

GENERATING SERIES FOR THE E -POLYNOMIALS OF $GL(n, \mathbb{C})$ -CHARACTER VARIETIES

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ABSTRACT. With $G = GL(n, \mathbb{C})$, let $\mathcal{X}_\Gamma G$ be the G -character variety of a given finitely presented group Γ , and let $\mathcal{X}_\Gamma^{irr} G \subset \mathcal{X}_\Gamma G$ be the locus of irreducible representation conjugacy classes. We provide a concrete relation, in terms of plethystic functions, between the generating series for E -polynomials of $\mathcal{X}_\Gamma G$ and the one for $\mathcal{X}_\Gamma^{irr} G$, generalizing a formula of Mozgovoy-Reineke [MR]. The proof uses a natural stratification of $\mathcal{X}_\Gamma G$ coming from affine GIT, the combinatorics of partitions, and the formula of MacDonal-Cheah for symmetric products; we also adapt it to the so-called Cartan brane in the moduli space of Higgs bundles. Combining our methods with arithmetic ones yields explicit expressions for the E -polynomials, and Euler characteristics, of the irreducible stratum of $GL(n, \mathbb{C})$ -character varieties of some groups Γ , including surface groups, free groups, and torus knot groups, for low values of n .

1. INTRODUCTION

Let G be a complex reductive algebraic group, Γ be a finitely presented group, such as the fundamental group of a compact manifold or a finite CW -complex, and let

$$\mathcal{X}_\Gamma G = \text{Hom}(\Gamma, G) // G$$

be the G -character variety of Γ : the (affine) geometric invariant theory quotient of the algebraic variety of representations of Γ into G . When the group Γ is the fundamental group of a Riemann surface (or more generally, a Kähler group) these varieties are homeomorphic to moduli spaces of G -Higgs bundles via the non-abelian Hodge correspondence (see, for example [Sim]), spaces which have been studied in connection to important problems in Mathematical-Physics in the context of mirror symmetry, and in the quantum field theory interpretation of the geometric Langlands correspondence [KW].

The study of geometric and topological properties of character varieties is an active topic and there are many recent advances in the computation of their Poincaré polynomials and other invariants, especially in the surface group case and for related groups Γ . With the introduction of arithmetic methods, Hausel and Rodriguez-Villegas [HRV1] showed that many of these varieties are of polynomial type, which allows, upon applying a theorem of N. Katz [HRV1, Appendix] to infer their E -polynomials by counting the number of points over finite fields. The fact that moduli spaces of Higgs bundles have pure cohomology allows the derivation of the Poincaré polynomial from the E -polynomial, and this approach was particularly successful in the case of smooth moduli spaces (see the works of Schiffmann, Mellit [Sc, Me], and references therein).

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However, explicitly computable formulae for these polynomials are very hard to obtain, in particular for many well known *singular* character varieties, as one can infer from the geometric methods of Logares, Muñoz, Newstead and Lawton [LMN], [LM] and from the arithmetic approach of Baraglia and Hekmati [BH], which become intractable for higher dimensional groups G .

In this article, we introduce another point of view in the computations of E -polynomials of $GL(n, \mathbb{C})$ -character varieties for *arbitrary* finitely presented Γ . In particular, our methods yield formulae for E -polynomials of character varieties which are not necessarily of polynomial type. The new approach is based on a stratification of $GL(n, \mathbb{C})$ -character varieties by partition type, and relates well with geometric and arithmetic techniques, relying also on the combinatorics of the plethystic functions, that have been previously used with success in connection with counting formulae for moduli spaces of polynomial type over finite fields.

This new perspective on E -polynomial calculations for character varieties, unveils another connection between the representation theory of $GL(n, \mathbb{C})$, and that of the symmetric group S_n . A similar approach may be possible for other reductive groups G , yielding a relation between effective E -polynomial computations for G -character varieties of an arbitrary Γ , and the representation theory of the Weyl group of G . Our approach is also intimately related to the plethystic program for counting gauge invariant operators in supersymmetric quantum field theories, where a fundamental role is played by symmetric products of the moduli spaces of vacua (see [FHH]). In another direction, by combining our approach with previous results on character varieties of free groups, we were able to prove (see [FNZ]) that the E -polynomials of $\mathcal{X}_\Gamma SL(n, \mathbb{C})$ and of $\mathcal{X}_\Gamma PGL(n, \mathbb{C})$ agree for all $n \in \mathbb{N}$, when Γ is a free group, an equality predicted in [LM, Rmk. 9] (and proved there for $n = 2, 3$).

We now outline the article, and some of the main results. In sections 2 and 3 we present the main properties of E -polynomials defined from mixed Hodge structures on complex quasi-projective varieties, and we describe natural methods for stratifying general G -character varieties in the context of affine geometric invariant theory (GIT). Since we always work over \mathbb{C} , we will abbreviate $GL(n, \mathbb{C})$ to GL_n . Let $E(X; u, v)$ denote the E -polynomial (in two variables u, v) of a quasi-projective complex variety X . In section 4 we introduce the stratification by partition type of the character varieties $\mathcal{X}_\Gamma GL_n$, for arbitrary Γ . Along with $\mathcal{X}_\Gamma GL_n$, we consider what we call the *irreducible character varieties*:

$$\mathcal{X}_\Gamma^{irr} GL_n \subset \mathcal{X}_\Gamma GL_n,$$

which are Zariski open subvarieties consisting of (equivalence classes of) irreducible representations $\rho : \Gamma \rightarrow GL_n$. Let us denote the plethystic exponential of a formal power series $f(x, y, z) \in \mathbb{Q}[x, y][[z]]$ by $\text{PExp}(f)$ (definition in Section 4). We prove:

Theorem 1.1. *Let Γ be a finitely generated group. Then, in $\mathbb{Q}[u, v][[t]]$:*

$$\sum_{n \geq 0} E(\mathcal{X}_\Gamma GL_n; u, v) t^n = \text{PExp} \left(\sum_{n \geq 1} E(\mathcal{X}_\Gamma^{irr} GL_n; u, u) t^n \right).$$

Unravelling the above power series, and the definitions and properties of the plethystic functions, we obtain a closed formula for each individual E -polynomial of $\mathcal{X}_\Gamma GL_n$ as a finite sum in the E -polynomials of the irreducible character varieties $\mathcal{X}_\Gamma^{irr} GL_n$ of lower dimension, indexed by what we call *rectangular partitions of n* (see Definition 4.14).

Corollary 1.2. *For every n and Γ as above,*

$$E(\mathcal{X}_\Gamma GL_n; u, v) = \sum_{[[k]] \in \mathcal{RP}_n} \prod_{l,h=1}^n \frac{E(\mathcal{X}_\Gamma^{irr} GL_l; u^h, v^h)^{k_{l,h}}}{k_{l,h}! h^{k_{l,h}}},$$

where \mathcal{RP}_n is the (finite) set of all rectangular partitions of n .

As a first application of these results, in Section 5 we write the E -polynomial of the abelian stratum $\mathcal{X}_\Gamma^{[1^n]} GL_n \subset \mathcal{X}_\Gamma GL_n$ in terms of usual partitions, generalizing a result in [FS]; we also apply the same methods to write the E -polynomial of the so-called Cartan brane on the moduli space of rank n and degree zero Higgs bundles, an algebraic variety which is generally not of polynomial type.

Theorem 1.1 and Corollary 1.2 work both ways so that, by knowing all polynomials $E(\mathcal{X}_\Gamma GL_m; u, v)$ for all $m \leq n$, we are able to determine $E(\mathcal{X}_\Gamma^{irr} GL_n; u, v)$. This is explored in the last subsection, where by using previous computations of E -polynomials of $\mathcal{X}_\Gamma GL_n$, for $n = 2$ and 3 , and for groups Γ other than the free group (mainly using [BH]), we determine E -polynomials of some *irreducible character varieties* that have not been calculated before: when Γ is the fundamental group of a compact surface (in both the orientable, and non-orientable cases) and when Γ is a torus knot group. From these formulæ, we readily obtain new results for these groups Γ : the number of irreducible components of $\mathcal{X}_\Gamma^{irr} GL_n$ and their Euler characteristics.

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2. MIXED HODGE STRUCTURES AND E -POLYNOMIALS

Let X be a quasi-projective variety over \mathbb{C} (possibly singular, not complete, and/or not irreducible). Denote by $H_c^k(X) := H_c^k(X, \mathbb{C})$ its degree k (singular) complex cohomology group, with compact support, for $k \in \{0, \dots, 2d\}$, where d is the complex dimension of X . Deligne defined natural and functorial mixed Hodge structures on the $H_c^k(X)$, which are subtle algebraic invariants of X (c.f. [De]). For the general theory of mixed Hodge structures on cohomology groups and its properties, see [De] and [PS]. Here, we review their most important features for our purposes, and introduce the notation.

2.1. Mixed Hodge polynomials. Numerically, mixed Hodge structures on X can be codified via the so-called mixed Hodge numbers

$$h^{k,p,q}(X) = \dim_{\mathbb{C}} H_c^{k,p,q}(X) \in \mathbb{N}_0,$$

where $p, q \in \{0, \dots, k\}$. We say that (p, q) are k -weights of X , when $h^{k,p,q} \neq 0$.

In general, mixed Hodge numbers verify $h^{k,p,q} = h^{k,q,p}$, and $\dim_{\mathbb{C}} H_c^k(X) = \sum_{p,q} h^{k,p,q}$, so they provide the (compactly supported) Betti numbers (and the usual Betti numbers, in the smooth case, by Poincaré duality). For some interesting classes of spaces, the above sum reduces to a one-variable sum. For example, when X is a compact Kähler manifold, the Hodge structure is called *pure*, which means that for each k , the only k -weights are of the form $(p, k-p)$ with $p \in \{0, \dots, k\}$. Another such case,

relevant for the present article, is when X is of *Hodge-Tate type* (also called *balanced type*), for which all the k -weights are of the form (p, p) with $p \in \{0, \dots, k\}$.

We can assemble all the $h^{k,p,q}(X)$ in the mixed Hodge polynomial

$$(2.1) \quad \mu(X; t, u, v) := \sum_{k,p,q \geq 0} h^{k,p,q}(X) t^k u^p v^q \in \mathbb{N}_0[t, u, v],$$

of three variables. The mixed Hodge polynomial specializes to the (compactly supported) Poincaré polynomial by setting $u = v = 1$, $P_t^c(X) = \mu(X; t, 1, 1)$. Again, this gives the usual Poincaré polynomial in the smooth situation.

2.2. The E -polynomial. Mixed Hodge polynomials are generally difficult to compute. However, by substituting $t = -1$ we obtain a certain Euler characteristic version, which is easier to compute due to its multiplicative and additive properties. We define the E -polynomial of X by

$$(2.2) \quad E(X; u, v) = \sum_{k,p,q} (-1)^k h^{k,p,q}(X) u^p v^q \in \mathbb{Z}[u, v],$$

which is also called the E -polynomial. Observe that

$$\chi^c(X) = E(X; 1, 1) = \mu(X; -1, 1, 1)$$

is the (compactly supported) Euler characteristic of X .

The Künneth theorem is valid for mixed Hodge structures (see [PS]) and so, μ verifies a multiplicative property with respect to Cartesian products:

$$\mu(X \times Y) = \mu(X)\mu(Y),$$

and induces analogous statements for P^c and E (we write simply $\mu(X)$, $P^c(X)$, $E(X)$ etc, in formulae where the variables of the polynomials are not relevant).

The big computational advantage of $E(X)$, as compared to $\mu(X)$ or $P^c(X)$ is that it satisfies both an *additive property with respect to stratifications* by locally closed (in the Zariski topology) strata and a *multiplicative property for fibrations* in at least three important situations that we summarize in the following statement.

Proposition 2.1. [DL, LMN] *If the quasi-projective variety X has a closed subvariety $Z \subset X$ (so that $X = Z \sqcup (X \setminus Z)$ is a stratification of X by locally closed subvarieties), then*

$$E(X) = E(Z) + E(X \setminus Z).$$

Also, if X is the total space of an algebraic fibration of quasi-projective varieties

$$F \rightarrow X \rightarrow B,$$

and either:

- (i) *it is locally trivial in the Zariski topology of B , or*
- (ii) *F , X and B are smooth, the fibration is locally trivial in the complex analytic topology, and $\pi_1(B)$ acts trivially on $H_c^*(F)$, or*
- (iii) *X , B are smooth and F is a complex connected Lie group.*

Then

$$E(X) = E(F) \cdot E(B).$$

Proof. The additive property is well known and can be found in [DL] or in the book [PS]. The multiplicative property presented here is a slight reformulation (in the non-equivariant case) of the one in Dimca-Lehrer [DL, Thm. 6.1] (and [DL, Remarks 6.2]), and also appears in [LMN, Prop. 1.9]; a more detailed proof has been recently presented in [FS], so we refer to those proofs, adding only a couple of comments that may serve to deduce the present statement.

The weight polynomial used by Dimca-Lehrer is equivalent to the E -polynomial in the case of Hodge-Tate type varieties, using the substitution $t^2 = uv$. So, this statement is a generalization of [DL, Thm. 6.1] to the 2 variable E -polynomial. Note also that the case (iii) actually follows from (ii) since the action of $\pi_1(B)$ on the cohomology of a connected Lie group F is always trivial. \square

Example 2.2. (1) Let $n \in \mathbb{N}_0$. Simple calculations give

$$\mu(\mathbb{C}^n) = t^{2n}u^n v^n, \quad \mu(\mathbb{C}^*; t, u, v) = t^2 uv + t.$$

This implies that $E(\mathbb{C}^n) = (uv)^n$ and $E(\mathbb{C}^*) = uv - 1$, a result compatible with the locally closed decomposition $\mathbb{C} = \mathbb{C}^* \sqcup \{0\}$. Note the absence of additivity for μ .

(2) The group $GL_n \mathbb{C}$ can be given as the fibration of smooth varieties,

$$SL_n \mathbb{C} \rightarrow GL_n \mathbb{C} \rightarrow \mathbb{C}^*,$$

whose projection map is the determinant. This is not locally trivial in the Zariski topology, but it is so in the analytic topology, and the fact that the complex Lie group $SL_n \mathbb{C}$ is connected implies that $\pi_1(\mathbb{C}^*)$ acts trivially on the cohomology of $SL_n \mathbb{C}$. Then, the Proposition 2.1 implies:

$$E(GL_n \mathbb{C}) = E(SL_n \mathbb{C})(uv - 1).$$

Note that all the groups in the fibration are of Hodge-Tate type.¹

In this article, if the E -polynomial of an algebraic variety X depends only on the product uv (for example, when X is of Hodge-Tate type, such as the cases in Example 2.2), we write $x = uv$ and use the notation:

$$E_x(X) := E(X; \sqrt{x}, \sqrt{x}) \in \mathbb{Z}[x].$$

For example, since $E(\mathbb{C}^*; u, v) = uv - 1$ we write $E_x(\mathbb{C}^*) = x - 1$. Then $E_x((\mathbb{C}^*)^l) = (x - 1)^l$, by the product formula.

3. AFFINE GIT AND CHARACTER VARIETIES

In this section we recall some aspects of Geometric Invariant Theory (GIT) and of character varieties of finitely presented groups.

3.1. Affine GIT. Consider an affine algebraic variety X over \mathbb{C} , and an affine algebraic reductive \mathbb{C} -group G . Given an algebraic action of G on X , we have an induced action of G on the ring $\mathbb{C}[X]$ of regular functions on X and we can define the (affine) GIT quotient by

$$X//G := \text{Spec}(\mathbb{C}[X]^G),$$

where $\mathbb{C}[X]^G$ denotes the subring of G -invariants in $\mathbb{C}[X]$. In many situations this quotient differs from the usual orbit quotient, since this one identifies G -orbits whose closures intersect. Nevertheless, with the notion of stability we can sometimes recover good properties of the GIT quotient. Let $G_x \subset G$ denote the stabilizer of a point $x \in X$ and let us call the subgroup $G_X := \bigcap_{x \in X} G_x$ the *center of the action*, since it acts trivially, and G/G_X acts effectively on X . Denote by ψ_x be the (effective) orbit map through x :

$$\begin{aligned} \psi_x : G/G_X &\rightarrow X \\ g &\mapsto g \cdot x. \end{aligned}$$

¹In fact, every complex algebraic reductive group G is of Hodge-Tate type (see, eg. [DL] and [Jo]).

Definition 3.1. In the situation above, we say that $x \in X$ is *polystable* if the orbit $G \cdot x$ is closed in X . We say that $x \in X$ is *stable* if it is polystable and ψ_x is a proper map.

Remark 3.2. This definition of stability differs from that of [MJK], being equivalent to the more common notion when G_X is finite (see [Ki, CF]). The above definition is more convenient in this article (as was the case in [Ki]) since, for character varieties, G_X always contains the center of G (see below).

By standard GIT results, one can show that the stable locus $X^s \subset X$ is a Zariski open (hence dense, when non-empty) set and one gets a better quotient for the stable locus. We say that a morphism

$$f : X \rightarrow Y$$

is a *geometric quotient* if f is G -invariant, induces the quotient topology on Y , and it is a bijection $Y = X/G$ which preserves rings of functions in the sense that $\mathbb{C}[f^{-1}(U)]^G = \mathbb{C}[U]$, for every $U \subset Y$ open. The following shows that the stable quotient is geometric.

Proposition 3.3. *The restriction $X^s \rightarrow X^s/G$ of the affine quotient map $\Phi : X \rightarrow X//G$ is a geometric quotient. Moreover, $\Phi(X^s)$ is Zariski open in $X//G$.*

Proof. See [Mu, Chap. 5]. □

3.2. Character varieties. Let G be as before, and let Γ be a finitely presented group. Denote by

$$\mathcal{R}_\Gamma G = \text{Hom}(\Gamma, G)$$

the algebraic variety of representations of Γ in G . An element $\rho \in \mathcal{R}_\Gamma G$ is defined by $\rho(\gamma)$, for γ in a generating set for Γ , and the elements $\rho(\gamma) \in G$, satisfy the algebraic relations of Γ . Consider also the algebraic action of G on $\mathcal{R}_\Gamma G$ by conjugation of representations. The corresponding GIT quotient is the G -character variety of Γ :

$$\mathcal{X}_\Gamma G := \text{Hom}(\Gamma, G)//G,$$

sometimes also called the moduli space of representations of Γ into G .

We will need to work also with an alternative description of this quotient, by using polystable representations which, according to Definition 3.1, are representations $\rho \in \text{Hom}(\Gamma, G) = \mathcal{R}_\Gamma G$ whose orbits $G \cdot \rho := \{g\rho g^{-1} : g \in G\}$ are (Zariski) closed. The subset of polystable representations in $\mathcal{R}_\Gamma G$ is denoted by $\mathcal{R}_\Gamma^{ps} G$, and it can be shown that $\mathcal{R}_\Gamma^{ps} G \subset \mathcal{R}_\Gamma G$ is a Zariski locally-closed subvariety (containing the stable locus $\mathcal{R}_\Gamma^s G \subset \mathcal{R}_\Gamma G$, but neither open nor closed in general). It can also be shown that a representation $\rho : \Gamma \rightarrow G$ is polystable if and only if it is completely reducible. This means that if $\rho(\Gamma)$ is contained in some proper parabolic $P \subset G$, then it is actually contained in a Levi subgroup of P (see [Si]).

Proposition 3.4. [FL1] *There is a bijective correspondence:*

$$\mathcal{X}_\Gamma G = \mathcal{R}_\Gamma G//G \cong \mathcal{R}_\Gamma^{ps} G/G,$$

where the right hand side is called the *polystable quotient*.

We also need the notion of irreducible representations, and consider the character varieties consisting of these “nicer” representations. For a given $\rho \in \mathcal{R}_\Gamma^{ps} G$, denote by $Z_\rho := G_\rho$ the centralizer of $\rho(\Gamma)$ inside G (coincides with the stabilizer of ρ). For character varieties, Z_ρ always contains ZG , the center of G . Hence, ZG is always contained in the center of the action (justifying our definition of stability, when $\dim ZG > 0$).

Definition 3.5. Let $\rho \in \mathcal{R}_\Gamma^{ps} G$. We say that ρ is *irreducible* if Z_ρ is a finite extension of ZG .

Remark 3.6. (1) This definition is equivalent to the usual definition, involving parabolic subgroups: indeed, $\rho \in \text{Hom}(\Gamma, G)$ is irreducible if and only if it is polystable and its image is not contained in a proper parabolic subgroup of G (see [CF, Si]).

(2) For character varieties, irreducibility is equivalent to stability in the sense of Definition 3.1 (see [CF, Prop. 5.11 (iii)]). So, the subset of irreducible representations, denoted $\mathcal{R}_\Gamma^{\text{irr}}G \subset \mathcal{R}_\Gamma^{\text{ps}}G$, equals the stable locus, and being a Zariski open subset of $\mathcal{R}_\Gamma G$, is a quasi-projective variety.

Since irreducibility is well defined on G -orbits, we define the G -irreducible character variety of Γ as

$$\mathcal{X}_\Gamma^{\text{irr}}G := \mathcal{R}_\Gamma^{\text{irr}}G/G$$

which is a geometric quotient. Hence, $\mathcal{X}_\Gamma^{\text{irr}}G$ is a Zariski open subvariety of $\mathcal{X}_\Gamma G$, by Proposition 3.3.

3.3. Stratification by stabilizer dimension. Let $G_{\mathcal{R}} := G_{\mathcal{R}_\Gamma G} \subset G$ be the center of the action of G on $\mathcal{R}_\Gamma G$, that is:

$$G_{\mathcal{R}} := \bigcap_{\rho \in \mathcal{R}_\Gamma G} Z_\rho,$$

where Z_ρ is the stabilizer of ρ . Then, $\dim Z_\rho \geq \dim G_{\mathcal{R}} \geq \dim ZG$, for all $\rho \in \mathcal{R}_\Gamma G$.

Proposition 3.7. *Let $m_0 := \dim G_{\mathcal{R}} \in \mathbb{N}_0$. Then, the character variety $\mathcal{X}_\Gamma G$ can be written as a union of locally closed quasi-projective varieties,*

$$\mathcal{X}_\Gamma G = \bigsqcup_{m \geq m_0} \mathcal{X}_\Gamma^m G,$$

where $\mathcal{X}_\Gamma^m G$ consists of equivalence classes of polystable representations ρ with $\dim Z_\rho = m$. Moreover, $\mathcal{X}_\Gamma^{m_0} G$ is precisely the open and dense stable locus $\mathcal{X}_\Gamma^s G = \Phi(\mathcal{R}_\Gamma^s G)$ as in Proposition 3.3.

Proof. Let $\mathcal{R}_\Gamma^m G \subset \mathcal{R}_\Gamma^{\text{ps}}G$ be the subset of all polystable representations $\rho \in \mathcal{R}_\Gamma^{\text{ps}}G$ such that $\dim Z_\rho = m$, and note that we have

$$(3.1) \quad \mathcal{R}_\Gamma^{\text{ps}}G = \bigsqcup_{m \geq m_0} \mathcal{R}_\Gamma^m G,$$

as a finite set-theoretic disjoint union. Since the stabilizer dimension is a conjugation invariant, denote their equivalence classes under conjugation by $\mathcal{X}_\Gamma^m G = \mathcal{R}_\Gamma^m G // G$. By Proposition 3.4, the character variety $\mathcal{X}_\Gamma G$ is isomorphic to the polystable quotient and, hence, equation (3.1) yields the set-theoretic disjoint union:

$$(3.2) \quad \mathcal{X}_\Gamma G \cong \bigsqcup_{m \geq m_0} \mathcal{R}_\Gamma^m G // G = \bigsqcup_{m \geq m_0} \mathcal{X}_\Gamma^m G.$$

To prove the locally closedness property, consider the following construction. Observe that the subset $R_\Gamma^{m_0} G = \{\rho \in \mathcal{R}_\Gamma^{\text{ps}}G \mid \dim Z_\rho = m_0 = \dim G_{\mathcal{R}}\}$ (which is non-empty by assumption) is precisely the subset of stable points, since the condition $\dim Z_\rho = m_0$ is equivalent to $\dim Z_\rho$ being minimal. This also means that the orbit map ψ_ρ is proper and conversely (see Definition 3.1), proving the last statement. Therefore, from Proposition 3.3, the restriction

$$\Phi_{m_0} : R_\Gamma^{m_0} G \rightarrow R_\Gamma^{m_0} G / G$$

is a geometric quotient, and the stable locus $R_\Gamma^{m_0} G \subset \mathcal{R}_\Gamma^{\text{ps}}G$ and $\mathcal{X}_\Gamma^{m_0} G := \Phi_{m_0}(R_\Gamma^{m_0} G) \subset \mathcal{X}_\Gamma G$ are Zariski open subsets. Now, let $R_\Gamma^{>m_0} G := \mathcal{R}_\Gamma^{\text{ps}}G \setminus R_\Gamma^{m_0} G$. Given that $R_\Gamma^{>m_0} G$

is Zariski closed in $\mathcal{R}_\Gamma^{ps}G$ and the action of G is well defined on it, we can repeat the argument for the subset:

$$R_\Gamma^{m_1}G := \{\rho \in R_\Gamma^{>m_0}G \mid \dim Z_\rho = m_1\} \subset R_\Gamma^{>m_0}G,$$

where $m_1 \in \mathbb{N}$ is the minimum of the dimensions of $\{Z_\rho \mid \rho \in R_\Gamma^{>m_0}G\}$; then $R_\Gamma^{m_1}G$ is a Zariski open (and non-empty) subset (of the Zariski closed set $R_\Gamma^{>m_0}G$) containing all stable representations in $R_\Gamma^{>m_0}G$. Hence, again, the restriction

$$\Phi_{m_1} : R_\Gamma^{m_1}G \rightarrow R_\Gamma^{m_1}G/G,$$

is a geometric quotient and $\mathcal{X}_\Gamma^{m_1}G := \Phi_{m_1}(R_\Gamma^{m_1}G)$ is also an open subset of the Zariski closed set $\mathcal{X}_\Gamma G \setminus \mathcal{X}_\Gamma^{m_0}G = R_\Gamma^{>m_0}G//G$, and therefore $\mathcal{X}_\Gamma^{m_1}G$ is locally closed. By repeating this procedure in a finite number of steps we obtain a stratification of the character variety $\mathcal{X}_\Gamma G$ by locally-closed quasi-projective varieties, which completes the proof. \square

4. THE LINEAR CASE: GL_n .

In this Section, we examine the linear case, the case of $G = GL_n$.

Let Γ be a finitely presented group. We now provide explicit formulae for the E -polynomials of GL_n -character varieties of Γ in terms of E -polynomials of all irreducible GL_m -character varieties of Γ , for $m \leq n$. These formulae present several interesting features: firstly, they are independent of the group Γ ; secondly they relate, not just the individual polynomials $E(\mathcal{X}_\Gamma GL_n)$, but their generating functions (as power series in a formal variable), to the corresponding generating functions of the $E(\mathcal{X}_\Gamma^{irr} GL_n)$; moreover, the relation between these two kinds of generating functions is the so-called *plethystic exponential* which plays a prominent role in the combinatorics of symmetric functions, and has applications in counting of gauge invariant operators in supersymmetric quantum theories (see eg. [FHH]).

Note that, besides their intrinsic relevance, irreducible character varieties often coincide, or are related with, the smooth locus of the full character varieties. For example, by [FL2] the irreducible character variety $\mathcal{X}_\Gamma^{irr} GL_n$ coincides precisely with the *smooth locus* of the full character variety $\mathcal{X}_\Gamma GL_n$, in the case of the free group $\Gamma = F_r$. Recently, this theme has been greatly expanded in [GLR].

4.1. The stratification by partition type. We start by describing what we call the *stratification by partition type* of our GL_n -character varieties, a convenient refinement of the stratification by stabilizer dimension of Proposition 3.7. Given the standard representation of GL_n in \mathbb{C}^n , we have a natural notion of direct sum $\rho_1 \oplus \rho_2 \in \mathcal{R}_\Gamma GL_{n_1+n_2}$ of representations $\rho_i : \Gamma \rightarrow GL_{n_i}$, $i = 1, 2$. This is clearly a commutative operation.

To proceed, we need to consider partitions of n , and employ the following ‘‘power’’ notation. A partition of $n \in \mathbb{N}$ is denoted by $[k] = [1^{k_1} \dots j^{k_j} \dots n^{k_n}]$ where the exponent k_j means that $[k]$ has $k_j \geq 0$ parts of size $j \in \{1, \dots, n\}$, so that $n = \sum_{j=1}^n j \cdot k_j$. The sum of the exponents $||k|| := \sum k_j$ will be called the *length* of $[k]$ and \mathcal{P}_n stands for the finite set of partitions of $n \in \mathbb{N}$. For example, $[1^2 4] \in \mathcal{P}_6$ is the partition $6 = 4 + 1 + 1$, whose length is 3.

Definition 4.1. Let $G = GL_n$ and $[k] \in \mathcal{P}_n$. We say that $\rho \in \mathcal{R}_\Gamma G = \text{Hom}(\Gamma, G)$ is $[k]$ -polystable if ρ is conjugated to

$$(4.1) \quad \bigoplus_{j=1}^n \rho_j$$

where each ρ_j is, in turn, a direct sum of $k_j > 0$ *irreducible* representations of $\mathcal{R}_\Gamma(GL_j)$, for $j = 1, \dots, n$ (by convention, if some $k_j = 0$, then ρ_j is not present in the direct sum). We denote $[k]$ -polystable representations by $\mathcal{R}_\Gamma^{[k]}G$ and use similar terminology/notation for equivalence classes under conjugation $\mathcal{X}_\Gamma^{[k]}G \subset \mathcal{X}_\Gamma G$.

Remark 4.2. We note that the trivial partition $[n] = [n^1]$ (of minimal length 1) corresponds exactly to the irreducible locus: $\mathcal{R}_\Gamma^{[n]}G = \mathcal{R}_\Gamma^{irr}G$ and $\mathcal{X}_\Gamma^{[n]}G = \mathcal{X}_\Gamma^{irr}G$. Moreover, $\mathcal{R}_\Gamma^{[k]}G \subset \mathcal{R}_\Gamma^{ps}G$ as every $[k]$ -polystable representation, being a sum of irreducibles, has indeed a closed G -orbit inside $\mathcal{R}_\Gamma G$.

Proposition 4.3. *Fix $n \in \mathbb{N}$, and let $G = GL_n$. The character variety $\mathcal{X}_\Gamma G$ can be written as a disjoint union, labelled by partitions $[k] \in \mathcal{P}_n$, of locally closed quasi-projective varieties of $[k]$ -polystable equivalence classes:*

$$\mathcal{X}_\Gamma G = \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{X}_\Gamma^{[k]}G,$$

and this stratification refines the one by stabilizer dimension (Proposition 3.7).

Proof. Let $[k] = [1^{k_1} \dots n^{k_n}]$ be a partition of n . As in the proof of Proposition 3.7 note that

$$(4.2) \quad \mathcal{R}_\Gamma^{ps}G = \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{R}_\Gamma^{[k]}G,$$

is a set theoretic disjoint union, and the analogous decomposition is valid for the polystable character variety $\mathcal{X}_\Gamma^{ps}G$. Indeed, every polystable representation is completely reducible and this means, for $G = GL_n$, that it is a direct sum of irreducibles. Thus, for every $\rho \in \mathcal{R}_\Gamma^{ps}G$ we have a *unique* partition $[k] = [1^{k_1(\rho)} \dots n^{k_n(\rho)}] \in \mathcal{P}_n$ so that $k_j(\rho)$ is the number (possibly zero) of representations in $\mathcal{R}_\Gamma^{irr}GL_j$ that appear in the decomposition of ρ in (4.1). In particular, the length of $[k]$, $\sum_j k_j(\rho) = |[k]|$, is well defined by $\rho \in \mathcal{R}_\Gamma^{ps}G$. Proposition 3.4 and equation (4.2) yield

$$(4.3) \quad \mathcal{X}_\Gamma G \cong \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{R}_\Gamma^{[k]}G // G = \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{X}_\Gamma^{[k]}G,$$

as a set theoretic disjoint union. The fact that this is a stratification by locally-closed quasi-projective varieties follows the same steps of the proof in Proposition 3.7, noting that, for every $\rho \in \mathcal{R}_\Gamma^{[k]}G$, we have

$$\dim Z_\rho = \sum k_j(\rho) = |[k]|.$$

Indeed, by Schur's lemma the stabilizer of $\rho_j \in \mathcal{R}_\Gamma^{irr}GL_j$ is the center of GL_j , which equals \mathbb{C}^* , independently of $j > 0$. Hence, this stratification refines the one in Proposition 3.7 and different partitions with the same length become (disjoint) irreducible components of each stratum by stabilizer dimension. \square

Corollary 4.4. *Let $G = GL_n \mathbb{C}$. If, for a given character variety $\mathcal{X}_\Gamma G$, all $[k]$ -polystable strata for $[k] \in \mathcal{P}_n$ are of Hodge-Tate type, then $\mathcal{X}_\Gamma G$ is of Hodge-Tate type.*

Proof. This follows at once, by combining Proposition 4.3 with Proposition 2.1. \square

Note that the converse statement is not valid in general. Moreover, there are character varieties which are not of Hodge-Tate type. Indeed, a recent article by I. Rapinchuk [Ra] showed that every irreducible affine variety, defined over \mathbb{Q} , can be written as an irreducible component of a character variety; this class certainly contains varieties which

are not of Hodge-Tate type, such as a smooth affine cubic in the plane isomorphic to an elliptic curve with one point removed².

4.2. Generating functions of E -polynomials. Recall that the partition $[k] = [1^{k_1} \dots n^{k_n}] \in \mathcal{P}_n$ has $k_j \geq 0$ parts of size $j \in \{1, \dots, n\}$. For each $[k] \in \mathcal{P}_n$, denote by $L_{[k]}$ the reductive subgroup:

$$(4.4) \quad L_{[k]} := GL_1^{k_1} \times \dots \times GL_n^{k_n} \subset GL_n,$$

which we call the $[k]$ -Levi of GL_n (in fact, all Levi subgroups of GL_n are conjugate to one obtained in this way).

Now, $L_{[k]}$ acts naturally, factorwise, on the space of polystable representations of type $[k]$, $\mathcal{R}_\Gamma^{[k]}G$, and the GIT quotient is a product of irreducible character varieties (recall that each block of polystable representations corresponds to irreducible ones):

$$(4.5) \quad \mathcal{R}_\Gamma^{[k]}G // L_{[k]} = (\mathcal{X}_\Gamma^{irr} GL_1)^{k_1} \times (\mathcal{X}_\Gamma^{irr} GL_2)^{k_2} \times \dots \times (\mathcal{X}_\Gamma^{irr} GL_n)^{k_n}.$$

Note, however, that this does not coincide with the $[k]$ -character variety $\mathcal{X}_\Gamma^{[k]}GL_n$ as defined in Proposition 4.3. Indeed, when some $k_j > 1$, there is a permutation group acting on $\mathcal{R}_\Gamma^{[k]}G$ by permuting the blocks of equal size. To obtain $\mathcal{X}_\Gamma^{[k]}G$ define, for each partition $[k] \in \mathcal{P}_n$, the finite subgroup

$$S_{[k]} := S_{k_1} \times S_{k_2} \times \dots \times S_{k_n} \subset S_n,$$

of the symmetric group S_n on n letters. For an algebraic variety X , we let $\text{Sym}^m(X) = X^m / S_m$ denote the m^{th} symmetric product of X .

Proposition 4.5. *Let $G = GL_n$ and let Γ be a finitely presented group. For every partition $[k] \in \mathcal{P}_n$, there are isomorphisms of algebraic varieties:*

$$\mathcal{X}_\Gamma^{[k]}G \cong \times_{j=1}^n \text{Sym}^{k_j}(\mathcal{X}_\Gamma^{irr} GL_j).$$

Proof. This follows directly from the construction above. Indeed, since:

$$\mathcal{X}_\Gamma^{[k]}G \cong \mathcal{R}_\Gamma^{[k]}G // G,$$

and the action of G on $\mathcal{R}_\Gamma^{[k]}G$ reduces to an action of $L_{[k]}$ and the action of permutation of blocks of equal size, we get from equation (4.5):

$$\mathcal{X}_\Gamma^{[k]}G \cong \left(\mathcal{R}_\Gamma^{[k]}G // L_{[k]} \right) / S_{[k]} \cong \left(\times_{j=1}^n (\mathcal{X}_\Gamma^{irr} GL_j)^{k_j} \right) / \left(\times_{j=1}^n S_{k_j} \right).$$

Moreover, since each subgroup $S_{k_j} \subset S_n$ only permutes the k_j blocks of size j , and does not act on other blocks, the result follows from $X^k / S_k = \text{Sym}^k X$. \square

By the above proposition, we need to consider symmetric products of irreducible character varieties. It is interesting to observe that the E -polynomials of symmetric products are intrinsically related to the so-called *plethystic exponential functions*, which we now recall. Given a power series $f \in \mathbb{Q}[x, y][[z]]$, formal in z , written in the form:

$$(4.6) \quad f(x, y, z) = \sum_{n \geq 0} f_n(x, y) z^n,$$

where $f_n(x, y) \in \mathbb{Q}[x, y]$ are polynomials in x, y , with rational coefficients³, the plethystic exponential, denoted PExp , is defined formally (in terms of the usual exponential)

²We thank Sean Lawton for providing us this reference.

³For our purposes, coefficients in \mathbb{Q} are enough, although the theory can be developed over any field or even ring.

as:

$$\text{PExp}(f) := e^{\Psi(f)} \in \mathbb{Q}[x, y][[z]],$$

where Ψ , called the (multi-variable) *Adams operator*, is the invertible \mathbb{Q} -linear operator on $\mathbb{Q}[x, y][[z]]$ acting on monomials in x, y and z as: $\Psi(x^i y^j z^k) = \sum_{l \geq 1} \frac{x^i y^j z^{lk}}{l}$, where $(i, j, k) \in \mathbb{N}_0^3 \setminus \{(0, 0, 0)\}$. Note that, from the additivity of Ψ , we get the property:

$$\text{PExp}(f_1 + f_2) = \text{PExp}(f_1) \text{PExp}(f_2), \quad \forall f_1, f_2 \in \mathbb{Q}[x, y][[z]].$$

Proposition 4.6. *Let X be a quasi-projective variety. Then, the generating function of the E -polynomial of its symmetric products $\text{Sym}^m(X)$, $m \in \mathbb{N}$, is a rational function, and can be written as:*

$$\sum_{n \geq 0} E(\text{Sym}^n(X); u, v) y^n = \text{PExp}(E(X; u, v)y).$$

Proof. We apply the generating function of J. Cheah who showed, in [Ch], the following formula:

$$\sum_{n \geq 0} \mu(\text{Sym}^n(X); t, u, v) y^n = \prod_{k, p, q \geq 0} (1 - (-t)^k u^p v^q y)^{(-1)^{k+1} h^{k, p, q}(X)},$$

(recall from equation (2.2) that $h^{k, p, q}(X)$ are the Hodge-Deligne numbers of X for cohomology with *compact support*). Since, by definition, $E(X; u, v) = \sum_{k, p, q \geq 0} (-1)^k h^{k, p, q}(X) u^p v^q$, the above equality becomes:

$$\sum_{n \geq 0} E(\text{Sym}^n(X); u, v) y^n = \prod_{k, p, q \geq 0} (1 - u^p v^q y)^{(-1)^{k+1} h^{k, p, q}(X)}.$$

The proof follows from the next Lemma, by using $a_{p, q} := \sum_{k \geq 0} (-1)^k h^{k, p, q}(X)$. \square

Recall that plethystic exponentials have also a *product form*. The following can be shown in much greater generality; we restrict to the case at need, for simplicity.

Lemma 4.7. *If $g(u, v) = \sum_{p, q \geq 0} a_{p, q} u^p v^q$ for some $a_{p, q} \in \mathbb{Z}$, then:*

$$\text{PExp}(g(u, v)y) = \prod_{p, q \geq 0} (1 - u^p v^q y)^{-a_{p, q}}.$$

Proof. Taking the logarithm of the left hand side, we get:

$$\begin{aligned} \Psi(g(u, v)y) &= \sum_{k \geq 1} \frac{g(u^k, v^k) y^k}{k} = \sum_{k \geq 1} \sum_{p, q \geq 0} \frac{a_{p, q} u^{kp} v^{kq} y^k}{k} = \\ &= \sum_{p, q \geq 0} a_{p, q} \sum_{k \geq 1} \frac{(u^p v^q y)^k}{k} = - \sum_{p, q \geq 0} a_{p, q} \log(1 - u^p v^q y) \\ &= \log \left(\prod_{p, q \geq 0} (1 - u^p v^q y)^{-a_{p, q}} \right), \end{aligned}$$

which is the logarithm of the right hand side. \square

Remark 4.8. As mentioned, the above product formula is valid more generally, and there are analogous formulae for formal power series in any number of variables.

We will also use the following property of formal power series.

Lemma 4.9. *Let R be a ring and let $g_n \in R[[t]]$ be a sequence of formal power series written as*

$$g_n(t) = \sum_{k \geq 0} a_k^{(n)} t^k, \quad n \in \mathbb{N}, \quad a_k^{(n)} \in R.$$

Then

$$\prod_{n \geq 1} g_n(t^n) = g_1(t) g_2(t^2) g_3(t^3) \cdots = \sum_{n \geq 0} \sum_{[k] \in \mathcal{P}_n} a_{k_1}^{(1)} \cdots a_{k_n}^{(n)} t^n.$$

Proof. This follows by expanding

$$\left(\sum_{k \geq 0} a_k^{(1)} t^k \right) \left(\sum_{k \geq 0} a_k^{(2)} t^{2k} \right) \left(\sum_{k \geq 0} a_k^{(3)} t^{3k} \right) \cdots = \sum_{n \geq 0} b_n t^n,$$

and noting that b_n collects all terms of the form $a_{k_1}^{(1)} \cdots a_{k_n}^{(n)}$ such that $n = \sum_{j=1}^n j k_j$. These are precisely the partitions of n . \square

We are now ready for the proof of Theorem 1.1, as follows.

Theorem 4.10. *Let Γ be any finitely presented group, and write $A_n^\Gamma(u, v) := E(\mathcal{X}_\Gamma GL_n; u, v)$, $B_n^\Gamma(u, v) := E(\mathcal{X}_\Gamma^{irr} GL_n; u, v)$. Then:*

$$\sum_{n \geq 0} A_n^\Gamma(u, v) t^n = \text{PExp} \left(\sum_{n \geq 1} B_n^\Gamma(u, v) t^n \right).$$

Proof. From Proposition 4.5, and the multiplicative property of E we get, for the $[k]$ -polystable stratum of $\mathcal{X}_\Gamma GL_n$:

$$E(\mathcal{X}_\Gamma^{[k]} GL_n; u, v) = a_{k_1}^{(1)}(u, v) \cdots a_{k_n}^{(n)}(u, v),$$

where we define the polynomials

$$a_k^{(j)}(u, v) := E(\text{Sym}^k(\mathcal{X}_\Gamma^{irr} GL_j); u, v) \in \mathbb{Z}[u, v].$$

Since the E -polynomial is also additive, we get by Proposition 4.3,

$$E(\mathcal{X}_\Gamma GL_n; u, v) = \sum_{[k] \in \mathcal{P}_n} E(\mathcal{X}_\Gamma^{[k]} GL_n; u, v) = \sum_{[k] \in \mathcal{P}_n} a_{k_1}^{(1)}(u, v) \cdots a_{k_n}^{(n)}(u, v).$$

Now, we form the generating function:

$$\sum_{n \geq 0} A_n^\Gamma(u, v) t^n = \sum_{n \geq 0} E(\mathcal{X}_\Gamma GL_n; u, v) t^n = \sum_{n \geq 0} \left(\sum_{[k] \in \mathcal{P}_n} a_{k_1}^{(1)}(u, v) \cdots a_{k_n}^{(n)}(u, v) \right) t^n$$

which, by Lemma 4.9 (with $R = \mathbb{Z}[u, v]$) equals to the product,

$$g_1(u, v)(t) g_2(u, v)(t^2) g_3(u, v)(t^3) \cdots = \prod_{n \geq 1} g_n(u, v)(t^n),$$

where:

$$\begin{aligned} g_n(u, v)(t) &:= \sum_{k \geq 0} a_k^{(n)}(u, v) t^k = \sum_{k \geq 0} E(\text{Sym}^k(\mathcal{X}_\Gamma^{irr} GL_n); u, v) t^k \\ &= \text{PExp}(E(\mathcal{X}_\Gamma^{irr} GL_n; u, v) t) = \text{PExp}(B_n^\Gamma(u, v) t), \end{aligned}$$

and the last line used Proposition 4.6. Finally, we use the multiplicative property of plethystic exponentials to obtain:

$$\sum_{n \geq 0} A_n^\Gamma(u, v) t^n = \prod_{n \geq 1} g_n(u, v)(t^n) = \prod_{n \geq 1} \text{PExp}(B_n^\Gamma(u, v) t^n) = \text{PExp}\left(\sum_{n \geq 1} B_n^\Gamma(u, v) t^n\right),$$

as wanted. \square

Corollary 4.11. *Assume that $\mathcal{X}_\Gamma^{irr} GL_n$ is of Hodge-Tate type. Then:*

$$\sum_{n \geq 0} A_n^\Gamma(x) t^n = \text{PExp} \left(\sum_{n \geq 1} B_n^\Gamma(x) t^n \right),$$

with $A_n^\Gamma(x) = E_x(\mathcal{X}_\Gamma GL_n)$ and $B_n^\Gamma(x) = E_x(\mathcal{X}_\Gamma^{irr} GL_n)$.

Remark 4.12. (1) When Γ is the free group, the above formula appears in the proof of [MR, Thm 2.5]. So Corollary 4.11 generalizes it to the general Hodge-Tate case.

(2) The case when $B_n^\Gamma(x) = 0$ for $n \geq 2$ is still interesting. For example, using $B_1^\Gamma(x) = (x-1)^r$ in Corollary 4.11, we recover the E -polynomials of the GL_n -character varieties of $\Gamma = \mathbb{Z}^r$, the free abelian group of rank r . See [FS] and Subsection 5.1 below.

(3) We thank S. Mogovoy for drawing our attention to his recent Preprint [Mo1], where another method of approaching this Corollary is suggested (cf, [Mo1, Thm. 1.2]), within a general framework for counting isomorphism classes of objects in additive categories over finite fields (which can be traced back to [Mo2]), using also Katz's Theorem [HRV1, Appendix]. However, our proof of Theorem (4.10), does not rely on counting points over finite fields, and hence remains valid for character varieties (over \mathbb{C}) which are not necessarily of Hodge-Tate type or of polynomial count. See Subsection 5.2 for an example.

4.3. Rectangular partitions and the E -polynomial of each individual strata.

A further combinatorial analysis of the plethystic exponential in Theorem 4.10 allows us to get an explicit formula relating the polynomials $A_n^\Gamma(u, v)$ and $B_n^\Gamma(u, v)$ through a *finite process*: indeed, for a fixed n , $A_n^\Gamma(u, v)$ only depends on $B_m^\Gamma(u, v)$ for $m \leq n$, and this can be given in a concrete way using what we call *rectangular partitions*. Moreover, this also allows to obtain a closed expression for the E -polynomials of each individual strata, which can easily be implemented algorithmically using standard computer software.

We start by noting that, in the particular case when f is of the form $f(x, y, z) = g(x, y)z$ (so that $f_1 = g$ in equation (4.6), the remaining terms being zero), the plethystic exponential can be written in yet another useful form, in terms of usual partitions.

Lemma 4.13. *For any $g(u, v) \in \mathbb{Q}[u, v]$, we have*

$$\text{PExp}(g(u, v)y) = \sum_{n \geq 0} \left(\sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{g(u^j, v^j)^{k_j}}{k_j! j^{k_j}} \right) y^n.$$

Proof. By direct computation, we have:

$$\begin{aligned} \text{PExp}(g(u, v)y) &= \exp(\Psi(g(u, v)y)) = \exp\left(\sum_{j \geq 1} \frac{g(u^j, v^j)y^j}{j}\right) \\ &= \prod_{j \geq 1} \exp\left(\frac{g(u^j, v^j)y^j}{j}\right) = \prod_{j \geq 1} \sum_{k \geq 0} \frac{g(u^j, v^j)^k y^{jk}}{k! j^k} \\ &= \sum_{n \geq 0} y^n \left(\sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{g(u^j, v^j)^{k_j}}{k_j! j^{k_j}} \right), \end{aligned}$$

where in the last expression we gather all terms that contribute to y^n . Since these correspond to writing $n = \sum_{j=1}^n j k_j$, they correspond to partitions of n . \square

To write PExp of an arbitrary series $f(x, y, z) \in \mathbb{Q}[x, y][[z]]$ in a similar way, we also need to develop a theory of *rectangular partitions* of a positive integer.

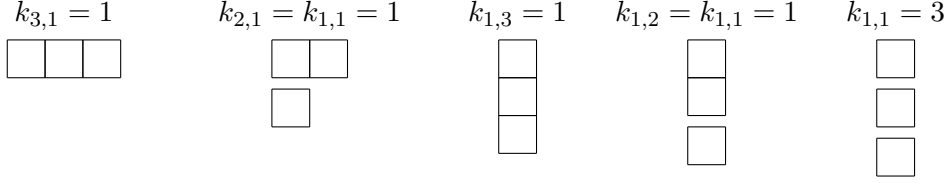


FIGURE 4.1. The five rectangular partitions of $n = 3$. The gluing map π takes the first one to the Young diagram of the partition $[3]$, the second one corresponds to $[1\ 2]$ and the last three to $[1^3]$.

Definition 4.14. Let $n \in \mathbb{N}$ be a natural number. A *rectangular partition* of n is a double sequence of non-negative integers $k_{l,h} \geq 0$ (and $k_{l,h} \leq n$) for each $l, h \in \{1, \dots, n\}$ satisfying

$$n = \sum_{l=1}^n \sum_{h=1}^n l h k_{l,h},$$

the finite set of rectangular partitions of n is denoted by \mathcal{RP}_n and such a rectangular partition is denoted by

$$[[k]] = [(1 \times 1)^{k_{1,1}} (1 \times 2)^{k_{1,2}} \dots (1 \times n)^{k_{1,n}} \dots (n \times n)^{k_{n,n}}] \in \mathcal{RP}_n.$$

There is a canonical “gluing map” sending a rectangular partition to a usual partition:

$$\begin{aligned} \pi : \mathcal{RP}_n &\rightarrow \mathcal{P}_n \\ [[k]] &\mapsto [m] = [1^{m_1} \dots n^{m_n}] \quad \text{defined by } m_l := \sum_{h=1}^n h \cdot k_{l,h}. \end{aligned}$$

The geometric interpretation of rectangular partitions is as follows: we are decomposing an initial set with area n , into a set of rectangles of each possible size $l \times h \leq n$ (of length l and height h), and each $l \times h$ rectangle appears with multiplicity $k_{l,h}$ (rectangles $l \times h$ and $h \times l$ are considered distinct). This explains the terminology “gluing map” as it is obtained by gluing all rectangles to form the usual Young diagram of a partition.

Example 4.15. For $n = 3$, Figure 4.1 shows the 5 possible rectangular partitions (all multiplicities $k_{l,h}$ not indicated are zero). Figure 4.2 shows the 11 cases for $n = 4$.

The following general formula may be useful in other situations.

Theorem 4.16. Given two sequences of polynomials $a_n(u, v), b_n(u, v) \in \mathbb{Q}[u, v]$, satisfying:

$$(4.7) \quad \sum_{n \geq 0} a_n(u, v) t^n = \text{PExp} \left(\sum_{n \geq 1} b_n(u, v) t^n \right)$$

we have:

$$a_n(u, v) = \sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{1}{k_j!} \left(\sum_{d|j} \frac{b_d(u^{j/d}, v^{j/d})}{j/d} \right)^{k_j} = \sum_{[[k]] \in \mathcal{RP}_n} \prod_{l,h=1}^n \frac{b_l(u^h, v^h)^{k_{l,h}}}{k_{l,h}! h^{k_{l,h}}}.$$

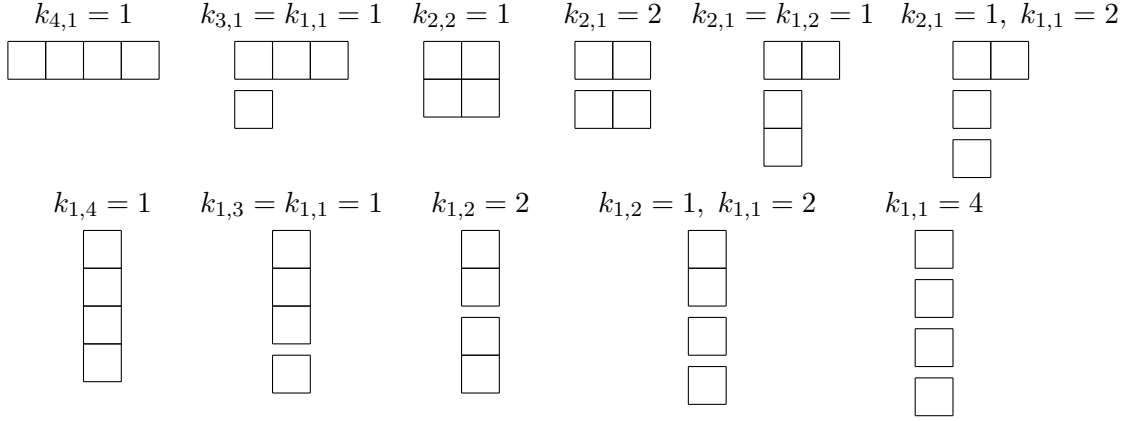


FIGURE 4.2. The eleven rectangular partitions of $n = 4$. The gluing map π takes the first rectangular partition to the Young diagram of the partition $[4]$, the second one corresponds to $[13]$, the third and fourth ones to $[2^2]$, the fifth and sixth to $[1^2 2]$ and the last five to $[1^4]$.

Remark 4.17. As with Lemma 4.7, both Lemma 4.13 and the above formulae are valid for an arbitrary number r of variables. For example, we have:

$$a_n(u_1, \dots, u_r) = \sum_{[[k]] \in \mathcal{RP}_n} \prod_{l,h=1}^n \frac{b_l(u_1^h, \dots, u_r^h)^{k_{l,h}}}{k_{l,h}! h^{k_{l,h}}},$$

when $\sum_{n \geq 0} a_n t^n = \text{PExp}(\sum_{n \geq 1} b_n t^n)$, and $a_n, b_n \in \mathbb{Q}[u_1, \dots, u_r]$.

Proof. In view of the above Remark, we consider the one variable case $x = uv$, the general case being analogous. By setting $C_j(x) := \sum_{d|j} \frac{b_d(x^{j/d})}{j/d}$, we first show that $a_n(x) = \sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{C_j(x)^{k_j}}{k_j!}$. This can be done by expanding the exponential:

$$\begin{aligned} \sum_{n \geq 0} a_n(x) t^n &= \exp(\Psi((b_1(x)t) + \Psi(b_2(x)t^2) + \dots)) \\ &= \exp(b_1(x)t + \frac{1}{2}b_1(x^2)t^2 + \dots + b_2(x)t^2 + \frac{1}{2}b_2(x^2)t^4 + \dots) \\ &= \exp(\sum_{n \geq 1} \sum_{d|n} \frac{d}{n} b_d(x^{n/d}) t^n) \\ &= \prod_{n \geq 1} \exp(C_n(x) t^n) = \prod_{n \geq 1} \sum_{k=0}^{\infty} \frac{1}{k!} C_n(x)^k t^{nk}. \end{aligned}$$

By collecting all terms contributing to a given $m = \sum j k_j$, we see that we are considering partitions of m , in the form $[k] = [1^{k_1} \dots m^{k_m}]$ and we get

$$\sum_{n \geq 0} a_n(x) t^n = \sum_{m \geq 1} \left(\sum_{[k] \in \mathcal{P}_m} \prod_{j=1}^m \frac{1}{k_j!} C_j(x)^{k_j} \right) t^m,$$

which finishes the proof of the first expression. To prove the second one, we need to collect all terms contributing to a given part of size j : we see that we are decomposing $j = lh$, as a rectangle of length l and height h , where the length appears as the subscript in the polynomials b_l , and the height appears as the power of the variable x . Moreover, the coefficient of each rectangle $l \times h$ is precisely $\frac{1}{k! h^k}$ if its multiplicity is

$k \geq 0$. So, we get a sum of rectangular partitions of n , where $[[k]] \in \mathcal{RP}_n$ contributes as:

$$\prod_{l,h=1}^n \frac{b_l(x^h)^{k_{l,h}}}{k_{l,h}! h^{k_{l,h}}},$$

as wanted. \square

Now, we can prove Corollary 1.2, and write the E -polynomial of each stratum (by partition type) in terms of the irreducible lower dimensional strata.

Theorem 4.18. *Let Γ be a finitely presented group. Then,*

$$E(\mathcal{X}_\Gamma GL_n; u, v) = \sum_{[[k]] \in \mathcal{RP}_n} \prod_{l,h=1}^n \frac{B_l^\Gamma(u^h, v^h)^{k_{l,h}}}{k_{l,h}! h^{k_{l,h}}},$$

Moreover, for a given $[m] \in \mathcal{P}_n$, the E -polynomial of the corresponding stratum is:

$$E(\mathcal{X}_\Gamma^{[m]} GL_n; u, v) = \sum_{[[k]] \in \pi^{-1}[m]} \prod_{l,h=1}^n \frac{B_l^\Gamma(u^h, v^h)^{k_{l,h}}}{k_{l,h}! h^{k_{l,h}}},$$

where $B_l^\Gamma(u, v) := E(\mathcal{X}_\Gamma^{irr} GL_l; u, v)$.

Proof. The first formula is just Theorem 4.16 for $B_l^\Gamma(u, v)$. The second formula is immediate from the above construction, as the only terms which contribute to a partition, i.e., to a given Young diagram, correspond to rectangular partitions whose image under π is that same Young diagram. \square

Example 4.19. The simplest case is $n = 2$, $G = GL_2$, where we get that the E -polynomial for the character variety is given by (dropping the superscript Γ in $A_i(u, v)$ and $B_i(u, v)$):

$$A_2(u, v) = E(\mathcal{X}_\Gamma GL_2; u, v) = \frac{1}{2}B_1(u^2, v^2) + \frac{1}{2}B_1(u, v)^2 + B_2(u, v),$$

since, for each stratum, we have:

$$\begin{aligned} E(\mathcal{X}_\Gamma^{[2]} GL_2; u, v) &= B_2(u, v), \\ E(\mathcal{X}_\Gamma^{[1^2]} GL_2; u, v) &= \frac{1}{2}B_1(u^2, v^2) + \frac{1}{2}B_1(u, v)^2. \end{aligned}$$

In the following examples, for brevity, we assume that $E(\mathcal{X}_\Gamma^{irr} GL_n; u, v)$ only depends on the product variable $x = uv$; the 2 variable E -polynomial is treated in exactly the same way.

Example 4.20. Next, with $n = 3$, $G = GL_3$, and using the same ordering as in Figure 4.1, we get 5 terms:

$$A_3(x) = E_x(\mathcal{X}_\Gamma GL_3) = B_3(x) + B_2(x)B_1(x) + \frac{B_1(x^3)}{3} + \frac{B_1(x^2)B_1(x)}{2} + \frac{B_1(x)^3}{6},$$

where the first term corresponds to $E_x(\mathcal{X}_\Gamma^{[3]} GL_3)$, the second to $E_x(\mathcal{X}_\Gamma^{[1^2]} GL_3)$, and remaining 3 terms to $E_x(\mathcal{X}_\Gamma^{[1^3]} GL_3)$.

Example 4.21. In a similar way, for $n = 4$, $G = GL_4$, we obtain the polynomials for each strata:

$$\begin{aligned} E_x(\mathcal{X}_\Gamma^{[4]}GL_4) &= B_4(x), \\ E_x(\mathcal{X}_\Gamma^{[1^3]}GL_4) &= B_3(x)B_1(x), \\ E_x(\mathcal{X}_\Gamma^{[2^2]}GL_4) &= \frac{B_2(x)^2}{2} + \frac{B_2(x^2)}{2}, \\ E_x(\mathcal{X}_\Gamma^{[1^22]}GL_4) &= \frac{B_2(x)B_1(x^2)}{2} + \frac{B_2(x)B_1(x)^2}{2}, \\ E_x(\mathcal{X}_\Gamma^{[1^4]}GL_4) &= \frac{B_1(x^4)}{4} + \frac{B_1(x^3)B_1(x)}{3} + \frac{B_1(x^2)^2}{8} + \frac{B_1(x^2)B_1(x)^2}{4} + \frac{B_1(x)^4}{24}, \end{aligned}$$

yielding $A_4(x) = E_x(\mathcal{X}_\Gamma GL_4)$ as the sum of these 5 strata (which comprise the 11 terms coming from the rectangular partitions in Figure 4.2).

5. SOME EXPLICIT COMPUTATIONS, FOR LOW n

In this last section, we collect several explicit computations of E -polynomials of GL_n -character varieties, and their partition type strata, for some classes of groups Γ , including surface groups, free groups and torus knot groups. We concentrate on the two extreme cases of these strata: the abelian stratum and the irreducible stratum. The abelian case allows some results for general n , but the E -polynomials of irreducible character varieties are typically very difficult to calculate; however, for low values of n , we can use some previous computations of E -polynomials (obtained in most cases by point counting over finite fields, see for example [MR, BH]) to determine the E -polynomials and the Euler characteristics of the irreducible character varieties, yielding new results for the irreducible stratum. As mentioned before the irreducible locus coincides, in many cases, with the smooth locus of the full character variety.

We also illustrate our methods with a simple computation of the E -polynomial of the Cartan brane inside the moduli space of GL_n -Higgs bundles, an object of interest in the geometric Langlands programme (see [FPN]).

5.1. E -polynomials of the abelian strata. We start by examining representations of finitely presented abelian groups. The following result, recently obtained in [FS], deals with the group $\Gamma = \mathbb{Z}^r$.

Theorem 5.1. *Let $r, n \in \mathbb{N}$. Then $\mathcal{X}_{\mathbb{Z}^r}GL_n$ is of Hodge-Tate type and*

$$E_x(\mathcal{X}_{\mathbb{Z}^r}GL_n) = \sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{(x^j - 1)^{r k_j}}{k_j! j^{k_j}}.$$

Proof. For the Hodge-Tate statement, see [FS] or Proposition 5.3, below. In that article, the formula is shown as a consequence of computing the mixed Hodge-Deligne polynomial of $\mathcal{X}_{\mathbb{Z}^r}GL_n$. For the E -polynomial we can also apply our main result obtaining a simpler proof. Being an abelian group, every irreducible representation of \mathbb{Z}^r is one dimensional, and for $n = 1$ we have

$$E_x(\mathcal{X}_{\mathbb{Z}^r}GL_1) = E_x(\text{Hom}(\mathbb{Z}^r, \mathbb{C}^*)) = E_x((\mathbb{C}^*)^r) = (x - 1)^r.$$

Then, the generating series $\sum_{n \in \mathbb{N}} B_n^{\mathbb{Z}^r}(x)t^n$ of the irreducible loci reduces to

$$B_1^{\mathbb{Z}^r}(x)t = (x - 1)^r t,$$

as $B_1^{\mathbb{Z}^r}(x) = E_x(\mathcal{X}_{\mathbb{Z}^r}^{irr} GL_1) = E_x(\mathcal{X}_{\mathbb{Z}^r} GL_1)$, and $B_n^{\mathbb{Z}^r}(x) \equiv 0$ for all $n > 1$. Then, the generating series for $E_x(\mathcal{X}_{\mathbb{Z}^r} GL_n)$, by Theorem 4.10, is

$$\sum_{n \geq 0} A_n^{\mathbb{Z}^r}(x) t^n = \text{PExp}(B_1^{\mathbb{Z}^r}(x)t) = \sum_{n \geq 0} \left(\sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{B_1^{\mathbb{Z}^r}(x^j)^{k_j}}{k_j! j^{k_j}} \right) t^n,$$

(where the last equality comes from Lemma 4.13) which immediately gives the result for $A_n^{\mathbb{Z}^r}(x) = E_x(\mathcal{X}_{\mathbb{Z}^r} GL_n)$. \square

Now, let us consider a general finitely presented group Γ , with abelianization

$$\Gamma_{Ab} := \Gamma / [\Gamma, \Gamma],$$

where $[\Gamma, \Gamma]$ is the normal subgroup generated by all commutators in Γ (words of the form $aba^{-1}b^{-1}$, $a, b \in \Gamma$). It is well known that $\Gamma_{Ab} \cong \mathbb{Z}^r \oplus F_N$, where $r \in \mathbb{N}_0$ is called the rank of Γ_{Ab} and the torsion F_N is a finite abelian group of order N . It is also clear that we have

$$(5.1) \quad \mathcal{R}_\Gamma GL_1 = \mathcal{R}_\Gamma^{irr} GL_1 \cong \mathcal{R}_{\Gamma_{Ab}} GL_1,$$

and the same applies to the corresponding character varieties. More generally, we have the following Lemma, that justifies calling $\mathcal{X}_\Gamma^{[1^n]} GL_n$ the *abelian stratum*.

Lemma 5.2. *For every $n \in \mathbb{N}$, the abelian stratum is isomorphic to the character variety of the abelianization of Γ :*

$$\mathcal{X}_\Gamma^{[1^n]} GL_n \cong \mathcal{X}_{\Gamma_{Ab}} GL_n.$$

Proof. This is a consequence of the analogous isomorphism of polystable loci:

$$\mathcal{R}_\Gamma^{[1^n]} GL_n \cong \mathcal{R}_{\Gamma_{Ab}}^{ps} GL_n,$$

which can be shown as follows. Every polystable representation of Γ_{Ab} into GL_n gives, by composition with the quotient $\Gamma \rightarrow \Gamma_{Ab}$, a representation of Γ which belongs to the $[1^n]$ stratum, since the only irreducible representations of an abelian group are one-dimensional. So, $\mathcal{R}_{\Gamma_{Ab}}^{ps} GL_n \subset \mathcal{R}_\Gamma^{[1^n]} GL_n$, and the inclusion is a morphism of algebraic varieties. Conversely, for a $[1^n]$ -polystable representation of Γ into GL_n , all the generators of Γ are sent to diagonal matrices (in some basis, being direct sums of one-dimensional representations); so, all commutators (the kernel of $\Gamma \rightarrow \Gamma_{Ab}$) are sent to the identity. Thus, it defines a unique GL_n representation of Γ_{Ab} . \square

Proposition 5.3. *Let Γ be a finitely generated group with abelianization $\Gamma_{Ab} = \mathbb{Z}^r \oplus F_N$, with $N = |F_N|$. Then, the abelian stratum is of Hodge-Tate type, and its E-polynomial satisfies:*

$$E_x(\mathcal{X}_\Gamma^{[1^n]} GL_n) = \sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{N^{k_j} (x^j - 1)^{rk_j}}{k_j! j^{k_j}}.$$

Proof. It follows from a formula of J. Cheah ([Ch]), that symmetric products of balanced varieties are balanced (see also [Sil]). Therefore, since

$$\mathcal{X}_\Gamma^{[1^n]} GL_n \cong \text{Sym}^n \mathcal{X}_\Gamma GL_1,$$

we only need to show that $\mathcal{X}_\Gamma GL_1$ is of Hodge-Tate type. Clearly,

$$\mathcal{X}_\Gamma GL_1 = \mathcal{X}_\Gamma^{irr} GL_1 \cong \mathcal{R}_\Gamma GL_1 \cong \mathcal{R}_{\Gamma_{Ab}} \mathbb{C}^* = \text{Hom}(\mathbb{Z}^r \oplus F_N, \mathbb{C}^*)$$

Since F_N is an abelian group of order $N \in \mathbb{N}$, it is a direct sum of cyclic groups \mathbb{Z}_m , $m \in \mathbb{N}$. The set $\text{Hom}(\mathbb{Z}_m, \mathbb{C}^*)$ is in bijection with the m^{th} roots of unity (sending the generator of \mathbb{Z}_m to each root). Therefore, $\text{Hom}(F_N, \mathbb{C}^*)$ has N elements and

$$\text{Hom}(\mathbb{Z}^r \oplus F_N, \mathbb{C}^*) \cong \text{Hom}(\mathbb{Z}^r, \mathbb{C}^*) \times \text{Hom}(F_N, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \times F_N,$$

which is clearly of Hodge-Tate type and we get:

$$B_1^\Gamma(x) = E_x(\mathcal{X}_{\Gamma}^{\text{irr}} GL_1) = N(x-1)^r.$$

Finally, using Theorem 4.18, the abelian stratum $[1^n]$ is obtained from all rectangular partitions with a single column (i.e. $k_{l,h} = 0$ unless $l = 1$, see Definition 4.14), and this corresponds to usual partitions $[k] \in \mathcal{P}_n$. So, we get:

$$(5.2) \quad E_x(\mathcal{X}_{\Gamma}^{[1^n]} GL_n) = \sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{B_1^\Gamma(x^j)^{k_j}}{k_j! j^{k_j}} = \sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{N^{k_j} (x^j - 1)^{r k_j}}{k_j! j^{k_j}}.$$

as wanted. \square

5.2. The Cartan brane of the moduli space of Higgs bundles. We now illustrate the method of computation of E -polynomials in a non-balanced case: the Cartan brane in the moduli space of Higgs bundles.

The non-abelian Hodge correspondence (c.f. [Sim]) establishes a homeomorphism between $\mathcal{X}_{\Gamma_g} GL_n$, for the surface group $\Gamma_g = \pi_1(\Sigma_g)$ (where Σ_g is a Riemann surface of genus g) and the moduli space $\mathcal{M}_n \Sigma_g$ of rank n Higgs bundles (E, φ) of degree zero, over Σ_g . This is a singular algebraic variety, whose singular stratification is again given in terms of partitions. More precisely, thinking of the moduli space as parametrizing *polystable Higgs bundles*, we have a disjoint union of locally closed subvarieties

$$\mathcal{M}_n \Sigma_g = \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{M}_{[k]} \Sigma_g,$$

where, for a partition $[k] = [1^{k_1} \dots n^{k_n}]$, $\mathcal{M}_{[k]} \Sigma_g$ is the locus of Higgs bundles of the form:

$$\bigoplus_{j=1}^n (E_j, \varphi_j)$$

where each (E_j, φ_j) is, in turn, a direct sum of $k_j > 0$ *stable* Higgs bundles of rank j (and degree zero), for $j = 1, \dots, n$ (again, if some $k_j = 0$, the Higgs summand (E_j, φ_j) is not present in the direct sum). Note that non-abelian Hodge correspondence matches precisely the strata labelled by the same partitions. That is, we have homeomorphisms

$$\mathcal{M}_{[k]} \Sigma_g \approx \mathcal{X}_{\Gamma_g}^{[k]} GL_n,$$

since *stable* Higgs bundles of a given rank $m \leq n$, correspond to *irreducible* representations of Γ_g into GL_m . However, these homeomorphisms are not holomorphic (on their smooth loci): in fact, the Hodge structure is *pure* on $\mathcal{M}_n \Sigma_g$ and *mixed* (of Hodge-Tate type) on $\mathcal{X}_{\Gamma_g}^{[k]} GL_n$.

Nevertheless, we can compute the E -polynomial of the is the lowest dimensional stratum, $\mathcal{M}_{[1^n]} \Sigma_g$, called the Cartan brane in [FPN], as follows. This stratum consists of direct sums of Higgs line-bundles of degree zero:

$$(L_1, \varphi_1) \oplus \dots \oplus (L_n, \varphi_n),$$

and so we have:

$$\mathcal{M}_{[1^n]} \Sigma_g \cong \text{Sym}^n \mathcal{M}_1 \Sigma_g,$$

where $\mathcal{M}_1\Sigma_g$ is isomorphic to the cotangent bundle of the Jacobian of Σ_g , $T^*(J\Sigma_g)$. Since we are dealing with symmetric products, the analogous method for obtaining the E -polynomial of $\mathcal{X}_{\Gamma_g}^{[1^n]}GL_n$, as in Proposition 5.3, gives:

$$E(\mathcal{M}_{[1^n]\Sigma_g}; u, v) = \sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{B_1(u^j, v^j)^{k_j}}{k_j! j^{k_j}},$$

where

$$B_1(u, v) = E(\mathcal{M}_1\Sigma_g; u, v) = E(T^*J\Sigma_g; u, v) = (uv)^g(1-u)^g(1-v)^g,$$

because the Jacobian $J\Sigma_g$ of Σ_g (an abelian variety of dimension g), has a pure Hodge structure and well known cohomology ring (the cotangent bundle of $J\Sigma_g$ is trivial, and the term $(uv)^g$ comes from the use of compactly supported cohomology). We have thus shown the following.

Theorem 5.4. *The E -polynomial of the Cartan brane of the moduli space of rank n Higgs bundles of degree zero is given by:*

$$E(\mathcal{M}_{[1^n]\Sigma_g}; u, v) = \sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{\left((u^j - u^{2j})(v^j - v^{2j}) \right)^{k_j g}}{k_j! j^{k_j}}.$$

5.3. E -polynomials of irreducible character varieties. In [MR], Mozgovoy and Reineke obtained formulae for the E -polynomials of GL_n -character varieties of the free group of rank r , $\Gamma = F_r$. Recently, for $n = 2$ and 3 , Baraglia and Hekmati derived explicit formulae for the E -polynomials of GL_n -character varieties of surface groups (both the orientable and non-orientable cases) and for torus knot groups (as well as the cases $G = SL_n$, for $n = 2, 3$ [BH]). These results were obtained by counting the number of points of a spreading out of these character varieties, over finite fields, and rely on a theorem of N. Katz ([HRV1, Appendix]). Briefly, the later proves that, if there is unique polynomial that encodes the number of points, over every finite field, of a spreading out of a given complex variety X , then this polynomial agrees with the E -polynomial of X .

In this final subsection, we use some of those formulae and our Theorem 4.10, to determine E -polynomials of the corresponding *irreducible* character varieties, deriving $E(\mathcal{X}_{\Gamma}^{irr}GL_n)$ from the knowledge of $E(\mathcal{X}_{\Gamma}GL_n)$. Explicit expressions are given in Theorems 5.5 and 5.7 below, and are new results, to the best of our knowledge. As a consequence, we obtain the numbers of irreducible components and Euler characteristics of $\mathcal{X}_{\Gamma}^{irr}GL_n$.

We consider the following classes of groups Γ . If Σ_g is a compact surface without boundary of genus $g \geq 1$, its fundamental group can be written as

$$\Gamma_g := \pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle,$$

and its abelianization is $(\Gamma_g)_{Ab} = \mathbb{Z}^{2g}$, since the unique relation is a product of commutators (belongs to $[\Gamma_g, \Gamma_g]$). A non-orientable compact surface (without boundary) of genus k is a connected sum of k copies of the real projective plane $\mathbb{R}\mathbb{P}^2$. Its fundamental group is denoted by

$$\hat{\Gamma}_k := \langle a_1, a_2, \dots, a_k | a_1^2 \cdots a_k^2 = 1 \rangle,$$

and in this case we have: $(\hat{\Gamma}_k)_{Ab} \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$, since this is the kernel of the map $\mathbb{Z}^k \rightarrow \mathbb{Z}$, sending $(b_1, \dots, b_k) \in \mathbb{Z}^k$ to $2(b_1 + \dots + b_k)$, whose vanishing corresponds to writing $a_1^2 \cdots a_k^2 = 1$ additively. As before, we let F_r denote the free group in

r generators. Note that Γ_g , $\hat{\Gamma}_k$ and F_r exhaust all fundamental groups of compact surfaces with a finite set of points removed. Finally, consider the fundamental group

$$\Gamma_{a,b} = \langle x, y | x^a = y^b \rangle$$

of the complement of a torus knot in S^3 of type (a, b) , where $a, b \in \mathbb{N}$ are relatively prime. Its abelianization coincides with the first homology group, which is \mathbb{Z} (rank 1 and no torsion).

5.3.1. *The case of GL_2 .* All the E -polynomials below depend on a single variable $x = uv$, so we again use the notation $E_x(X) := E(X; \sqrt{x}, \sqrt{x})$.

Theorem 5.5. *The following are the E -polynomials of irreducible GL_2 -character varieties for the given groups Γ :*

(1) For $\Gamma = F_{s+1}$, we have

$$\frac{E_x(\mathcal{X}_{F_{s+1}}^{irr} GL_2)}{(x-1)^{s+1}} = (x-1)^s x^s ((x+1)^s - 1) - \frac{1}{2}(x+1)^s + \frac{1}{2}(x-1)^s.$$

(2) For $\Gamma_g = \pi_1(\Sigma_g)$, with $c = 2g - 2$,

$$\frac{E_x(\mathcal{X}_{\Gamma_g}^{irr} GL_2)}{(x-1)^{c+2}} = (x^2 - 1)^c (x^c + 1) + \frac{(x^{c+1} - x - 1)}{2} (x+1)^c - \frac{(x^{c+1} - x + 1)}{2} (x-1)^c - x^c.$$

(3) For $\hat{\Gamma}_k = \pi_1(\hat{\Sigma}_g)$, with $h = k - 2$,

$$\begin{aligned} \frac{E_x(\mathcal{X}_{\hat{\Gamma}_k}^{irr} GL_2)}{(x-1)^{h+1}} &= 2(x^h + 1)(x^2 - 1)^h + x^h(x-1) \frac{(x-1)^h + (x+1)^h}{2} + \\ &+ (2 - 4x^h)(x-1)^h - (x+1)^h - 2x^h. \end{aligned}$$

(4) For $\Gamma_{a,b}$ we have:

$$\frac{E_x(\mathcal{X}_{\Gamma_{a,b}}^{irr} GL_2)}{x-1} = \begin{cases} \frac{1}{4}(a-1)(b-1)(x-2), & a, b \text{ both odd} \\ \frac{1}{4}(b-1)(ax-3a+4), & a \text{ even, } b \text{ odd.} \end{cases}$$

Proof. In all cases, the character varieties $\mathcal{X}_\Gamma GL_2$ were shown to be of polynomial type, so their E -polynomials equal their counting polynomials computed in [BH]. Thus, to obtain the formulae above, we consider the stratification:

$$\mathcal{X}_\Gamma GL_2 = \mathcal{X}_\Gamma^{[2]} GL_2 \sqcup \mathcal{X}_\Gamma^{[1^2]} GL_2 \cong \mathcal{X}_\Gamma^{irr} GL_2 \sqcup \mathcal{X}_{\Gamma_{Ab}} GL_2.$$

So, $E_x(\mathcal{X}_\Gamma^{irr} GL_2)$ is obtained by subtracting, from $E_x(\mathcal{X}_\Gamma GL_2)$, the E -polynomial of the abelian stratum using Proposition 5.3, given the rank and torsion of Γ_{Ab} .

In the free group case $\Gamma = F_r$, we have $\Gamma_{Ab} = \mathbb{Z}^r$ (no torsion), so

$$E_x(\mathcal{X}_r^{[1^2]} GL_2) = \frac{1}{2}(x^2 - 1)^r + \frac{1}{2}(x-1)^{2r},$$

and the above formula comes from [BH, Section 6.2], where it was shown, using $r = s + 1$:

$$E_x(\mathcal{X}_{s+1} GL_2) = (x-1)^{s+1} \left((x^3 - x)^s - (x^2 - x)^s + x \frac{(x+1)^s + (x-1)^s}{2} \right).$$

For the surface group case, we have $(\Gamma_g)_{Ab} = \mathbb{Z}^{2g}$, and Proposition 5.3 gives:

$$E_x(\mathcal{X}_{\Gamma_g}^{[1^2]} GL_2) = \frac{1}{2}(x^2 - 1)^{2g} + \frac{1}{2}(x-1)^{4g}.$$

Thus, to obtain (2), we subtract it from the formula in [BH, Section 6.5], which can be rewritten, letting $c = 2g - 2$, as:

$$\frac{E_x(\mathcal{X}_{\Gamma_g} GL_2)}{(x-1)^{c+2}} = (x^2-1)^c(x^c+1) + \frac{(x^{c+1}+x^2+x)}{2}(x+1)^c - \frac{(x^{c+1}-x^2+x)}{2}(x-1)^c - x^c.$$

For (3), we have $(\hat{\Gamma}_k)_{Ab} \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$, so Proposition 5.3 gives us,

$$E_x(\mathcal{X}_{\Gamma_k}^{[1^2]} GL_2) = (x^2-1)^{k-1} + 2(x-1)^{2k-2} = (x-1)^{k-1}[(x+1)^{k-1} + 2(x-1)^{k-1}],$$

which is subtracted from [BH, Section 6.8], using $h = k - 2$:

$$\begin{aligned} \frac{E_x(\mathcal{X}_{\Gamma_k} GL_2)}{(x-1)^{h+1}} &= 2(x^h+1)(x^2-1)^h + x^h(x-1) \frac{(x-1)^h + (x+1)^h}{2} + \\ &\quad + x[(x+1)^h + 2(x-1)^h] - 4(x^2-x)^h - 2x^h. \end{aligned}$$

For (4), the E -polynomial of the character variety of the torus knot appears in [BH, Section 6.10]:

$$E_x(\mathcal{X}_{\Gamma_{a,b}} GL_2) = \begin{cases} (x-1)(x + \frac{1}{4}(a-1)(b-1)(x-2)), & a, b \text{ both odd} \\ (x-1)(x + \frac{1}{4}(b-1)(ax-3a+4)), & a \text{ even, } b \text{ odd.} \end{cases}$$

So, to get the irreducible part, we again subtract the abelian stratum:

$$E_x(\mathcal{X}_{\Gamma_{a,b}}^{[1^2]} GL_2) = \frac{1}{2}(x^2-1) + \frac{1}{2}(x-1)^2 = x^2 - x,$$

using $(\Gamma_{a,b})_{Ab} = \mathbb{Z}$ in Proposition 5.3. \square

Corollary 5.6. *The variety $\mathcal{X}_{\Gamma}^{irr} GL_2$ has respectively 1, 1, 2, $\frac{1}{4}(a-1)(b-1)$ and $\frac{1}{4}a(b-1)$ irreducible components, for the groups F_r , Γ_g , $\hat{\Gamma}_k$, $\Gamma_{a,b}$ (ab odd), and $\Gamma_{a,b}$ (a even, b odd) respectively. The Euler characteristics of all these character varieties are zero.*

Proof. The number of irreducible components equals the leading coefficient of the corresponding E -polynomial. For the Euler characteristic, just substitute $x = 1$ in the appropriate formulae. \square

5.3.2. *The case of GL_3 .* In the case $n = 3$, the stratification by partition type is:

$$\mathcal{X}_{\Gamma} GL_3 = \mathcal{X}_{\Gamma}^{[3]} GL_3 \sqcup \mathcal{X}_{\Gamma}^{[1^2]} GL_3 \sqcup \mathcal{X}_{\Gamma}^{[1^3]} GL_3,$$

and $E_x(\mathcal{X}_{\Gamma}^{[1^2]} GL_3) = E_x(\mathcal{X}_{\Gamma}^{irr} GL_1) E_x(\mathcal{X}_{\Gamma}^{irr} GL_2)$. For the orientable surface group Γ_g , $E(\mathcal{X}_{\Gamma_g} GL_3)$ has been computed in [BH, Section 7.1]. The abelian stratum is also easy to get: in this case (from Example 4.20), we have

$$E_x(\mathcal{X}_{\Gamma_g}^{[1^3]} GL_3) = \frac{B_1(x^3)}{3} + \frac{B_1(x^2)B_1(x)}{2} + \frac{B_1(x)^3}{6},$$

with $B_1(x) = (x-1)^{2g}$ (as $(\Gamma_g)_{Ab} = \mathbb{Z}^{2g}$). Therefore, we can obtain the E -polynomial of the irreducible stratum as:

$$E_x(\mathcal{X}_{\Gamma_g}^{irr} GL_3) = E_x(\mathcal{X}_{\Gamma_g} GL_3) - E_x(\mathcal{X}_{\Gamma_g}^{irr} GL_1) E_x(\mathcal{X}_{\Gamma_g}^{irr} GL_2) - E_x(\mathcal{X}_{\Gamma_g}^{[1^3]} GL_3)$$

Here, we just present the final result.

Theorem 5.7. *The irreducible GL_3 -character variety of a compact orientable surface of genus g has zero Euler characteristic, is an irreducible variety, and its E polynomial, setting $c = 2g - 2$, is given by:*

$$\begin{aligned} \frac{E_x(\mathcal{X}_{\Gamma_g}^{irr} GL_3)}{(x-1)^{c+2}} &= (x-1)^{2c+2} \left[x^{3c} - \frac{x^{c+1}}{2} - (x+1)^c(x^c+1) + \frac{1}{3} \right] \\ &+ (x-1)^{2c+1} (x-2x^{2c}) \left[\frac{x^c(x-2)}{2} + (x+1)^c(x^c+1) \right] \\ &+ (x-1)^{2c} (x^2+x+1)^c \left[(x+1)^c(x^{3c}+1) + x^{2c} \right] \\ &+ (x-1)^{2c} (x-2)x^{2c} \left[(x+1)^c(x^c+1) + \frac{x^c(x-3)}{6} \right] \\ &+ \frac{(x-1)^{c+1}(x+1)^c}{2} [x^{c+1} - x^{3c+1}] \\ &+ (x-1)^c(x^c-1)[x^{c-2} + x^{c+1} - 2] + (x-1)^{c+2}[x^{2c-2} - x^{c-2}] \\ &+ \frac{(x^2+x+1)^c}{3} [x^{3c+1}(x+1) - (x^2+x+1)] - x^{3c}. \end{aligned}$$

Remark 5.8. The case of a free group of rank r , $\Gamma = F_r$, can be treated in a completely explicit way for every n . In the article [FNZ], where we relate the E -polynomials of $\mathcal{X}_{F_r} G$, for $G = GL_n, SL_n$ and PGL_n , we have obtained explicit formulae for the E -polynomials and Euler characteristics of all partition type strata, for all values of r and n .

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