rangle\,\overline{dz}, \tag{8.5}$$

to a single one by using the change of variables property (8.3), and the Ad_ρ-invariance $\langle\gamma\cdot f, \gamma\cdot h\rangle = \langle f, h\rangle$ for all $\gamma \in \pi_1$. Employing the notation $f^\gamma \equiv f \circ \gamma$, and using

$R_{j-1}\alpha_j\beta_j\alpha_j^{-1} = R_j\beta_j$, the expression (8.5) equals:

$$\int_{R_{j-1}y}^{R_{j-1}\alpha_j y} \langle f,h\rangle\,\overline{dz} - \int_{R_j\beta_j y}^{R_j\beta_j\alpha_j y} \langle f,h\rangle\,\overline{dz} =$$

$$= \int_y^{\alpha_j y} \left(\left\langle f^{R_{j-1}}, h^{R_{j-1}}\right\rangle \overline{R'_{j-1}} - \left\langle f^{R_j\beta_j}, h^{R_j\beta_j}\right\rangle \overline{(R_j\beta_j)'}\right)\overline{dz}$$

$$= \int_y^{\alpha_j y} \left(\langle \phi_1(R_{j-1}) + R_{j-1}\cdot f,\, R_{j-1}\cdot h\rangle - \langle \phi_1(R_j\beta_j) + R_j\beta_j\cdot f,\, R_j\beta_j\cdot h\rangle\right)\overline{dz}$$

$$= \int_y^{\alpha_j y} \left(\langle \phi_1(R_{j-1}),\, R_{j-1}\cdot h\rangle - \langle \phi_1(R_j\beta_j),\, R_j\beta_j\cdot h\rangle\right)\overline{dz}$$

$$= \int_y^{\alpha_j y} \left(-\left\langle \phi_1(R_{j-1}^{-1}),\, h\right\rangle + \left\langle \phi_1(\beta_j^{-1}R_j^{-1}),\, h\right\rangle\right)\overline{dz}$$

$$= -\left\langle \phi_1(R_{j-1}^{-1}),\, h\right\rangle + \left\langle \phi_1(\beta_j^{-1}R_j^{-1}),\, h\right\rangle = -\left\langle \phi_1\left(\sharp\tfrac{\partial R}{\partial \alpha_j}\right),\, \phi_2(\alpha_j)\right\rangle$$

where we also used the cocycle property $\phi_1(\gamma) = -\operatorname{Ad}_\rho(\gamma)\cdot\phi_1(\gamma^{-1}) = -\gamma\cdot\phi_1(\gamma^{-1})$. An analogous computation for the integrals $\int_{R_{j-1}\alpha_j y}^{R_{j-1}\alpha_j\beta_j y}$ and $\int_{R_{j-1}\alpha_j\beta_j\alpha_j^{-1}y}^{R_j y}$, and a substitution into Equation (8.4) provides the desired formula. $\square$

Theorem 8.4 may be called the bilinear relations for periods of $\operatorname{Ad}(E_\rho)$ since it reduces to the classical Riemann's bilinear relations in the one dimensional case, that is, when $\rho \in \operatorname{Hom}(\pi_1, \mathbb{C}^*)$ (see [Flo01]).

8.2. **Derivative at unitary representations.** From Theorem 7.9 and (7.1), we know that the kernel of the derivative of the Schottky map at a good Schottky representation $\rho \in \operatorname{Hom}(\pi_1, G)$ is given by

$$\ker \mathrm{d}_{[\rho]}\mathbf{V} \cong T_{[\rho]}\mathbb{S} \bigcap \operatorname{Im} d_0 Q_\rho \cong \left(\mathfrak{z}^g \oplus H^1(F_g,\, \mathfrak{g}_{\operatorname{Ad}\rho_2})\right) \bigcap \operatorname{Im} d_0 Q_\rho$$

where $\rho = (\rho_1, \rho_2) : F_g \to Z \times G$, as in Section 2. According to Proposition 6.13, since $d_0 Q_\rho$, coincides with $P_{\operatorname{Ad}_\rho}$, we can write the kernel as the following intersection

$$\ker \mathrm{d}_{[\rho]}\mathbf{V} \cong \left(\mathfrak{z}^g \oplus H^1(F_g,\, \mathfrak{g}_{\operatorname{Ad}\rho_2})\right) \bigcap \operatorname{Im} P_{\operatorname{Ad}_\rho}.$$

Note that we are identifying the cohomology space, given by $\mathfrak{z}^g \oplus H^1(F_g,\, \mathfrak{g}_{\operatorname{Ad}\rho_2})$, with its image under the natural inclusion $\mathfrak{z}^g \oplus H^1\left(F_g,\, \mathfrak{g}_{\operatorname{Ad}_{\rho_2}}\right) \subset H^1\left(\pi_1,\, \mathfrak{g}_{\operatorname{Ad}_\rho}\right)$.

In the case ρ is strict, $T_{[\rho]}\mathbb{S}_s \cong H^1(F_g,\, \mathfrak{g}_{\operatorname{Ad}\rho_2})$ and we can identify the cohomology space $H^1\left(F_g,\, \mathfrak{g}_{\operatorname{Ad}_{\rho_2}}\right)$ with its image under the natural inclusion $H^1\left(F_g,\, \mathfrak{g}_{\operatorname{Ad}_{\rho_2}}\right) \subset H^1\left(\pi_1,\, \mathfrak{g}_{\operatorname{Ad}_\rho}\right)$.

Lemma 8.5. *Let ρ be a unitary and good strict Schottky representation. Consider $\omega \in H^0\left(X, \operatorname{Ad}(E_\rho)\otimes\Omega^1_X\right)$ such that $P_{\operatorname{Ad}_\rho}(\omega) \in H^1\left(F_g, \mathfrak{g}_{\operatorname{Ad}_{\rho_2}}\right)$ (in particular, the component of $P_{\operatorname{Ad}_\rho}(\omega)$ in $\mathfrak{z}^g$ vanishes). Then $\omega = 0$. In other words, under the stated conditions:*

$$H^1\left(F_g,\, \mathfrak{g}_{\operatorname{Ad}_{\rho_2}}\right) \bigcap \operatorname{Im} P_{Ad_\rho} = 0.$$

Proof. According to Theorem 8.4, the hermitian inner product of ω verifies $(\omega,\,\omega) = \left\langle\!\left\langle P_{\operatorname{Ad}_\rho}(\omega),\, P_{\operatorname{Ad}_\rho}(\omega)\right\rangle\!\right\rangle$. In this case the cup product of this class with itself is $P_{Ad_\rho}(\omega) \cup P_{Ad_\rho}(\omega) \in H^2\left(F_g, \mathfrak{g}_{\operatorname{Ad}_{\rho_2}}\right)$. Since for a free group F_g, $H^2\left(F_g,\, \mathfrak{g}_{\operatorname{Ad}_{\rho_2}}\right) = 0$, we obtain $P_{Ad_\rho}(\omega) \cup P_{Ad_\rho}(\omega) = 0$ and by Theorem 8.4, $\omega = 0$ since the Hermitian product is non-degenerate. $\square$

We can now prove our main result of this section (Theorem B of the introduction). Let $\mathbf{V}_s : \mathbb{S}_s \to \mathcal{M}_G$ be the restriction of the Schottky moduli map (Definition 7.8) to strict Schottky space.

Theorem 8.6. *Let ρ be a good and unitary Schottky representation, and suppose that $[E_\rho] \in \mathcal{M}_G$ is a smooth point. If ρ is strict, the derivative of the Schottky moduli map, $d_{[\rho]}\mathbf{V}_s : T_{[\rho]}\mathbb{S}_s \to T_{[E_\rho]}\mathcal{M}_G$, is an isomorphism. In the general case, the derivative of the Schottky moduli map $\mathbf{V} : \mathbb{S}^* \to \mathcal{M}_G$ has maximal rank at $[\rho]$. In particular, $\mathbf{V}$ is a local submersion so that, locally around $[\rho]$, it is a projection with* $\dim\left(\mathbf{V}^{-1}([E_\rho])\right) = g \dim Z^\circ$.

Proof. In the case ρ is strict,

$$\ker d_{[\rho]}\mathbf{V}_s \cong H^1\left(F_g,\, \mathfrak{g}_{\mathrm{Ad}_{\rho_2}}\right) \bigcap \mathrm{Im}\left(P_{\mathrm{Ad}_\rho}\right)$$

and by Lemma 8.5, $\dim\ker d_{[\rho]}\mathbf{V}_s = 0$. Since, by Theorem 7.1, $\dim T_{[\rho]}\mathbb{S}_s = (g-1)\dim G + \dim Z$ and by [Ram96, Theorem 5.9], $\dim \mathcal{M}_G = (g-1)\dim G + \dim Z$, thus

$$\dim T_{[\rho]}\mathbb{S}_s = \dim \mathcal{M}_G,$$

and we conclude that $d_{[\rho]}\mathbf{V}_s$ is an isomorphism at $[\rho]$, where ρ is a good and unitary strict Schottky representation.

In the general case, by (7.1), we have $T_{[\rho]}\mathbb{S} \cong \mathfrak{z}^g \oplus T_{[\rho_2]}\mathbb{S}_s$, where ρ_2 is a good and unitary strict Schottky representation. The tangent space $T_{[\rho_2]}\mathbb{S}_s$ can be identified with a subspace of $T_{[\rho]}\mathbb{S}$. By the previous case, $d_{[\rho_2]}\mathbf{V}$ is an isomorphism, so if we take as domain $T_{[\rho]}\mathbb{S}$, $d_{[\rho]}\mathbf{V}$ remains surjective with $\dim\ker d_{[\rho]}\mathbf{V} = g\dim Z^\circ$, because by Corollary 7.2 $\dim T_{[\rho]}\mathbb{S} = \dim T_{[\rho_2]}\mathbb{S}_s + g\dim Z^\circ$. □

Remark 8.7. If ρ is a unitary representation of $\mathrm{Hom}(\pi_1, G)$, the corresponding G-bundle is semistable, by the main result in Ramanathan [Ram75]. Assuming that $g \geq 3$ and ρ is good and unitary, then $[E_\rho]$ is stable and smooth in $\mathcal{M}_G$, by Biswas-Hoffmann [BH12, Lemma 2.2].

In the case G is semisimple, the previous theorem implies the following.

Corollary 8.8. *Let G be semisimple. Then, at a good and unitary Schottky representation ρ, the derivative of the Schottky map, $d_{[\rho]}\mathbf{V} : T_{[\rho]}\mathbb{S} \to T_{[E_\rho]}\mathcal{M}_G$, is an isomorphism.*

Proof. First of all notice that the dimension of both spaces is the same. Indeed, since G is semisimple, $\dim Z = 0$. Moreover, applying Corollary 7.2 to $T_{[\rho]}\mathbb{S}$ we get $\dim T_{[\rho]}\mathbb{S} = (g-1)\dim G$ and by [Ram96, Theorem 5.9], $\dim \mathcal{M}_G = (g-1)\dim G$. By Theorem 8.6, $\ker d_{[\rho]}\mathbf{V} = 0$, so the result follows. □

9. Some Special Classes of Schottky Bundles

In this section, we consider two special classes of Schottky G-bundles over a compact Riemann surface X: the case when G is a connected algebraic torus (over a general X); and general G-bundles over an elliptic curve (X has genus 1). Recall that, by slight abuse of terminology, we say that a bundle is flat if it admits a holomorphic flat connection.

9.1. **Abelian Schottky G-bundles.** Let G be a complex connected algebraic torus. Then, it is well known that G is isomorphic to $(\mathbb{C}^*)^n$, for some $n \in \mathbb{N}$. So, we fix $G = (\mathbb{C}^*)^n$, and note that, in this situation, Schottky spaces are smooth varieties for any g. Indeed, the space of strict Schottky representations becomes

$$\mathbb{S}_s = \mathrm{Hom}\left(F_g,\, (\mathbb{C}^*)^n\right) \cong (\mathbb{C}^*)^{ng}$$

and $\mathbb{S} = \mathrm{Hom}\left(F_g,\, (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n\right) \cong (\mathbb{C}^*)^{2ng}$. We now generalize the result of [Flo01], stating that all flat line bundles are strict Schottky $\mathbb{C}^*$-bundles.

Proposition 9.1. *Let E be a $(\mathbb{C}^*)^n$-bundle over a compact Riemann surface X. Then E is a* strict *Schottky bundle if and only if it is flat.*

Proof. If E is Schottky then it is induced by a representation, so E is flat, by definition. Assume now that E is a flat G-bundle, with $G = (\mathbb{C}^*)^n$. As in Proposition 4.4, we can view E as an ordered n-tuple of $\mathbb{C}^*$-bundles $(E_1, \cdots, E_n)$, and then each E_i admits a flat connection. On the other hand, it is well known that $\mathbb{C}^*$-bundles are equivalent to line bundles, i.e., vector bundles of rank one. So, each E_i is a line bundle of degree zero (since E_i is flat). According to [Flo01], every line bundle with degree 0 is a Schottky vector bundle, that is, a *strict* Schottky $\mathbb{C}^*$-bundle. So, this implies that E_i is strict Schottky for every $i = 1, \cdots, n$. Hence, by Proposition 4.4, E is also a strict Schottky bundle. □

Remark 9.2. (1) Replacing $(\mathbb{C}^*)^n$ by an arbitrary reductive abelian group G, not necessarily connected, one can show that the previous result is still valid.
(2) It has been shown in [FL14] that unipotent bundles (arising from successive extensions of $\mathbb{C}^*$-bundles) are also Schottky, and in fact, there is an equivalence of categories between flat unipotent bundles over X, and unipotent representations of free groups.

For $G = (\mathbb{C}^*)^n$, it is well known that all G-bundles, considered as (ordered) n-tuples of line bundles, are semistable. Thus, the moduli space of semistable $(\mathbb{C}^*)^n$-bundles coincides with the space of all $(\mathbb{C}^*)^n$-bundles:

$$\mathcal{M}_{(\mathbb{C}^*)^n} \cong H^1(X, (\mathcal{O}_X^*)^n) \cong H^1(X, \mathcal{O}_X^*)^n.$$

It is well known that this sits in an exact sequence

$$H^1(X, \mathcal{O}_X)^n \to H^1(X, \mathcal{O}_X^*)^n \to \mathbb{Z}^n,$$

whose last morphism is the multi-degree, or first Chern class. So, the space of flat $(\mathbb{C}^*)^n$-bundles coincides with the kernel of the degree map, that is, with

$$(H^1(X, \mathcal{O}_X^*)^n)^0 \cong J(X)^n,$$

where $J(X)$ is the Jacobian of X. In this context the strict Schottky moduli map looks as follows

$$\mathbf{V}_s : \mathrm{Hom}\,(F_g, (\mathbb{C}^*)^n) \to J(X)^n,$$

and Proposition 9.1 implies that $\mathbf{V}_s$ is onto (then, of course $\mathbf{V} : \mathbb{S} \to J(X)^n$). Also note that $\dim \mathbb{S}_s = \dim J(X)^n = ng$. So this description reproduces the line bundle case, for $n = 1$, treated in [Flo01].

9.2. **Schottky G-bundles over elliptic curves.** In this section, we consider principal Schottky bundles over an elliptic curve X, the case $g = 1$, which was excluded in previous sections[4]. Firstly, we consider the case of vector bundles over an elliptic curve and recall some results relating flat connections, semistability and the Schottky property. Then, we relate G-bundles with the corresponding adjoint bundle in order to translate some of the previous properties to this case.

We begin by recalling the following theorem, due to Atiyah and Tu [Ati57, Tu93], which relates semistability with the indecomposable property.

Theorem 9.3. [Tu93] *Every indecomposable vector bundle over an elliptic curve is semistable; it is stable if and only if its rank and degree are relatively prime.*

[4]Note that the case $X = \mathbb{P}^1$ ($g = 0$) is irrelevant, as π_1 is trivial and so are Schottky representations.

To relate flatness with semistability we now use Weil's theorem [Wei38, Theorem 10], which states that a vector bundle is flat if and only all its indecomposable components have degree zero.

Proposition 9.4. *Let V be a vector bundle over an elliptic curve X. Then, V is flat if and only if V is semistable of degree zero.*

Proof. By the Krull-Remak-Schmidt Theorem, we can write V as a direct sum of indecomposable subbundles

$$V = \oplus_{i=1}^{n} V_i.$$

Suppose that V is flat. By Weil's theorem mentioned above, $\deg(V_i) = 0$ and, by Theorem 9.3, each one of V_i's are semistable. Since the sum of semistable vector bundles of the same slope ($\mu(V_i) = \deg(V_i)/\mathrm{rk}(V_i) = 0$) is semistable (of the same slope), V is semistable with $\deg(V) = 0$. Conversely, let V be semistable of degree 0. Then $0 = \deg(V) = \sum \deg(V_i)$, and if some V_i has degree $\deg(V_i) \neq 0$ then, at least, there is one V_j with $\deg(V_j) > 0 = \deg(V)$. By definition, this contradicts the hypothesis that V is semistable. Therefore, all of these V_i's have degree zero which implies, by [Wei38, Theorem 10], that every summand V_i is flat. Since a direct sum of flat bundles admits a natural flat connection, V is itself flat. □

In [Flo01, Thm. 6], it is shown that all flat vector bundles over elliptic curves are Schottky. By considering adjoint bundles, we now establish similar conclusions for G-bundles over elliptic curves.

Proposition 9.5. *Let X be an elliptic curve, G a connected reductive algebraic group and E a G-bundle over X. Then the following are equivalent:*

(1) *E is semistable;*
(2) $\mathrm{Ad}(E)$ *is semistable;*
(3) $\mathrm{Ad}(E)$ *is flat.*

If G is semisimple, then all conditions above are equivalent to:

(4) *E is flat.*

Proof. [AB01, Proposition 2.10] states that E is semistable if and only if $\mathrm{Ad}\,(E)$ is also semistable; thus we obtain the equivalence between the two first assertions. The statements (2) and (3) are equivalent by Proposition 9.4. Finally, we can use [AB03, Proposition 2.2], in the case that G is semisimple, to conclude that E admits a flat connection if and only if $\mathrm{Ad}(E)$ admits one (see also [BG96]). □

Remark 9.6. When G is reductive, although the equivalence (3) $\Leftrightarrow$ (4) is not generally valid, we still can say that if E is flat, then $Ad(E)$ is flat (see [AB03, Proposition 3.1]).

Theorem 9.7. *Let X be an elliptic curve, and let E be a G-bundle over X, for a connected reductive algebraic group G. Then, E is flat if and only if E is Schottky. In other words, for $g = 1$, the Schottky uniformization map $\mathbf{W} : \mathbb{S} \to M_G$ is surjective.*

Proof. A Schottky G-bundle E is, by definition, flat. If the G-bundle E admits a flat connection then it induces a flat connection in $\mathrm{Ad}\,(E)$. Using [Flo01, Theorem 6], $\mathrm{Ad}\,(E)$ is strict Schottky, because it is a flat vector bundle of degree 0. By Proposition 4.6, since $\mathrm{Ad}\,(E)$ is Schottky and E is flat we obtain that E is a Schottky G-bundle. □

Remark 9.8. When G has a connected center the above result, together with Proposition 6.4, implies that, on an elliptic curve, E is flat if and only if it is a *strict* Schottky G-bundle.

The following Corollary follows directly from Proposition 9.5 and Theorem 9.7.

Corollary 9.9. *Let X be an elliptic curve and let G be a semisimple algebraic group. Then every semistable G-bundle over X is Schottky and it is strict Schottky if Z is connected. In particular, the Schottky moduli map $\mathbf{V} : \mathbb{S}^* \to \mathcal{M}_G$ is surjective.*

Remark 9.10. (1) A statement that includes both cases in Sections 9.1 and 9.2 is the following: Let X be a compact Riemann surface, G a connected reductive group, and E a G-bundle on X. If either π_1 or G are abelian, then E is flat if and only if E is Schottky.
(2) In the case $g = 1$, since $\pi_1 \cong \mathbb{Z}^2$, there are no irreducible representations (nor good representations) $\rho : \pi_1 \to G$, for non-abelian G. However, the moduli space of semistable G-bundles is non-empty, and is generally a weighted projective space (see for example [FMW98]).

Schottky vector bundles over elliptic curves, have been applied to an analytic construction of non-abelian theta functions for $G = SL_n\mathbb{C}$, which is completely analogous to the abelian classic case, [FMN03, FMN04], in the context of geometric quantization of the moduli space of vector bundles. In a future work, we plan to give a generalization of these results to Schottky G-bundles over an elliptic curve, for a general reductive algebraic group G.

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