SERRE POLYNOMIALS OF SL_n- AND PGL_n-CHARACTER VARIETIES OF FREE GROUPS

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ABSTRACT. Let G be a complex reductive group and $\mathcal{X}_r G$ denote the G-character variety of the free group of rank r. Using geometric methods, we prove that $E(\mathcal{X}_r SL_n) = E(\mathcal{X}_r PGL_n)$, for any $n, r \in \mathbb{N}$, where $E(X)$ denotes the Serre (also known as E-) polynomial of the complex quasi-projective variety X, settling a conjecture of Lawton-Muñoz in [LM]. The proof involves the stratification by polystable type introduced in [FNZ], and shows moreover that the equality of E-polynomials holds for every stratum and, in particular, for the irreducible stratum of $\mathcal{X}_r SL_n$ and $\mathcal{X}_r PGL_n$. We also present explicit computations of these polynomials, and of the corresponding Euler characteristics, based on our previous results and on formulas of Mozgovoy-Reineke for GL_n-character varieties over finite fields.

1. INTRODUCTION

Given a complex reductive algebraic group G, and a finitely presented group Γ, the G-character variety of Γ is the (affine) geometric invariant theory (GIT) quotient

$$\mathcal{X}_\Gamma G = \mathrm{Hom}(\Gamma, G) /\!\!/ G.$$

The most well studied families of character varieties include the cases when the group Γ is the fundamental group of a Riemann surface Σ, and its "twisted" variants. These varieties correspond, via non-abelian Hodge theory, to certain moduli spaces of G-Higgs bundles which play an important role in the quantum field theory interpretation of the geometric Langlands correspondence (see, for example [Si], [KW]).

In the context of SYZ mirror symmetry [SYZ], the hyperkähler nature of the Hitchin systems allowed a topological criterion for mirror symmetry: as a coincidence between certain Hodge numbers of moduli spaces of G-Higgs bundles over Σ, for Langlands dual groups G and G^L (see [Th]). The Hodge structure of these moduli spaces is *pure*, and topological mirror symmetry has been established in the smooth/orbifold case for the pair of Langlands dual groups $SL_n \equiv SL(n, \mathbb{C})$ and $PGL_n \equiv PGL(n, \mathbb{C})$ (see [HT, GWZ]).

Such topological mirror symmetries are also expected for *mixed* Hodge structures on the cohomology of other classes of moduli spaces. In the case of character varieties, whose Hodge structure is *not pure*, and which are generally, neither projective nor smooth, information on the Hodge numbers is encoded in the Serre polynomial (also called E-polynomial), which also provides interesting arithmetic properties (see [HRV]).

In this article, we consider character varieties of the free group $\Gamma = F_r$ of rank $r \in \mathbb{N}$, and prove the following equality. We denote the E-polynomial of a complex quasi-projective variety X by $E(X)$.

Theorem 1.1. *Let $r \in \mathbb{N}$, and $\Gamma = F_r$. Then*

$$E(\mathcal{X}_\Gamma SL_n) = E(\mathcal{X}_\Gamma PGL_n), \quad \forall n \in \mathbb{N}.$$

This result was proved for $n = 2$ and 3 in Lawton-Muñoz [LM], using complex geometry methods; these computations increase substantially for higher n, making it practically impossible to proceed explicitly. In [LM, Remark 6] an arithmetic argument by Mozgovoy is mentioned, showing the equality for all odd n (by using a theorem of Katz in [HRV, Appendix], see also subsection 4.5), as further evidence for the conjectured equality for all n, which we hereby confirm.

Denote by $\mathcal{X}_\Gamma^{irr} G \subset \mathcal{X}_\Gamma G$ the Zariski open subvariety of irreducible representation classes. The following result, together with the stratification\ by polystable type introduced in [FNZ] for arbitrary GL_n-character varieties, are the key ingredients in the proof of Theorem 1.1.

Theorem 1.2. *Let $r \in \mathbb{N}$, and $\Gamma = F_r$. Then*

$$E(\mathcal{X}_\Gamma^{irr} SL_n) = E(\mathcal{X}_\Gamma^{irr} PGL_n). \quad \forall n \in \mathbb{N}.$$

For the proof of this result, we use crucially the fact that free group character varieties admit a strong deformation retraction to analogous spaces of representations into $SU(n)$ and $PU(n)$, that follow from [FL1].

The outline of the article is as follows. In Section 2, we review standard facts on mixed Hodge structures and polynomial invariants, including their equivariant versions. Section 3 deals with character varieties in the context of affine GIT, which is enough to relate the E-polynomials of $\mathcal{X}_{F_r} PGL_n$ with those of $\mathcal{X}_{F_r} GL_n$. The polystable type stratification from [FNZ] is recalled in Section 4 for the $GL_n \equiv GL(n, \mathbb{C})$ case, and defined for the cases SL_n and PGL_n; then we provide the proof of the main theorem, assuming Theorem 1.2. This second theorem is proved in Section 5, by examining the action of $\mathrm{Hom}(F_r, Z)$ in the cohomology of $\mathrm{Hom}(F_r, SL_n)$, where $Z \cong \mathbb{Z}_n$ is the center of SL_n. Finally, Section 6 is devoted to describing a finite algorithm to obtain all the Serre polynomials and the Euler characteristics of all the strata $\mathcal{X}_{F_r}^{[k]} G$, for all partitions $[k] \in \mathcal{P}_n$ and $G = GL_n$, SL_n or PGL_n. The algorithm uses formulae of Mozgovoy-Reineke for $E(\mathcal{X}_{F_r}^{irr} GL_n)$, the irreducible GL_n case [MR]. Our main results were announced at the ISAAC conference 2019 [FNSZ].

1.1. **Comparison with related results in the literature.** For the benefit of the reader, we outline here some related previous results, without pretending to be exhaustive, and summarize the main novelties in our approach. For the surface group case (where $\Gamma = \pi_1(\Sigma)$ with Σ a compact orientable surface, and related groups) the calculations of Poincaré polynomials of $\mathcal{X}_\Gamma G$ started with Hitchin and Gothen (for $G = SL_n$, $n = 2, 3$, see [Hi, Got]), and have been pursued more recently by García-Prada, Heinloth, Schmitt, Hausel, Letellier, Mellit, Rodriguez-Villegas, Schiffmann and others, who also considered the parabolic version of these character varieties (see [GPH, GPHS, HRV, Le, Me, Sc]).

Many of those recent results also use arithmetic methods: it is shown that the number of points of the corresponding moduli space over finite fields is given by a polynomial

which, by Katz's theorem mentioned above, coincides with the E-polynomial of $\mathcal{X}_\Gamma G$. Then, in the smooth case, this allows the derivation of the Poincaré polynomial from the E-polynomial.

The equality of *stringy E-polynomials* (in the sense of Batyrev-Dais [BD]) of moduli spaces of G-Higgs bundles, for SL_n and PGL_n, in the coprime case, was established by Hausel-Thaddeus, for $n = 2, 3$, in [HT], and by Groechenig-Wyss-Ziegler for all n in [GWZ]. More recently, Gothen-Oliveira considered the parabolic version in [GO]. In another direction, the *full Hodge-Deligne* polynomials of *free abelian* group character varieties were computed in [FS].

Most of the above results were obtained only in the smooth/orbifold cases of the corresponding moduli spaces. On the other hand, for many important classes of *singular character varieties*, such as *free groups* or surface groups $\pi_1\Sigma$ *without twisting* (corresponding to degree zero bundles) explicitly computable formulas for the E-polynomials are very hard to obtain. In the articles of Lawton, Logares, Martínez, Muñoz and Newstead (by using geometric methods, see [LMN, Ma, MM] for surface groups and [LM] for the free group) and of Cavazos-Lawton and Baraglia-Hekmati ([CL] and [BH], using arithmetic methods), the Serre polynomials were computed for several character varieties, with $G = GL_n$, SL_n and PGL_n for small values of n, but the computations quickly become intractable for n higher than 3. Recently, SL_n-character varieties of surface groups were found to give rise to a Lax monoidal topological quantum field theory (see [GPLM]). In the case of free group character varieties, the E-polynomials were obtained in [MR], by point counting over finite fields, for all GL_n, and in [BH] for SL_n with $n = 2, 3$.

Our present results and methods differ from the previous literature in the following aspects. We consider the singular character variety of the free group, for $G = SL_n$ and *every* $n \in \mathbb{N}$, and consider the standard compactly supported E-polynomial (not the stringy one). We extend the stratification by polystable type (described in [FNZ] for GL_n) to the cases of PGL_n and SL_n, and carefully examine the action of the centre of SL_n on the cohomology of several spaces, to prove the main results. Our Theorems 1.1 and 1.2 do not use point counting methods over finite fields, and we only use the formulas of Mozgovoy-Reineke for GL_n-character varieties of F_r (see [MR]) in the last section, to provide explicit formulas for Serre polynomials and Euler characteristics for all $\mathcal{X}_{F_r}SL_n$ and all polystable strata.

2. Mixed Hodge structures and Serre polynomials

This section recalls some standard facts on mixed Hodge structures and polynomial invariants, introducing terminology and notation that will be used throughout.

Let X be a quasi-projective variety over $\mathbb{C}$, of complex dimension d, which may be singular, not complete, and/or not irreducible. Following Deligne (c.f. [De], [PS]), the

compactly supported cohomology $H_c^*(X) := H_c^*(X, \mathbb{C})$ is endowed with a mixed Hodge structure. Denote the corresponding mixed Hodge numbers by

$$h^{k,p,q}(X) = \dim_{\mathbb{C}} H_c^{k,p,q}(X) \in \mathbb{N}_0,$$

for $k \in \{0, \cdots, 2d\}$ and $p, q \in \{0, \cdots, k\}$. We say that (p, q) are k-weights of X, when $h^{k,p,q} \neq 0$.

Mixed Hodge numbers verify $h^{k,p,q}(X) = h^{k,q,p}(X)$, and $\dim_{\mathbb{C}} H_c^k(X) = \sum_{p,q} h^{k,p,q}(X)$, so they provide the (compactly supported) Betti numbers, which are easily translated to the usual Betti numbers, in the smooth case, by Poincaré duality.

Hodge numbers yield the so-called mixed Hodge polynomial of X,

$$(2.1) \qquad \mu(X; t, u, v) := \sum_{k,p,q} h^{k,p,q}(X)\ t^k u^p v^q \in \mathbb{N}_0[t, u, v],$$

on three variables. The mixed Hodge polynomial specializes to the (compactly supported) Poincaré polynomial by setting $u = v = 1$, $P_t^c(X) = \mu(X; t, 1, 1)$. Again, this provides the usual Poincaré polynomial in the smooth situation.

By substituting $t = -1$, mixed Hodge polynomials become a very useful generalization of the Euler characteristic, called the *Serre polynomial* or *E-polynomial* of X:

$$E(X; u, v) := \sum_{k,p,q} (-1)^k h^{k,p,q}(X)\ u^p v^q \in \mathbb{Z}[u, v].$$

From the E-polynomial we can compute the Euler characteristic of X as $\chi(X) = E(X; 1, 1) = \mu(X; -1, 1, 1)$ (recall that the compactly supported Euler characteristic equals the usual one for complex quasi-projective varieties).

When X is smooth and projective, its Hodge structure is *pure* (for each k, the only k-weights are of the form $(p, k - p)$ with $p \in \{0, \cdots, k\}$), and the E-polynomial actually determines the Poincaré polynomial $P_t = P_t^c$.

In the present article, we deal with varieties which are neither smooth nor complete, but often are of *Hodge-Tate type* (also called *balanced* type), for which all the k-weights are of the form (p, p) with $p \in \{0, \cdots, k\}$. This restriction on weights holds for complex (affine) algebraic groups (see, for example [DL, Jo]) and smooth toric varieties, among others classes.

Poincaré, mixed Hodge and Serre polynomials satisfy a multiplicative property with respect to Cartesian products $\mu(X \times Y) = \mu(X)\mu(Y)$ (coming from Künneth theorem, see [PS]). The big computational advantage of $E(X)$, as compared to $\mu(X)$ or $P^c(X)$ is that it also satisfies an *additive property with respect to stratifications* by locally closed (in the Zariski topology) strata: whenever X has a closed subvariety $Z \subset X$ we have

$$E(X) = E(Z) + E(X \setminus Z),$$

(see, eg. [PS]) which generalizes the well known analogous statement for χ.

Remark 2.1. The terminology for E-polynomials is not standard, and some authors refer to them as *Hodge-Deligne* (mostly when dealing with pure Hodge structures), and others as *virtual Poincaré or Serre polynomials*. After a literature review process, we are using *Serre polynomial*, following the oldest references (see, for example [GL]), and paying tribute to the ideas of Serre on the role of virtual dimensions for additivity in the weight filtration on mixed Hodge structures (see [To]).

2.1. **Equivariant Serre polynomials and fibrations.** We will also need the multiplicative property of the E-polynomial under certain algebraic fibrations, and more generally, for the *equivariant E-polynomial*, when these fibrations are acted by a finite group.

Definition 2.2. Let $\pi : X \to B$ be a morphism of quasi-projective varieties and W be a finite group acting algebraically on X, and preserving the fibers of π. Assume also that all fibers $\pi^{-1}(b)$, $b \in B$, are W-isomorphic to a given variety F. In this situation, we call

(2.2) $$F \to X \to B$$

an *algebraic W-fibration*.

Given an algebraic W-fibration $F \to X \to B$, the mixed Hodge structures on the (compactly supported) cohomology groups of F and X are representations of W, which we denote by $[H_c^{k,p,q}(F)]$ and similarly for X. This allows us to define the W-equivariant E-polynomials:

$$E^W(X; u, v) = \sum_{k,p,q} (-1)^k [H_c^{k,p,q}(X)]\ u^p v^q \in R(W)[u, v],$$

where $R(W)$ denotes the representation ring of W. By taking the dimensions of the fixed subspaces $[H_c^{k,p,q}(X)]^W$, under W, we recover the E-polynomial of the quotient variety X/W (see [DL], [FS]).

We have the following fundamental result.

Theorem 2.3. [DL, LMN] *Let W be a finite group and consider an algebraic W-fibration as above:*

$$F \to X \to B.$$

Assume either:
(i) the fibration is locally trivial in the Zariski topology of B, or
(ii) F, X and B are smooth, the fibration is locally trivial in the complex analytic topology, and $\pi_1(B)$ acts trivially on $H_c^*(F)$ (ie, the monodromy is trivial), or
(iii) X, B are smooth and F is a complex connected Lie group.
Then

$$E^W(X) = E^W(F) \cdot E(B).$$

Moreover, if F and B are of Hodge-Tate type, then so is X.

Proof. See [DL, Thm. 6.1] or [FS]. For (iii) and for the result on Hodge-Tate (balanced) type see also [LMN]. □

Of course, when $W = 1$ or the W action is trivial, the hypothesis imply the product formula $E(X) = E(F) \cdot E(B)$.

Example 2.4. The condition of F being connected in (iii) is necessary for the multiplicative property to hold, even in the case of trivial W. Indeed, the action of $F = \mathbb{Z}_2$ on $X = \mathbb{P}^1 \times \mathbb{P}^1$ by permuting the entries provides a fibration onto $\mathbb{P}^2$, the second symmetric power of $\mathbb{P}^1$:

$$\mathbb{Z}_2 \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathsf{Sym}^2(\mathbb{P}^1) = \mathbb{P}^2 \ ,$$

By elementary methods, the E-polynomials are $E(\mathbb{Z}_2) = 2$, $E(\mathbb{P}^1 \times \mathbb{P}^1) = (1 + uv)^2$ and $E(\mathbb{P}^2) = 1 + uv + u^2v^2$, which do not satisfy the multiplicative property. One can also check that this algebraic fibration is not locally trivial in the Zariski topology, but only in the analytic (strong) topology.

2.2. **Special fibrations.** The notion of special group according to Serre and Grothendieck (c.f. [Gr, Se]) provides a useful criterion for applying Theorem 2.3 to some algebraic fibrations.

By definition, a *special group* is an algebraic group H such that every principal H-bundle is locally trivial in the Zariski topology (c.f. [Gr, p.11]). In this context, a principal H-bundle $\pi : X \to B$, will be called a *special fibration*, whenever H is special. If, furthermore, a finite group W acts on X as in Definition 2.2, we call π a *special W-fibration*. Thus, Given a special W-fibration $H \to X \to B$, the following is an immediate consequence of Theorem 2.3(i).

Corollary 2.5. *Let $X \to B$ be a special W-fibration with fiber isomorphic to H. Then*

$$E^W(X) = E^W(H) \cdot E(B).$$

Now, let G be a connected complex algebraic reductive group, with center $Z \subset G$, and denote by $PG = G/Z$ its adjoint group. In case Z is connected, we obtain a simple relation between the Serre polynomials of G, Z and PG.

Proposition 2.6. *Let G be a complex reductive group with connected center Z. Then $E(G) = E(PG) \cdot E(Z)$.*

Proof. Since PG is defined to be a quotient by a subgroup, the following fibration

$$(2.3) \qquad\qquad Z \to G \to PG,$$

is a principal Z-bundle. Since G is complex, its center Z is the product of a group of multiplicative type and a unipotent subgroup. Given that G is reductive, the unipotent subgroup is trivial. Since Z is connected by hypothesis, Z is a torus, that is $Z \cong (\mathbb{C}^*)^l$, $l = \dim Z$. Since a torus is a special group by [Se], (2.3) is a special fibration and the result follows from Corollary 2.5 (with W being the trivial group). $\qquad\square$

Example 2.7. Since $E(\mathbb{C}^*) = uv - 1$, the formula $E(GL_n) = (uv - 1)E(PGL_n)$ follows from the special fibration

$$\mathbb{C}^* \to GL_n \to PGL_n,$$

which will be very important later on, and can be directly shown to be locally trivial in the Zariski topology. By contrast, the determinant fibration

$$SL_n \to GL_n \overset{\det}{\to} \mathbb{C}^*$$

is not trivial in the Zariski topology, although we still have $E(GL_n) = (uv - 1)E(SL_n)$.

Remark 2.8. As mentioned before, complex (affine) algebraic groups G are of Hodge-Tate type. Hence, their mixed Hodge polynomials $\mu(G; t, u, v)$ reduce to a two variable polynomial (with variables t and uv), and their E-polynomials depend only on the product uv. On the other hand, given a variety X whose E-polynomial is a function of only $x = uv$, it is not necessarily true that X is of Hodge-Tate type. For example, if we had $\mu(X; t, u, v) = 1 + tu + t^2u + t^2uv$, then X could not be of Hodge-Tate type, but $E(X) = 1 + uv = E(\mathbb{P}^1)$ is a one-variable polynomial in $x := uv$. We strongly believe that the character varieties studied in this paper are also of Hodge-Tate (balanced) type. Indeed, the methods in [LMN, LM] show this is the case for $n = 2$ and 3. However, as far as we know, the general case seems to be an open problem (see also Conjecture 4.10).

3. Character Varieties for a group and its adjoint

As before, let G be a connected complex reductive algebraic group with center Z. In this section, we recall some aspects of character varieties and their construction via affine GIT (Geometric Invariant Theory). This is applied to provide a simple relation between the Serre polynomials of $\mathcal{X}_\Gamma GL_n$ and of $\mathcal{X}_\Gamma PGL_n$, for the free group $\Gamma = F_r$.

3.1. Character varieties as GIT quotients.
Let Γ be a finitely presented group, and denote by
$$\mathcal{R}_\Gamma G = \mathrm{Hom}(\Gamma, G),$$
the algebraic variety of representations of Γ in G, where $\rho \in \mathcal{R}_\Gamma G$ is defined by the image of the generators of Γ, $\rho(\gamma)$, satisfying the relations in Γ. The moduli space of representations of Γ into G is the G-character variety of Γ
$$\mathcal{X}_\Gamma G := \mathrm{Hom}(\Gamma, G) /\!\!/ G,$$
defined as the affine GIT quotient under the algebraic action of G on $\mathcal{R}_\Gamma G$ by conjugation of representations. Given that a GIT quotient identifies those orbits whose closure has a non-empty intersection, we describe the quotient by the unique closed point in each equivalence class, called the *polystable representations*, which we define below. Given a representation $\rho \in \mathcal{R}_\Gamma G$, denote by Z_ρ the centralizer of $\rho(\Gamma)$ inside G. Note that $ZG \subset Z_\rho$, since the center commutes with any representation. Let us call the subgroup $G_{\mathcal{R}_\Gamma G} := \bigcap_{\rho \in \mathcal{R}_\Gamma G} Z_\rho$ the *center of the action*, since it acts trivially, and $G/G_{\mathcal{R}_\Gamma G}$ acts effectively on $\mathcal{R}_\Gamma G$. Denote by ψ_ρ the (effective) orbit map through ρ:
$$\begin{aligned} \psi_\rho : G/G_{\mathcal{R}_\Gamma G} &\to \mathcal{R}_\Gamma G \\ g &\mapsto g \cdot \rho \end{aligned}$$

Definition 3.1. In the situation above, we say that $\rho \in \mathcal{R}_\Gamma G$ is *polystable* if the orbit $G \cdot \rho := \{g\rho g^{-1} : g \in G\}$ is closed (Zariski) in $\mathcal{R}_\Gamma G$. We say that $\rho \in \mathcal{R}_\Gamma G$ is *stable* if it is polystable and ψ_ρ is a proper map.

It can be shown that the subset of polystable representations in $\mathcal{R}_\Gamma G$, denoted by $\mathcal{R}_\Gamma^{ps} G \subset \mathcal{R}_\Gamma G$, is stratified by locally-closed subvarieties, yielding a quotient which can be identified with the affine GIT quotient (see [FL2]):
$$\mathcal{X}_\Gamma G = \mathcal{R}_\Gamma G /\!\!/ G \cong \mathcal{R}_\Gamma^{ps} G / G.$$

Definition 3.2. We say that $\rho \in \mathcal{R}_\Gamma G$ is irreducible if ρ is polystable and Z_ρ is a finite extension of ZG, and call ρ a good representation if it is irreducible and $Z_\rho = ZG$.

Denote by $\mathcal{R}_\Gamma^{irr} G \subset \mathcal{R}_\Gamma^{ps} G$ and by $\mathcal{R}_\Gamma^g G$ the subset of irreducible and good representations, respectively, and set $\mathcal{X}_\Gamma^{irr} G := \mathcal{R}_\Gamma^{irr} G / G$ and $\mathcal{X}_\Gamma^g G := \mathcal{R}_\Gamma^g G / G$. It is known that $\mathcal{R}_\Gamma^{irr} G$ is a quasi-projective variety, since it is a Zariski open subset of $\mathcal{R}_\Gamma G$. The character variety of good representations is a smooth algebraic variety, by [Sik]. For further details, we refer the reader to [FLR], [GLR] or [FNZ, Section 3].

In the case when $G = GL_n$ or $G = SL_n$, Schur's lemma easily implies the equivalence between good and irreducible representations.

Lemma 3.3. *Let $G = GL_n$ or $G = SL_n$ and $\rho \in \mathrm{Hom}(\Gamma, G)$ be polystable. Then ρ is a good representation if and only if it is irreducible. In particular $\mathcal{X}_\Gamma^g G = \mathcal{X}_\Gamma^{irr} G$.*

Proof. By definition, a good representation is irreducible, and the converse follows from [FL3]; we can also proceed as follows. First let $G = GL_n$. If ρ is irreducible, Schur's lemma states that any $g \in G$ commuting with every $\rho(\gamma)$, $\gamma \in \Gamma$, is central, so that $Z_\rho = ZGL_n = \mathbb{C}^*$. Therefore, ρ is good.

Now let $G = SL_n$. For any representation $\rho : \Gamma \to SL_n$ we get a representation of GL_n by composition $i \circ \rho : \Gamma \to SL_n \overset{i}{\hookrightarrow} GL_n$. Since SL_n is the derived group of GL_n, we have

$$\frac{Z_{i\circ\rho}}{ZGL_n} = \frac{Z_{i\circ\rho} \cap SL_n}{ZGL_n \cap SL_n}, \text{ and } Z_\rho = Z_{i\circ\rho} \cap SL_n.$$

Hence, if ρ is an irreducible representation of SL_n then $i \circ \rho$ is an irreducible representation of GL_n and hence, by the previous argument, $i \circ \rho$ is a good GL_n-representation. Therefore by using the fact that $Z_\rho = Z_{i\circ\rho} \cap SL_n$ we get $Z_\rho = ZSL_n$ which says that ρ is a good representation into SL_n. □

Proposition 3.4. *Let $G = G_1 \times G_2$, be a product of two reductive groups. We have the following isomorphism of smooth algebraic varieties:*

$$\mathcal{X}_\Gamma^g(G) = \mathcal{X}_\Gamma^g(G_1) \times \mathcal{X}_\Gamma^g(G_2).$$

Proof. We have $\mathcal{X}_\Gamma(G) = \mathcal{X}_\Gamma(G_1) \times \mathcal{X}_\Gamma(G_2)$ as algebraic varieties, since conjugation by $G = G_1 \times G_2$ works independently on each factor of (cf. also [FL1]):

$$\mathrm{Hom}(\Gamma, G) = \mathrm{Hom}(\Gamma, G_1) \times \mathrm{Hom}(\Gamma, G_2).$$

Now suppose that a given representation $\rho \in \mathrm{Hom}(\Gamma, G)$ has a trivial stabilizer (the center of G). Then it is a product of the centers of G_1 and of G_2, so ρ is of the form $\rho = (\rho_1, \rho_2)$ with good factors $\rho_i \in \mathrm{Hom}(\Gamma, G_i)$, $i = 1, 2$. The converse is also clear, so the above isomorphism restricts to the good loci. □

Put together, the previous 2 statements show that, for GL_n and for SL_n-character varieties, the good locus coincides with both the irreducible locus and with the stable locus, and these are multiplicative, under products of reductive groups.

3.2. GL_n- and PGL_n-character varieties of the free group.

When considering $\Gamma = F_r$, the free group of rank r, we use the simplified notations $\mathcal{R}_r G := \mathrm{Hom}(F_r, G)$ and $\mathcal{X}_r G := \mathrm{Hom}(F_r, G) /\!/ G$, for the G-representation and G-character varieties, respectively.

There is a natural action of $\mathcal{R}_r \mathbb{C}^* := \mathrm{Hom}(F_r, \mathbb{C}^*) \cong (\mathbb{C}^*)^r$ on $\mathcal{R}_r GL_n$, which, in terms of the fixed generators $\gamma_1, \cdots, \gamma_r$ of F_r is given by scalar multiplication of representations:

$$(3.1) \qquad\qquad \sigma \cdot \rho(\gamma_i) := \sigma(\gamma_i)\rho(\gamma_i), \quad \sigma \in \mathcal{R}_r \mathbb{C}^*, \ \rho \in \mathcal{R}_r GL_n.$$

The quotient of this action is the representation variety for PGL_n, $\mathcal{R}_r PGL_n$. Since the central $\mathbb{C}^*$ action commutes with conjugation of representations, this sequence descends to the character varieties

$$(3.2) \qquad\qquad (\mathbb{C}^*)^r \to \mathcal{X}_r GL_n \to \mathcal{X}_r PGL_n.$$

Note that, because $\Gamma = F_r$, all PGL_n representations can be lifted to GL_n. The following generalization to other groups G and their adjoints PG is immediate from Corollary 2.5.

Proposition 3.5. *Let G be a connected reductive group with connected center Z. Then, the natural quotient map $\mathcal{X}_r G \to \mathcal{X}_r PG$ is a special fibration, with fiber $\mathcal{X}_r Z \cong Z^r$. Hence,*

$$E(\mathcal{X}_r G) = (uv - 1)^{rl} E(\mathcal{X}_r PG),$$

with $l = \dim Z$. In particular, for $G = GL_n$ we get $E(\mathcal{X}_r GL_n) = (uv - 1)^r E(\mathcal{X}_r PGL_n)$.

4. THE STRICTLY POLYSTABLE CASE

In this section and the next one, we prove Theorem 1.1: the equality of the Serre polynomials of $\mathcal{X}_r SL_n$ and $\mathcal{X}_r PGL_n$. If we tried to imitate the fibration methods of Section 3, we would consider the quotient morphism $SL_n \to PGL_n$, with kernel $\mathbb{Z}_n$, the center of SL_n, to obtain a fibration of character varieties:

$$\mathbb{Z}_n^r \to \mathcal{X}_r SL_n \to \mathcal{X}_r PGL_n.$$

However, since the fiber is not connected, we cannot apply Corollary 2.5; instead we proceed by stratifying this fibration by polystable type, in analogy to the stratification used in [FNZ] (recalled below) and examine the $\mathbb{Z}_n^r$ action on the cohomology of each individual stratum. From now on, our E-polynomials will only depend on a single variable; to emphasize this property, we adopt the substitution $x \equiv uv$ and use the notation:

$$(4.1) \qquad e(X) := E(X; \sqrt{x}, \sqrt{x}) \in \mathbb{Z}[x].$$

4.1. **Stratifications by polystable type.** Any character variety admits a stratification by the dimension of the stabilizer of a given representation. When dealing with the general linear group GL_n as well as the related groups SL_n and PGL_n, there is a more refined stratification which gives a lot more information on the corresponding character varieties $\mathcal{X}_\Gamma G$. In this subsection, we recall this stratification, following [FNZ, Section 4.1], and describe its analogous versions for SL_n and PGL_n.

A *partition* of $n \in \mathbb{N}$ is denoted by $[k] = [1^{k_1} \cdots j^{k_j} \cdots n^{k_n}]$ where the exponent k_j means that $[k]$ has $k_j \geq 0$ parts of size $j \in \{1, \cdots, n\}$, with $n = \sum_{j=1}^n j \, k_j$. The *length* of the partition is given by the sum of the exponents $|[k]| := \sum k_j$ and call $\mathcal{P}_n$ the set of all partitions of $n \in \mathbb{N}$. As an example, $[1^2 \, 2 \, 4] \in \mathcal{P}_8$ is the partition $8 = 1 \cdot 2 + 2 + 4$, with length equal to 4.

Definition 4.1. Let $n \in \mathbb{N}$, $[k] \in \mathcal{P}_n$ and Γ be a finitely presented group. We say that $\rho \in \mathcal{R}_\Gamma GL_n = \mathrm{Hom}(\Gamma, GL_n)$ is $[k]$-*polystable* if ρ is conjugated to $\bigoplus_{j=1}^n \rho_j$ where each ρ_j is, in turn, a direct sum of $k_j > 0$ *irreducible* representations of $\mathcal{R}_\Gamma(GL_j)$, for $1 \leq j \leq n$ (by convention, if some $k_j = 0$, then ρ_j is not present in the direct sum). We denote the $[k]$-polystable representations by $\mathcal{R}_\Gamma^{[k]} GL_n$ and $\mathcal{X}_\Gamma^{[k]} GL_n \subset \mathcal{X}_\Gamma GL_n$ refers to the $[k]$-polystable locus of the character variety.

Remark 4.2. The abelian stratum, i.e, the stratum of representations factoring as $\Gamma \to \Gamma/[\Gamma, \Gamma] \to G$, corresponds to the partition $[1^n]$ of maximal length n (see [FNZ]); on the other hand, irreducible representations correspond to the partition $[n]$ of minimal length 1. By Lemma 3.3, this irreducible stratum is also the smooth (and the stable) locus of the representation varieties $\mathcal{R}_\Gamma^{irr} GL_n := \mathcal{R}_\Gamma^{[n]} GL_n$.

The following summarizes the situation and is proved in [FNZ, Proposition 4.3].

Proposition 4.3. *Fix $n \in \mathbb{N}$. The character variety $\mathcal{X}_\Gamma GL_n$ can be written as a disjoint union of of locally closed quasi-projective varieties, labelled by partitions $[k] \in \mathcal{P}_n$*

$$\mathcal{X}_\Gamma GL_n = \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{X}_\Gamma^{[k]} GL_n,$$

where $\mathcal{X}_\Gamma^{[k]} GL_n$ consists of equivalence classes of $[k]$-polystable representations. Moreover, $\mathcal{X}_\Gamma^{[n]} GL_n$ is precisely the open locus $\mathcal{X}_\Gamma^{irr} GL_n$ of irreducible classes of representations.

4.2. **The free group case.** From now on, we restrict ourselves to the case $\Gamma = F_r$, the free group of rank $r \in \mathbb{N}$, and use the notations

$$\mathcal{X}_r GL_n, \qquad \mathcal{X}_r SL_n, \qquad \mathcal{X}_r PGL_n$$

for the corresponding character varieties. We will now define the $[k]$-polystable loci $\mathcal{X}_r^{[k]} SL_n \subset \mathcal{X}_r SL_n$ and $\mathcal{X}_r^{[k]} PGL_n \subset \mathcal{X}_r PGL_n$, and the corresponding stratifications, in analogy to Proposition 4.3.

Recall the action in (3.1) and (3.2) which clearly preserves the polystable type stratification of GL_n, so we define

$$\mathcal{X}_r^{[k]} PGL_n := \mathcal{X}_r^{[k]} GL_n / \mathcal{R}_r \mathbb{C}^* = \mathcal{X}_r^{[k]} GL_n / (\mathbb{C}^*)^r.$$

The next result is proved in the same way as Proposition 3.5. We observe that the E-polynomials of all strata $\mathcal{X}_r^{[k]} GL_n$ are 1-variable polynomials (see [FNZ]). Hence, we use the notation in (4.1) for E-polynomials in the variable $x = uv$.

Proposition 4.4. *Let F_r be a free group of rank r. For every $[k] \in \mathcal{P}_n$, the fibration*

$$\mathcal{R}_r \mathbb{C}^* \to \mathcal{X}_r^{[k]} GL_n \to \mathcal{X}_r^{[k]} PGL_n$$

is special. In particular, $e(\mathcal{X}_r^{[k]} GL_n) = (x-1)^r \, e(\mathcal{X}_r^{[k]} PGL_n)$.

4.3. **The action of the symmetric group on strictly polystable strata.** For a partition $[k] \in \mathcal{P}_n$, we define the $[k]$-stratum of $\mathcal{X}_r SL_n$ by restriction of the corresponding one for GL_n:

$$\mathcal{X}_r^{[k]} SL_n := \{\rho \in \mathcal{X}_r^{[k]} GL_n \mid \det \rho = 1\},$$

where the determinant of a (conjugacy class of a) representation is an element of $\mathcal{R}_r \mathbb{C}^*$. The action of $\mathcal{R}_r \mathbb{C}^*$ on $\mathcal{X}_r^{[k]} GL_n$ does not preserve $\mathcal{X}_r^{[k]} SL_n$ because of the determinant condition. On the other hand, from the behaviour of the Zariski topology under closed inclusions and quotients, the following is clear.

Proposition 4.5. *Fix $n \in \mathbb{N}$. The character varieties $\mathcal{X}_r SL_n$ and $\mathcal{X}_r PGL_n$ can be written as a disjoint unions of of locally closed quasi-projective varieties, labelled by partitions $[k] \in \mathcal{P}_n$*

$$\mathcal{X}_r SL_n = \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{X}_r^{[k]} SL_n, \qquad \mathcal{X}_r PGL_n = \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{X}_r^{[k]} PGL_n$$

Moreover, $\mathcal{X}_r^{irr} SL_n = \mathcal{X}_r^{[n]} SL_n$ and $\mathcal{X}_r^{irr} PGL_n = \mathcal{X}_r^{[n]} PGL_n$ are Zariski open.

It turns out that the irreducible strata $\mathcal{X}_r^{irr} SL_n$ and $\mathcal{X}_r^{irr} PGL_n$ are the most difficult cases to compare, so we start by studying the ones given by partitions with at least 2 parts.

Theorem 4.6. *Fix r and $n \in \mathbb{N}$. For a partition $[k] \in \mathcal{P}_n$ with length $s > 1$, we have:*

$$e(\mathcal{X}_r^{[k]} GL_n) = (x-1)^r e(\mathcal{X}_r^{[k]} SL_n)$$

Proof. We start with a relation between E-polynomials of cartesian products of irreducible character varieties. Let $s \in \mathbb{N}$ and $\mathbf{n} = (n_1, \cdots, n_s)$ be a sequence of s positive integers with $n = \sum_{i=1}^s n_i$. Denote:

$$\mathsf{X}_r^{\mathbf{n}} := \times_{i=1}^s \mathcal{X}_r^{irr} GL_{n_i},$$

and

$$SX_r^{\mathbf{n}} := \left\{ \rho = (\rho_1, \cdots, \rho_s) \in X_r^{\mathbf{n}} \mid \prod_{i=1}^{s} \det \rho_i = 1 \right\} \subset X_r^{\mathbf{n}}.$$

It is clear that the previous constructions can be carried out in this setting. For example, letting $J := (\mathcal{R}_r \mathbb{C}^*)^s$, there is a natural action of J on $X_r^{\mathbf{n}}$:

$$(\sigma_1, \cdots, \sigma_s) \cdot (\rho_1, \cdots, \rho_s) = (\sigma_1 \rho_1, \cdots, \sigma_s \rho_s), \qquad \sigma \in J = (\mathcal{R}_r \mathbb{C}^*)^s, \rho_i \in \mathcal{X}_r^{irr} GL_{n_i},$$

since a scalar multiple of an irreducible representation is again irreducible.

Define the multiplication map $m : J \to \mathcal{R}_r \mathbb{C}^* \cong (\mathbb{C}^*)^r$ as follows:

$$m(\sigma_1, \cdots, \sigma_s) = \sigma_1^{m_1} \cdots \sigma_s^{m_s},$$

where $m_i := n_i / d$ with $d = gcd(n_1, \cdots, n_s)$ (a power of a representation into $\mathbb{C}^*$ is just the power of every generator). It turns out that

$$H := \ker m = \{ (\sigma_1, \cdots, \sigma_s) \in (\mathcal{R}_r \mathbb{C}^*)^s \mid \sigma_1^{m_1} \cdots \sigma_s^{m_s} = 1 \},$$

is abelian, connected (because $s > 1$, and the m_i are coprime) and reductive. Then, it follows that H is isomorphic to an algebraic torus of the appropriate dimension, $H \cong (\mathbb{C}^*)^{r(s-1)}$, since $J = (\mathcal{R}_r \mathbb{C}^*)^s \cong (\mathbb{C}^*)^{rs}$. This allows us to obtain a diagram of algebraic fibrations (vertical arrows):

$$
\begin{array}{ccccc}
H & \subset & (\mathcal{R}_r \mathbb{C}^*)^s & \overset{m}{\to} & \mathcal{R}_r \mathbb{C}^* \\
\downarrow & & \downarrow & & \\
SX_r^{\mathbf{n}} & \subset & X_r^{\mathbf{n}} & & \\
\downarrow & & \downarrow & & \\
SX_r^{\mathbf{n}}/H & = & X_r^{\mathbf{n}}/J, & &
\end{array}
$$

where both vertical fibrations (the left fibration being the restriction to $SX_r^{\mathbf{n}}$) are special.

Now we note that, using an action of a finite group, we can obtain all strata of the stratification of $\mathcal{X}_r^{[k]} GL_n$. To be concrete, denote by Q_n the finite set:

$$Q_n := \{ \mathbf{n} = (n_1, \cdots, n_s) \in \mathbb{N}^s \mid \sum_{i=1}^{s} n_i = n \}.$$

To every element of Q_n we associate a unique partition of n as follows:

$$
\begin{array}{ccc}
p : Q_n & \to & \mathcal{P}_n \\
\mathbf{n} = (n_1, \cdots, n_s) & \mapsto & [k] := [1^{k_1} \cdots n^{k_n}]
\end{array}
$$

where k_j is the number of entries in the sequence $\mathbf{n}$ equal to j. For every $\mathbf{n} \in Q_n$, let $S_{\mathbf{n}} := S_{[k]} \subset S_n$ be the subgroup of the symmetric group defined by this partition $[k] = p(\mathbf{n})$. Moreover, we have isomorphisms of varieties:

$$X_r^{\mathbf{n}}/S_{\mathbf{n}} \cong \mathcal{X}_r^{[k]} GL_n, \qquad SX_r^{\mathbf{n}}/S_{\mathbf{n}} \cong \mathcal{X}_r^{[k]} SL_n,$$

as can be easily checked. Now, for every $\mathbf{n} \in Q_n$, and taking $(X_r^{\mathbf{n}}/J)/S_{\mathbf{n}}$ at the bottom, the above diagram becomes a special algebraic $S_{\mathbf{n}}$-fibration (with trivial action on the bottom and the top row is still the same), so we can apply Theorem 2.3 and Corollary 2.5 to both vertical fibrations to obtain:

$$e^{S_{\mathbf{n}}}(X_r^{\mathbf{n}}) = e((X_r^{\mathbf{n}}/J)/S_{\mathbf{n}}) \, e^{S_{\mathbf{n}}}(J), \qquad e^{S_{\mathbf{n}}}(SX_r^{\mathbf{n}}) = e((X_r^{\mathbf{n}}/J)/S_{\mathbf{n}}) \, e^{S_{\mathbf{n}}}(H).$$

and to the top horizontal fibration (which is also special) to get

$$e^{S_{\mathbf{n}}}(J) = e^{S_{\mathbf{n}}}(H)\,(x-1)^r,$$

and putting the formulae together:

$$e^{S_{\mathbf{n}}}(\mathsf{X}_r^{\mathbf{n}}) = e((\mathsf{X}_r^{\mathbf{n}}/J)/S_{\mathbf{n}})\,e^{S_{\mathbf{n}}}(H)\,(x-1)^r = e^{S_{\mathbf{n}}}(S\mathsf{X}_r^{\mathbf{n}})\,(x-1)^r$$

Finally, we just need to take the invariant parts of the equivariant polynomials:

$$e(\mathcal{X}_r^{[k]}GL_n) = e(\mathsf{X}_r^{\mathbf{n}}/S_{\mathbf{n}}) = e(S\mathsf{X}_r^{\mathbf{n}}/S_{\mathbf{n}})\,(x-1)^r,$$

whenever $[k] = p(\mathbf{n}) \in \mathcal{P}_n$ (noting that $(x-1)^r = (x-1)^r T \in R(S_{\mathbf{n}})[x]$, where T is the trivial $S_{\mathbf{n}}$ one-dimensional representation, as elements of the ring $R(S_{\mathbf{n}})[x]$).   □

Remark 4.7. Even though the above formulas prove these E-polynomials are only functions of $x = uv$, one can not conclude that strata $\mathcal{X}_r^{[k]}SL_n$ are of Hodge-Tate type without further arguments (see Remark 2.8).

4.4. **Proof of the main Theorem.** We now outline the proof of the main theorem, which relies crucially on the following.

Theorem 4.8. *The central action of $\mathbb{Z}_n^r$ on $\mathcal{X}_r^{irr}SL_n$ giving the quotient map*

$$\mathcal{X}_r^{irr}SL_n \to \mathcal{X}_r^{irr}PGL_n$$

induces an isomorphism of mixed Hodge structures $H^*(\mathcal{X}_r^{irr}SL_n) \cong H^*(\mathcal{X}_r^{irr}PGL_n)$.

The proof of Theorem 4.8 is delayed to Section 5. We will use differential geometric techniques, taking advantage of the fact that $\mathcal{X}_r^{irr}SL_n$ is a smooth variety and $\mathcal{X}_r^{irr}PGL_n$ is an orbifold (see [FL2, Sik]).

Assuming Theorem 4.8, we can now prove the main result of the article.

Theorem 4.9. *Let $\Gamma = F_r$. Then, for all $[k] \in \mathcal{P}_n$, we have $e(\mathcal{X}_r^{[k]}SL_n) = e(\mathcal{X}_r^{[k]}PGL_n)$. Consequently, $e(\mathcal{X}_rSL_n) = e(\mathcal{X}_rPGL_n)$.*

Proof. From Theorem 4.8, we have $H^*(\mathcal{X}_r^{irr}SL_n) \cong H^*(\mathcal{X}_r^{irr}PGL_n)$ as mixed Hodge structures, so that their E-polynomials coincide. For any other stratum $[k] = [1^{k_1} \cdots n^{k_n}]$, which has more than one part, the equality $e(\mathcal{X}_r^{[k]}SL_n) = e(\mathcal{X}_r^{[k]}PGL_n)$ follows from Theorem 4.6. Finally, the last statement follows from the additivity of the E-polynomial applied to the locally-closed stratifications of Proposition 4.5.   □

Noting that SL_n and PGL_n are Langlands dual groups, and that our proof seems well adapted to more general actions of finite subgroups, we put forward the following conjecture (answered here in the case $G = SL_n$), and plan to address the general statement in a future work.

Conjecture 4.10. *Let $\Gamma = F_r$ and G, G^L be complex reductive Langlands dual groups. Then both $\mathcal{X}_rG$ and $\mathcal{X}_rG^L$ are of Hodge-Tate type, and $e(\mathcal{X}_rG) = e(\mathcal{X}_rG^L)$.*

Remark 4.11. The part of the conjecture claiming Hodge-Tate type is still largely open, even for $G = SL_n$ (see also Remark 2.8). To the best of our knowledge, the only free group character varieties that are known to be balanced are for $G = SL_n$ and $G = PGL_n$ with $n = 2, 3$ (see [LM]).

4.5. **Katz's theorem and the case n odd.** When n is odd, there is an alternative method to prove that the Serre polynomials of $\mathcal{X}_r SL_n$ and $\mathcal{X}_r PGL_n$ coincide. The argument was mentioned in [LM, Remark 9], and we detail it here, for convenience. Denote by $\mathbb{F}_q$ a finite field with q elements and characteristic p, so that $q = p^s$, for some $s \in \mathbb{N}$. A scheme X, defined over $\mathbb{Z}$, is called of *polynomial type* if there is a polynomial $C_X(t) \in \mathbb{Z}[t]$ (called the *counting polynomial* for X) such that the number of $\mathbb{F}_{q^s}$ points of X is precisely

$$|X/\mathbb{F}_{q^s}| = C_X(q^s),$$

for every s and almost every prime p. In [HRV, Appendix] (see also [BH]) N. Katz showed that if such a scheme X is of polynomial type, with counting polynomial C_X, then the Serre polynomial of the complex variety $X(\mathbb{C}) = X \otimes_{\mathbb{Z}} \mathbb{C}$ coincides with the counting polynomial:

$$E_x(X(\mathbb{C})) = C_{X(\mathbb{C})}(x).$$

To apply this to our character varieties, note that the natural surjective morphism of algebraic groups

$$SL_n(\mathbb{F}_p) \to PGL_n(\mathbb{F}_p),$$

has as kernel the scalar matrices aI of determinant 1, so that $a^n = 1$. By Dirichlet's Theorem on arithmetic progressions, for a fixed n odd, there exists an infinite number of primes such that $(n, p-1) = 1$, in which cases there are no non-trivial roots of unity, so that $SL_n(\mathbb{F}_p) \simeq PGL_n(\mathbb{F}_p)$. This implies that the representation spaces $\mathcal{R}_r SL_n(\mathbb{F}_p)$ and $\mathcal{R}_r PGL_n(\mathbb{F}_p)$ are in bijective correspondence, and the same holds for the number of points of the character varieties over $\mathbb{F}_q$:

$$|\mathcal{X}_r SL_n(\mathbb{F}_p)| = |\mathcal{X}_r PGL_n(\mathbb{F}_p)|.$$

In [MR, Corollary 2.6] it was shown that PGL_n-character varieties are of polynomial type. Therefore, by this result, the SL_n-character varieties are also polynomial-count, for n odd, with the same counting polynomial.

5. The Irreducible Case

In this section we prove that $E(\mathcal{X}_r^{irr} SL_n) = E(\mathcal{X}_r^{irr} PGL_n)$ for all $r, n \geq 1$, as stated in Theorem 4.8, thus completing the proof of the main result. Our methods are geometric in the sense that we mainly use complex algebraic and differential geometry. In particular, we will use the compact versions of all the character varieties that we have defined before.

5.1. **Compact representation spaces and their irreducible subspaces.** Consider the compact groups $U(n)$, $SU(n)$ and $PU(n)$, which are related through the fibrations:

$$\begin{array}{ccccc} SU(n) & \to & U(n) & \to & U(1) = S^1 \\ S^1 & \to & U(n) & \to & PU(n), \end{array}$$

so that $PU(n) \cong U(n)/S^1 \cong SU(n)/\mathbb{Z}_n$, where we identify the cyclic group with the group of nth roots of unity

$$\mathbb{Z}_n = \{e^{\frac{2\pi i k}{n}} : k \in \mathbb{Z}\} \subset S^1,$$

and with the center of $SU(n)$. For $r \in \mathbb{N}$, consider the *compact representation spaces*,

$$U_{r,n} := \mathrm{Hom}(F_r, U(n)) \cong U(n)^r,$$

and similarly $S_{r,n} := \mathrm{Hom}(F_r, SU(n)) \cong SU(n)^r$ and $P_{r,n} := \mathrm{Hom}(F_r, PU(n)) \cong PU(n)^r$, where the isomorphisms are obtained when fixing a set of r generators of the free group

F_r. Because we are dealing with the free group, representations can be multiplied componentwise: all spaces $U_{r,n}$, $S_{r,n}$ and $P_{r,n}$ are in fact Lie groups. The first fibration above defines a determinant map, which is actually a homomorphism of groups:

$$\det : U_{r,n} \to U_{r,1},$$

whose kernel is precisely $S_{r,n}$.

Now consider the *irreducible representation* spaces,

$$U_{r,n}^* := \mathrm{Hom}^{irr}(F_r, U(n)), \qquad S_{r,n}^* := \mathrm{Hom}^{irr}(F_r, SU(n)), \qquad P_{r,n}^* := \mathrm{Hom}^{irr}(F_r, PU(n)),$$

which are open subsets in the compact representation spaces:

$$U_{r,n}^* \subset U_{r,n}, \qquad SU_{r,n}^* \subset SU_{r,n}, \qquad PU_{r,n}^* \subset PU_{r,n}.$$

Also note that $SU_{r,n}^* = U_{r,n}^* \cap SU_{r,n}$. These irreducible subspaces $U_{r,n}^*, SU_{r,n}^*, PU_{r,n}^*$, are not Lie groups, but we observe the following straightforward properties of the natural multiplication action of $U_{r,1}$ on $U_{r,n}$:

(5.1) $\sigma \cdot \rho = (\sigma_1, \cdots, \sigma_r) \cdot (\rho_1, \cdots, \rho_r) := (\sigma_1 \rho_1, \cdots, \sigma_r \rho_r), \qquad \sigma_i \in U(1),\ \rho_i \in U(n),$

where $\rho_i = \rho(\gamma_i)$ and $\sigma_i = \sigma(\gamma_i)$ for $\gamma_1, \cdots, \gamma_r$ the fixed chosen generators of F_r.

Proposition 5.1. *The action of $U_{r,1}$ on $U_{r,n}$ is such that:*
(i) the subspace $U_{r,n}^*$ is preserved under the action;
(ii) $PU_{r,n}$ and $PU_{r,n}^*$ are, respectively, the orbit spaces of $U_{r,n}$ and $U_{r,n}^*$ under $U_{r,1}$;
(iii) $SU_{r,n}^* = \det^{-1}(1)$ for the restriction $\det : U_{r,n}^* \to U_{r,1}$, where $1 \in U_{r,1}$ denotes the trivial one dimensional representation.

Now define:

$$C_{r,n} := \mathrm{Hom}(F_r, \mathbb{Z}_n) \subset \mathrm{Hom}(F_r, U(1)) = U_{r,1},$$

so that $C_{r,n} \cong (\mathbb{Z}_n)^r$ is a subgroup of $U_{r,n}$ and can be identified with the center of $SU_{r,n}$. In the same way as $PU(n) = SU(n)/\mathbb{Z}_n$, we can also get $PU_{r,n}$ and $PU_{r,n}^*$ as finite quotients of $SU_{r,n}$ and $SU_{r,n}^*$:

$$PU_{r,n} = SU_{r,n}/C_{r,n}, \qquad PU_{r,n}^* = SU_{r,n}^*/C_{r,n}.$$

5.2. **Central action on representation spaces.** We now define a *stratification by polystable type* of $U_{r,n}$, $SU_{r,n}$ and $PU_{r,n}$ in complete analogy with the stratifications in [FNZ, Section 4.1] (for GL_n) and in Section 4 above (for PGL_n and SL_n).

Given a partition $[k] = [1^{k_1} \cdots j^{k_j} \cdots n^{k_n}]$, we say that $\rho \in U_{r,n}$ is of type $[k]$ if ρ is conjugated to $\bigoplus_{j=1}^n \rho_j$, where each ρ_j is a direct sum of k_j *irreducible* representations of $U_{r,j}^*$, for each $j = 1, \cdots, n$. We denote representations of type $[k]$ by $U_r^{[k]} \subset U_{r,n}$ and let:

$$SU_r^{[k]} = U_r^{[k]} \cap SU_{r,n}, \qquad PU_r^{[k]} = U_r^{[k]}/U_{r,1} \subset PU_{r,n}.$$

Note that all strata are locally closed, and that the irreducible strata $[k] = [n]$ are the only ones that are open (in the respective representation spaces), corresponding to the partition into one single part.

Proposition 5.2. *Fix $n \in \mathbb{N}$. The representation spaces $U_{r,n}$, $SU_{r,n}$ and $PU_{r,n}$ can be written as disjoint unions, labelled by partitions $[k] \in \mathcal{P}_n$, of locally closed submanifolds:*

$$U_{r,n} = \bigsqcup_{[k] \in \mathcal{P}_n} U_r^{[k]}, \qquad SU_{r,n} = \bigsqcup_{[k] \in \mathcal{P}_n} SU_r^{[k]}, \qquad PU_{r,n} = \bigsqcup_{[k] \in \mathcal{P}_n} PU_r^{[k]}.$$

Proof. The proof for $U_{n,r}$ is analogous to the proof in [FNZ, Proposition 4.3] for $\mathcal{R}_\Gamma GL_n$, observing that dealing with the usual Euclidean topology on $U_{n,r}$ works *ipsis verbis* as with the Zariski topology on $\mathcal{R}_\Gamma GL_n$. The cases $SU_{r,n}$ and $PU_{r,n}$ follow as in Proposition 4.5. $\qquad\square$

Now, we state the main result of this subsection.

Theorem 5.3. *The action of $C_{r,n}$ on the compactly supported cohomology of $SU^*_{r,n} = \mathrm{Hom}^{irr}(F_r, SU_n)$ is trivial. In particular, the natural quotient under $C_{r,n}$ induces an isomorphism $H^*_c(SU^*_{r,n}) \cong H^*_c(PU^*_{r,n})$.*

Note that we use compactly supported cohomology as dealing with non-compact spaces. To prove Theorem 5.3, we need to show two results: (i) under a open/closed decomposition of a compact space $X = U \sqcup Y$, the triviality of the action on 2 spaces implies the same for the third one; (ii) the actions of $C_{r,n}$ on the cohomology of every $[k]$-stratum are trivial, for partitions $[k]$ of length $l > 1$. (iii) the action on the whole $SU_{r,n}$ is trivial. We start with (iii), which uses the following standard lemma (see, for instance, [CFLO]).

Lemma 5.4. *If J is a finite group acting on a space X and, for every $g \in J$, the induced map $\hat{g} : X \to X$, $x \mapsto g \cdot x$ is homotopic to id_X, then J acts trivially on $H^*(X)$. In particular, if F is a finite subgroup of a connected group G acting on X, then F acts trivially on $H^*(X)$.*

Since the action of $C_{r,n}$ is the restriction of the action of the path connected group $SU(n)^r$ acting by left multiplication the following is clear.

Corollary 5.5. *The action of $C_{r,n} \cong (\mathbb{Z}_n)^r \subset SU(n)^r$ on $H^*(SU_{r,n}) = H^*(SU(n)^r)$, is trivial.*

Remark 5.6. Although the action of $C_{r,n}$ preserves the stratification, the argument above cannot be applied to the individual strata $SU^{[k]}_{r,n}$; indeed, it is not clear what *connected* group could interpolate the action of the discrete group $C_{r,n}$ on the irreducible stratum. For example, using the left multiplication by the whole group, the action of $(A^{-1}, A^{-1}) \in SU(n)^2$ on an irreducible pair $(A, B) \in SU(n)^2$ gives the pair $(I, A^{-1}B)$, which clearly belongs to the $[1^n]$-stratum. Other attempts at using smaller groups $H \subset SU(n)$, such as maximal tori, may still fail for the irreducible stratum. Given this difficulty, we resort to an argument analogous to the one of Theorem 4.6, for strata associated with partitions of length more than one.

Lemma 5.7. *Let $[k] \in \mathcal{P}_n$ be a partition with length $l > 1$. Then, the action of $C_{r,n}$ on the cohomology of $SU^{[k]}_r$ is trivial.*

Proof. We first consider cartesian products of irreducible representation spaces satisfying a determinant condition. Let $s > 1$, $\mathbf{n} := (n_1, \cdots, n_s) \in \mathbb{N}^s$, with $n = n_1 + n_2 + \cdots + n_s$ and denote by

$$\mathcal{S}^{\mathbf{n}} := \left\{ \rho = (\rho_1, \cdots, \rho_s) \in \times^s_{j=1} U^*_{r,n_j} \mid \prod^s_{j=1} \det \rho_j = 1 \right\}$$

which is a smooth manifold. Note that, in contrast to partitions, $\mathbf{n} := (n_1, \cdots, n_s) \in \mathbb{N}^s$ is an ordered s-tuple of elements of $\mathbb{N}$. There is an action of $C_{r,n}$ on $\mathcal{S}^{\mathbf{n}}$ given by:

$$(5.2) \quad \sigma \cdot \rho = \sigma \cdot (\rho_1, \cdots, \rho_s) := (\sigma\rho_1, \cdots, \sigma\rho_s), \qquad \sigma \in \mathrm{Hom}(F_r, \mathbb{Z}_n) = C_{r,n}, \ \rho_j \in U^*_{r,n_j}.$$

It is easy to check the product of determinants condition, so that $\sigma \cdot \rho \in \mathcal{S}^{\mathbf{n}}$.

Now, assume that the greatest common divisor of all $n_1, \cdots, n_s$ is 1. Then, there is at least one n_j prime with n, and without loss of generality, we can take $p := n_1$ prime with n. Denote for simplicity, $q = n - p = \sum_{i=2}^{s} n_i > 0$. Define the following elements of $U_{r,1}$:

$$\sigma_t = (e^{-2\pi it\frac{q}{n}}, 1, \cdots, 1), \qquad \tilde{\sigma}_t = (e^{2\pi it\frac{p}{n}}, 1, \cdots, 1)$$

parametrised by $t \in [0,1]$, and where each factor corresponds to a generator of F_r. Consider the following homotopy:

$$[0,1] \times \mathcal{S}^{\mathbf{n}} \quad \to \quad \mathcal{S}^{\mathbf{n}}$$
$$(t; \rho_1, \rho_2 \cdots, \rho_s) \quad \mapsto \quad (\sigma_t \rho_1, \tilde{\sigma}_t \rho_2, \cdots, \tilde{\sigma}_t \rho_s).$$

This is well defined since the product of determinants on the right side, for the first generator γ_1, is:

$$\left(e^{-2\pi it\frac{q}{n}}\right)^p \left(e^{2\pi it\frac{p}{n}}\right)^q \prod_{j=1}^{s} \det \rho_j(\gamma) = 1,$$

(the representation on the other generators does not change under this homotopy). Then, for $t = 0$ the map $\mathcal{S}^{\mathbf{n}} \to \mathcal{S}^{\mathbf{n}}$ is the identity, and for $t = 1$ the map is identified with the action, on the first generator γ_1, of multiplication by the scalar $e^{2\pi i\frac{p}{n}}$:

$$e^{-2\pi i\frac{q}{n}} = e^{-2\pi i\frac{q-n}{n}} = e^{2\pi i\frac{p}{n}},$$

which is a primitive n^{th} root of unity. Thus, we have found a homotopy between the identity and σ^* for the element $\sigma = (e^{2\pi i\frac{p}{n}}, 1, \cdots, 1) \in C_{r,n}$ (c.f. Lemma 5.4). Since all elements of $C_{r,n}$ are obtained as compositions of elements σ of this form (with non-trivial elements on the other entries), we obtain the necessary homotopies to apply Lemma 5.4, and finish the proof, in this case. If the greatest common divisor of all $n_1, \cdots, n_s$ is $d > 1$, then we can use the same map but with $t \in [0, \frac{1}{d}]$ (since there is some $p = n_j$ that verifies now $(\frac{p}{d}, n) = 1$).

So, the action of $\mathbb{Z}_n^r$ is trivial on the cohomology of $\mathcal{S}^{\mathbf{n}}$. Finally, let $[k]$ be the partition determined by the tuple $\mathbf{n} = (n_1, \cdots, n_s) \in \mathbb{N}^s$. Observe that the length of $[k]$ is s. Then, we note that

$$SU_r^{[k]} = \mathcal{S}^{\mathbf{n}}/S_{[k]},$$

where $S_{[k]} = \times_{j=1}^{n} S_{k_j} \subset S_n$ acts by permutation of the blocks of equal size. Since the action of $C_{r,n}$ commutes with the one of $S_{[k]}$, we get

$$H^*(SU_r^{[k]})^{C_{r,n}} = \left(H^*(\mathcal{S}^{\mathbf{n}})^{S_{[k]}}\right)^{C_{r,n}} = \left(H^*(\mathcal{S}^{\mathbf{n}})^{C_{r,n}}\right)^{S_{[k]}} = H^*(\mathcal{S}^{\mathbf{n}})^{S_{[k]}} = H^*(SU_r^{[k]}),$$

as wanted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For the next Proposition, we let X be a compact Hausdorff topological space acted by finite group F, and we have a closed F-invariant subspace $Y \subset X$, with complement $U := X \setminus Y$. Since we need compactly supported cohomology for the open case, and it coincides with usual cohomology for X and for Y, below we can use H_c^* uniformly.

Proposition 5.8. *If F acts trivially on the compactly supported cohomology of two of the spaces X, Y and U, then it acts trivially on the cohomology of the third one.*

Proof. This follows from the 5-lemma applied to the long exact sequence for cohomology with compact support associated to the decomposition $X = U \sqcup Y$. More precisely, let

$g \in F$, and denote by g_Z^* the associated morphisms in the cohomology for $Z = X, Y$ or U. Then, we can form the ladder:

$$
\begin{array}{ccccccccccc}
\cdots & \to & H_c^{k-1}(X) & \to & H_c^{k-1}(Y) & \to & H_c^k(U) & \to & H_c^k(X) & \to & H_c^k(Y) & \to & \cdots \\
& & \downarrow g_X^* & & \downarrow g_Y^* & & \downarrow g_U^* & & \downarrow g_X^* & & \downarrow g_Y^* & & \\
\cdots & \to & H_c^{k-1}(X) & \to & H_c^{k-1}(Y) & \to & H_c^k(U) & \to & H_c^k(X) & \to & H_c^k(Y) & \to & \cdots .
\end{array}
$$

By hypothesis, two of g_X^*, g_Y^* and g_U^* are isomorphisms. Then, by the 5-lemma, the third map is also an isomorphism. Since g_U^* is an isomorphism for every $g \in F$, the action of F on $H_c^*(U)$ is trivial. The same argument holds for the other 2 spaces X, Y. $\qquad\square$

The stratification $SU_{r,n} = \bigsqcup_{[k] \in \mathcal{P}_n} SU_r^{[k]}$ has the nice property that the closure of every stratum is a union of other strata. In fact, for every $a, b \in \mathbb{N}$, the direct sum of irreducible representations of sizes a and b is in the closure of the irreducible ones of size $a + b$. This means that the closure of $SU_r^{[k]}$ is the disjoint union of all strata $SU_r^{[l]}$ where $[l]$ is obtained by any subdivision of $[k] \in \mathcal{P}_n$. For example, with $n = 5$, $\overline{X_{[23]}} = X_{[23]} \sqcup X_{[1^2 3]} \sqcup X_{[1 2^2]} \sqcup X_{[1^3 2]} \sqcup X_{[1^5]}$, where we used the abbreviated notation $X_{[k]} := SU_r^{[k]}$.

Now, fix $p \in \{1, \cdots, n\}$ and consider the closed subset of $SU_{r,n}$ defined by

$$
Y_p := \bigcup_{|[k]|=p} \overline{SU_r^{[k]}} = \bigcup_{|[k]| \geq p} SU_r^{[k]},
$$

where $|[k]|$ is the length of $[k] \in \mathcal{P}_n$. Since each $SU_r^{[k]}$ is the only open stratum in $\overline{SU_r^{[k]}}$, it is easy to see that: $Y_p \setminus Y_{p+1} = \bigsqcup_{|[k]|=p} SU_r^{[k]}$, and this last union is disjoint.

Lemma 5.9. *Fix n and let $2 \leq p \leq n-1$. If $C_{r,n}$ acts trivially on the cohomology of Y_{p+1}, then the same holds for Y_p. Thus, $C_{r,n}$ acts trivially on the cohomology of Y_2.*

Proof. Since $p \geq 2$, by Lemma 5.7, $C_{r,n}$ acts trivially on

$$
H_c^*\left(\bigsqcup_{|[k]|=p} SU_r^{[k]} \right) = \bigoplus_{|[k]|=p} H_c^*(SU_r^{[k]}).
$$

We can use compactly supported cohomology since each $SU_r^{[k]}$ is a quotient of a smooth manifold by a finite group, so that it verifies Poincaré duality. Assuming $C_{r,n}$ acts trivially on the cohomology of Y_{p+1}, then the same holds for $Y_p = Y_{p+1} \sqcup (\sqcup_{|[k]|=p} SU_r^{[k]})$, by Proposition 5.8. Noting that $Y_n = SU_r^{[1^n]}$ (already a closed stratum) we have that $C_{r,n}$ acts trivially on $H^*(Y_n)$ again by Lemma 5.7. So, the last sentence follows by finite induction. $\qquad\square$

Finally, Theorem 5.3 follows by another application Proposition 5.8 and Corollary 5.5, to the spaces $X = Y_1 = SU_{r,n}$, the closed subset $Y = Y_2$ and $U = X \setminus Y_2 = SU_{r,n}^* = SU_r^{[n]}$.

5.3. **Cohomology of quotient spaces.** Now, we consider the action of $PU(n)$ on all three representation spaces by simultaneous conjugation, and define the compact irreducible "character varieties" $\mathcal{X}_r^* SU_n := SU_{r,n}^*/PU(n)$ and $\mathcal{X}_r^* PU_n := PU_{r,n}^*/PU(n)$.

We will need the notion of *equivariant cohomology*, which we now recall. Let X be a topological space endowed with the action of a topological group G, and $p : E_G \to B_G$ be

the universal principal G-bundle, where $B_G = E_G/G$ is the classifying space of G. The equivariant cohomology of X is defined by (for more details see, for example, [Br])

$$H_G^*(X) := H^*(X \times_G E_G),$$

and note that, if G acts freely on X, then

$$H_G^*(X) \cong H^*(X/G).$$

Proposition 5.10. *There are isomorphisms* $H^*(SU_{r,n}^*) \cong H^*(PU_{r,n}^*)$, $H^*(\mathcal{X}_r^* SU_n) \cong H^*(\mathcal{X}_r^* PU_n)$ *and* $H^*(\mathcal{X}_r SU_n) \cong H^*(\mathcal{X}_r PU_n)$.

Proof. The action of $C_{r,n}$ on the compactly supported cohomology of $SU_{r,n}^* = \operatorname{Hom}^{irr}(F_r, SU_n)$ is trivial, by Theorem 5.3, so:

$$H_c^*(SU_{r,n}^*) = H_c^*(SU_{r,n}^*)^{C_{r,n}} = H_c^*(SU_{r,n}^*/C_{r,n}) = H_c^*(PU_{r,n}^*).$$

Since $SU_{r,n}^*$ and $PU_{r,n}^*$ are either smooth manifolds or orbifolds (and are orientable), their cohomologies satisfy Poincaré duality, so the same isomorphism holds for the usual cohomologies. Now, as in [CFLO, p.12], because the quotient map $\pi : SU_{r,n}^* \to SU_{r,n}^*/C_{r,n} = PU_{r,n}^*$ induces an isomorphism in cohomology, and π is equivariant with respect to the conjugation $PU(n)$ action, we have that their equivariant cohomologies are the same:

$$H_{PU(n)}^*(SU_{r,n}^*) \cong H_{PU(n)}^*(PU_{r,n}^*).$$

Since the $PU(n)$ action is free on the irreducible representation spaces, we get

$$H^*(\mathcal{X}_r^* SU_n) \cong H_{PU(n)}^*(SU_{r,n}^*) \cong H_{PU(n)}^*(PU_{r,n}^*) \cong H^*(\mathcal{X}_r^* PU_n),$$

completing the proof of the second isomorphism. This also means that the action of $C_{r,n}$ on $H^*(\mathcal{X}_r^* SU_n)$ is trivial.

Now, for $[k] = [1^{k_1} \cdots n^{k_n}] \in \mathcal{P}_n$, we can write every stratum as a finite quotient $\mathcal{X}_r^{[k]} SU_n = (\times_{j=1}^n \mathcal{X}_r^* SU_{k_j})/S_{[k]}$ (with $S_{[k]} = S_{k_1} \times \cdots \times S_{k_n}$), and since there is a trivial action of $C_{r,n}$ on $H^*(\times_{j=1}^n \mathcal{X}_r^* SU_{k_j}) = \otimes_{j=1}^n H^*(\mathcal{X}_r^* SU_{k_j})$, the same also happens for the subspace fixed by $S_{[k]}$:

$$H^*(\mathcal{X}_r^{[k]} SU_n) = \left[\otimes_{j=1}^n H^*(\mathcal{X}_r^* SU_{k_j}) \right]^{S_{[k]}}.$$

Now, the stratification $\mathcal{X}_r SU_n = \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{X}_r^{[k]} SU_n$ has also the property that the closure of a stratum is a union of strata. For $p \in \{2, \cdots, n\}$ consider the closed subset:

$$X_p := \bigcup_{|[k]|=p} \overline{\mathcal{X}_r^{[k]} SU_n} = \bigcup_{|[k]| \ge p} \mathcal{X}_r^{[k]} SU_n,$$

and note that $X_p \setminus X_{p+1} = \bigsqcup_{|[k]|=p} \mathcal{X}_r^{[k]} SU_n$. Therefore, Lemma 5.9 applies to X_p in place of Y_p (here, note that $X_n = \mathcal{X}_r^{[1^n]} SU_n$ is already a closed stratum), to show that the cohomology of X_2 has a trivial $C_{r,n}$ action. Finally, the third isomorphism is shown by applying Proposition 5.8 to the triple: $X := X_1 = \mathcal{X}_r SU_n$, $Y := X_2$ and $U := X_1 \setminus X_2 = \mathcal{X}_r^* SU_n = \mathcal{X}_r^{[n]} SU_n$. The action of $C_{r,n}$ on the cohomology of U and Y being trivial, the same holds for X. $\square$

We are finally ready for the completion of the main result.

Theorem 5.11. *There are isomorphisms of mixed Hodge structures $H^*(\mathcal{X}_r SL_n) \cong H^*(\mathcal{X}_r PGL_n)$ and $H_c^*(\mathcal{X}_r^{irr} SL_n) \cong H_c^*(\mathcal{X}_r^{irr} PGL_n)$. In particular, the E-polynomials of $\mathcal{X}_r^{irr} SL_n$ and of $\mathcal{X}_r^{irr} PGL_n$ coincide.*

Proof. For every reductive group G, there is a strong deformation retraction from $\mathcal{X}_r G$ to the orbit space $\mathrm{Hom}(F_r, K)/K$, where K is a maximal compact subgroup of G (see [FL1]), acting by conjugation. Since the homotopy is defined by the polar decomposition, in the case $G = SL_n$, the strong deformation retraction from $\mathcal{X}_r SL_n$ to $\mathcal{X}_r SU_n$ commutes with the action of $C_{r,n} = \mathrm{Hom}(F_r, \mathbb{Z}_n)$. Then, from Proposition 5.10 we get isomorphisms in usual cohomology:

$$H^*(\mathcal{X}_r SL_n) \cong H^*(\mathcal{X}_r SU_n) \cong H^*(\mathcal{X}_r PU_n) \cong H^*(\mathcal{X}_r PGL_n).$$

Since the quotient $\mathcal{X}_r SL_n \to \mathcal{X}_r PGL_n$ is algebraic, the above isomorphism $H^*(\mathcal{X}_r SL_n) \cong H^*(\mathcal{X}_r PGL_n)$ is an isomorphism of mixed Hodge structures. Moreover, the strong deformation retraction from $X := \mathcal{X}_r SL_n$ to $SU_{r,n}$ restricts to a strong deformation retraction from the strictly polystable locus $Y := \bigsqcup_{|[k]| \geq 2} \mathcal{X}_r^{[k]} SL_n$ to $\bigsqcup_{|[k]| \geq 2} SU_n^{[k]}/PU(n)$ (because the polar decomposition preserves reducible representations, see [FL1]). Thus, in the same fashion, we obtain isomorphisms of mixed Hodge structures $H^*(Y) \cong H^*(\hat{Y})$, where $\hat{Y} := \bigsqcup_{|[k]| \geq 2} \mathcal{X}_r^{[k]} PGL_n$.

Now, considering the natural open inclusion $j : X \setminus Y = \mathcal{X}_r^{irr} SL_n \hookrightarrow X$ and the closed inclusion $i : Y \hookrightarrow X = \mathcal{X}_r SL_n$, we can consider the long exact sequence:

$$\cdots \to H^{k-1}(X) \to H^{k-1}(Y) \to H_c^k(X \setminus Y) \to H^k(X) \to H^k(Y) \to \cdots$$

which comes from the short exact sequence of sheaves $0 \to j_! j^* \mathbb{C} \to \mathbb{C} \to i_* i^* \mathbb{C} \to 0$ where $\mathbb{C}$ is the locally constant $\mathbb{C}$-valued sheaf (a standard reference is [Iv], or see [Gor, pag. 31], for a summarized account). By applying the same sequence to the closed subvariety $\hat{Y} \subset \hat{X} := \mathcal{X}_r PGL_n$, we get a sequence of isomorphisms:

$$
\begin{array}{ccccccccccc}
\cdots & \to & H^{k-1}(X) & \to & H^{k-1}(Y) & \to & H_c^k(X \setminus Y) & \to & H^k(X) & \to & H^k(Y) & \to & \cdots \\
 & & \| \wr & & \| \wr & & \uparrow & & \| \wr & & \| \wr & & \\
\cdots & \to & H^{k-1}(\hat{X}) & \to & H^{k-1}(\hat{Y}) & \to & H_c^k(\hat{X} \setminus \hat{Y}) & \to & H^k(\hat{X}) & \to & H^k(\hat{Y}) & \to & \cdots
\end{array}
$$

which provide, using the 5-lemma, the wanted isomorphisms:

$$H_c^k(\mathcal{X}_r^{irr} SL_n) = H_c^k(X \setminus Y) \cong H_c^k(\hat{X} \setminus \hat{Y}) = H_c^k(\mathcal{X}_r^{irr} PGL_n).$$

Finally, since the finite quotient $\mathcal{X}_r^{irr} SL_n \to \mathcal{X}_r^{irr} PGL_n$ is algebraic, this implies the isomorphism of mixed Hodge structures on the corresponding compactly supported cohomology groups, and the equality of E-polynomials. $\qquad\square$

6. Explicit Computations in the free group case

When $\Gamma = F_r$, the free group in r generators, Mozgovoy and Reineke obtained a general formula for the count of points in $\mathcal{X}_r^{irr} GL_n$ over finite fields, showing that these schemes are of polynomial type (see [MR]). In this section, we explore these formulae in detail and, by using the *plethystic exponential* correspondence proved in [FNZ], we provide a finite algorithm to obtain the Serre polynomials and the Euler characteristics of all the strata $\mathcal{X}_r^{[k]} G$, for all partitions $[k] \in \mathcal{P}_n$ and $G = GL_n$, SL_n or PGL_n.

6.1. **Serre polynomials of irreducible GL_n-character varieties $\mathcal{X}_r^{irr}G$.** Let us recall the definition of the plethystic functions, and the correspondence proved in [FNZ]. Define the *Adams operator* Ψ as the invertible $\mathbb{Q}$-linear operator acting on $\mathbb{Q}[x][[t]]$ by $\Psi(x^i t^k) := \sum_{l\geq 1} \frac{x^{li} t^{lk}}{l}$, where $(i,k) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, with inverse given by $\Psi^{-1}(x^j t^k) = \sum_{l\geq 1} \frac{\mu(l)}{l} x^{jl} t^{kl}$, and μ is the Möbius function $\mu : \mathbb{N} \to \{0,\pm 1\}$ ($\mu(n) = (-1)^k$ if n is square free with k primes in its factorization; $\mu(n) = 0$ otherwise).

Given a power series $f \in \mathbb{Q}[x][[t]]$, formal in t:

$$f(x,t) = 1 + \sum_{n\geq 1} f_n(x)\, t^n \ ,$$

where $f_n(x) \in \mathbb{Q}[x]$ are polynomials in x, define the plethystic exponential, PExp : $\mathbb{Q}[x][[t]] \to 1 + t\mathbb{Q}[x][[t]]$, and plethystic logarithm, PLog as:

$$\mathrm{PExp}(f) := e^{\Psi(f)}, \qquad \mathrm{PLog}(f) := \Psi^{-1}(\log f).$$

As established in [FNZ, Theorem 1.1 and Corollary 1.2], GL_n-character varieties can be expressed in terms of irreducible character varieties of lower dimension, by means of the plethystic exponential.

Theorem 6.1. [FNZ, Theorem 1.1] *Let Γ be a finitely presented group. Then, in $\mathbb{Q}[u,v][[t]]$:*

$$\sum_{n\geq 0} E(\mathcal{X}_\Gamma GL_n; u.v)\, t^n = \mathrm{PExp}\left(\sum_{n\geq 1} E(\mathcal{X}_\Gamma^{irr} GL_n; u,v)\, t^n \right) \ .$$

Using results of [MR] for the character varieties of the free group F_r, this relationship can be made explicit, in terms of partitions $[k] = [1^{k_1} \cdots d^{k_d}] \in \mathcal{P}_d$, with length denoted by $\|[k]\| = k_1 + \cdots + k_d$. Let $\binom{m}{k_1,\cdots,k_d} = m!(k_1! \cdots k_d!)^{-1}$ be the multinomial coefficients.

Proposition 6.2. *Let $r,n \geq 2$. The E-polynomials of the irreducible character varieties $B_n^r(x) := e(\mathcal{X}_r^{irr} GL_n)$ are explicitly given by:*

$$B_n^r(x) = (x-1) \sum_{d|n} \frac{\mu(n/d)}{n/d} \sum_{[k]\in\mathcal{P}_d} \frac{(-1)^{\|[k]\|}}{\|[k]\|} \binom{\|[k]\|}{k_1,\cdots,k_d} \prod_{j=1}^d b_j (x^{n/d})^{k_j} x^{\frac{n(r-1)k_j}{d}\binom{j}{2}} \ ,$$

where $b_j(x)$ are given by $F^{-1}(t) = 1 + \sum_{n\geq 1} b_n t^n$, for the series:

(6.1) $$F(t) = 1 + \sum_{n\geq 1} \left((x-1)(x^2-1)\ldots(x^n-1) \right)^{r-1} t^n.$$

Proof. As mentioned above, the varieties $\mathcal{X}_r^{irr} GL_n$ are of polynomial type by [MR, Thm. 1.1]. So, by Katz's theorem [HRV, Appendix], $B_n^r(x) := e_x(\mathcal{X}_r^{irr} GL_n)$ is obtained by replacing q by x in the counting polynomial $P_n(q) := |\mathcal{X}_r^{irr} GL_n/\mathbb{F}_q|$ which in [MR, Theorem 1.2] is shown to have generating series:

$$\sum_{n\geq 1} B_n^r(x)\, t^n = \sum_{n\geq 1} P_n(x)\, t^n = (1-x)\, \mathrm{PLog}(S \circ F^{-1}(t)) \ ,$$

with $F(t)$ as in (6.1), and S is a $\mathbb{Q}[x]$-linear shift operator defined on $\mathbb{Q}[x][[t]]$ by $S(t) = t$, and $S(t^n) := x^{(r-1)\binom{n}{2}} t^n$, for $n \geq 2$. Hence, we get

$$S \circ F^{-1}(t) = 1 + \sum_{n\geq 1} b_n(x)\, x^{(r-1)\binom{n}{2}}\, t^n \ .$$

So, the Proposition follows from Lemma 6.3 below, using

$$f_n(x) = b_n(x)\, x^{(r-1)\binom{n}{2}}$$

since, for a partition $[k] \in \mathcal{P}_d$, we have $f_j(x^{n/d})^{k_j} = b_j(x^{n/d})^{k_j}\, x^{\frac{n}{d}(r-1)k_j\binom{j}{2}}$, $j = 1, \ldots, d$, as wanted. $\square$

To complete the proof of Proposition 6.2, we need the following

Lemma 6.3. *Given $f_n(x) \in \mathbb{Q}[x]$, $n \in \mathbb{N}$, the coefficient of t^n in $\mathrm{PLog}(1 + \sum_{n\geq 1} f_n(x)\, t^n)$ is*

$$\sum_{d|n} \sum_{[k]\in\mathcal{P}_d} \frac{\mu(n/d)}{n/d} \frac{(-1)^{|[k]|-1}}{|[k]|} \binom{|[k]|}{k_1, \cdots, k_d} \prod_{j=1}^{d} f_j(x^{n/d})^{k_j} .$$

Proof. From the $\mathbb{Q}$-linearity of Ψ^{-1}, we can write, for a sequence of polynomials $g_m(x) \in \mathbb{Q}[x]$, $m \in \mathbb{N}$,

$$
\begin{aligned}
\Psi^{-1}\Big(\sum_{m\geq 1} g_m(x)t^m\Big) &= \sum_{m\geq 1} \Psi^{-1}(g_m(x)t^m) = \\
&= \sum_{l\geq 1} \frac{\mu(l)}{l} g_1(x^l)t^l + \sum_{l\geq 1} \frac{\mu(l)}{l} g_2(x^l)t^{2l} + \cdots = \\
&= \sum_{l\geq 1} \frac{\mu(l)}{l} \sum_{d\geq 1} g_d(x^l)t^{dl} = \\
(6.2) \qquad &= \sum_{n\geq 1}\sum_{d|n} \frac{\mu(n/d)}{n/d} g_d(x^{n/d})t^n.
\end{aligned}
$$

Now, from the series development of $\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$, we compute, using the multinomial theorem:

$$
\begin{aligned}
\log\Big(1 + \sum_{n\geq 1} f_n(x)\, t^n\Big) &= \Big(\sum_{n\geq 1} f_n(x)\, t^n\Big) - \frac{1}{2}\Big(\sum_{n\geq 1} f_n(x)\, t^n\Big)^2 + \frac{1}{3}\Big(\sum_{n\geq 1} f_n(x)\, t^n\Big)^3 - \cdots \\
&= \sum_{m\geq 1}\left[\sum_{[k]\in\mathcal{P}_m} \frac{(-1)^{|[k]|-1}}{|[k]|} \binom{|[k]|}{k_1, \cdots, k_m} \prod_{j=1}^{m} f_j(x)^{k_j}\right] t^m.
\end{aligned}
$$

Finally, we apply formula (6.2) to

$$g_m(x) := \sum_{[k]\in\mathcal{P}_m} \frac{(-1)^{|[k]|-1}}{|[k]|} \binom{|[k]|}{k_1, \cdots, k_m} \prod_{j=1}^{m} f_j(x)^{k_j} ,$$

proving the Lemma. $\square$

Remark 6.4. When n is a prime number the formula in Proposition 6.2 simplifies to:

$$B_n^r(x) = (x-1)\left[\frac{b_1(x^n)}{n} + \sum_{[k]\in\mathcal{P}_n} \frac{(-1)^{|[k]|}}{|[k]|} \binom{|[k]|}{k_1, \cdots, k_n} \prod_{j=1}^{n} b_j(x)^{k_j}\, x^{(r-1)k_j\binom{j}{2}}\right] .$$

Using Proposition 6.2, we can write down explicitly the E-polynomials of $\mathcal{X}_r^{irr} GL_n$, for any value of n in a recursive way, the first four cases being as follows.

Lemma 6.5. *The E-polynomials of the irreducible character varieties $B_n^r(x) = e(\mathcal{X}_r^{irr} GL_n)$, for $n = 1, 2, 3$ and 4, using the substitution $s = r - 1$, are given by:*

$$\frac{B_1^r(x)}{x-1} = (x-1)^{r-1} = (x-1)^s,$$

$$\frac{B_2^r(x)}{x-1} = \frac{1}{2}b_1(x^2) + \frac{1}{2}b_1(x)^2 - b_2(x)x^s,$$

$$= (x-1)^s \Big((x-1)^s x^s ((x+1)^s - 1) + \frac{1}{2}(x-1)^s - \frac{1}{2}(x+1)^s \Big),$$

$$\frac{B_3^r(x)}{x-1} = \frac{1}{3}b_1(x^3) - \frac{1}{3}b_1(x)^3 + b_1(x)b_2(x)x^s - b_3(x)x^{3s}$$

$$= (x-1)^s \Big(-\frac{1}{3}(x^2 + x + 1)^s + (x-1)^{2s}(\frac{1}{3} - x^s + x^s(x+1)^s$$

$$+ x^{3s} + x^{3s}(x+1)^s(x^2 + x + 1)^s - 2x^{3s}(x+1)^s) \Big),$$

$$\frac{B_4^r(x)}{x-1} = (x-1)^{2s} \Big(\frac{1}{4}(x-1)^{2s} - \frac{1}{4}(x+1)^{2s} + (x^2 - 1)^s x^s (1 - (x+1)^s)$$

$$+ \frac{1}{2}(x+1)^{2s} x^{2s} (1 - (x^2 + 1)^s) + \frac{1}{2}(x-1)^{2s} x^{2s} (1 - (x+1)^s)^2$$

$$- (x-1)^{2s} x^{3s} (-(x+1)^s (x^2 + x + 1)^s + 2(x+1)^s - 1)$$

$$- (x-1)^{2s} x^{6s} (-(x+1)^s (x^2 + x + 1)^s (x^3 + x^2 + x + 1)^s$$

$$+ 2(x+1)^s (x^2 + x + 1)^s + (x+1)^{2s} - 3(x+1)^s + 1) \Big).$$

Proof. We will make the formulae in Proposition 6.2 explicit, by inverting formal power series. If

$$F(t) = 1 + \sum_{n \geq 1} a_n t^n, \quad \text{and } F^{-1}(t) = 1 + \sum_{n \geq 1} b_n t^n$$

are formal inverses, the relation between a_n and b_n can be obtained from $\sum_{k \geq 0} a_k b_{n-k} = 0$ $(a_0 = b_0 = 1)$, recursively (valid for power series over any ring) as

(6.3) $\qquad b_1 = -a_1 \qquad ; b_2 = a_1^2 - a_2; \qquad b_3 = -a_1^3 + 2a_1 a_2 - a_3, \quad \text{etc.}$

Now, employing $a_n(x) := \big((x-1)(x^2 - 1) \ldots (x^n - 1) \big)^{r-1}$ as in (6.1), we get:

$$b_1(x) = -a_1(x) = -(x-1)^{r-1},$$

$$b_2(x) = a_1^2(x) - a_2(x) = (x-1)^{2r-2} - (x-1)^{r-1}(x^2 - 1)^{r-1}$$

$$= (x-1)^{2r-2}\big(-(x+1)^{r-1} + 1 \big),$$

$$b_3(x) = -a_1^3(x) + 2a_1(x)\,a_2(x) - a_3(x) =$$

$$= (x-1)^{3r-3}\big(-(x+1)^{r-1}(x^2 + x + 1)^{r-1} + 2(x+1)^{r-1} - 1 \big), \text{ etc},$$

which, by substitution in Proposition 6.2, completes the proof. $\qquad \square$

Recall from [FNZ, Definition 4.14] the notion of rectangular partition of n: a sequence of non-negative integers $0 \leq k_{l,h} \leq n$, $l, h \in \{1, \cdots, n\}$ satisfying $n = \sum_{l=1}^n \sum_{h=1}^n l\,h\,k_{l,h}$,

interpreted as a collection of $k_{l,h}$ rectangles of size $l \times h$, with total area n. An element of the set $\mathcal{RP}_n$, of rectangular partitions of n, is denoted:

$$[[k]] = [(1 \times 1)^{k_{1,1}} (1 \times 2)^{k_{1,2}} \cdots (1 \times n)^{k_{1,n}} \cdots (n \times n)^{k_{n,n}}] \in \mathcal{RP}_n \ ,$$

and the "gluing map" sends a rectangular partition to a usual partition

$$\begin{aligned} \pi : \mathcal{RP}_n \ &\to \ \mathcal{P}_n \\ [[k]] \ &\mapsto \ [m] = [1^{m_1} \cdots n^{m_n}] \quad \text{where } m_l := \sum_{h=1}^{n} h \cdot k_{l,h}. \end{aligned}$$

With the notion of rectangular partitions, we rephrase [FNZ, Corollary 1.2] for the free group $\Gamma = F_r$ and $G = GL_n$ case.

Theorem 6.6. [FNZ, Corollary 1.2] *The E-polynomial of the GL_n-character variety of the free group in r generators is*

$$e(\mathcal{X}_r GL_n) = \sum_{[[k]] \in \mathcal{RP}_n} \prod_{l,h=1}^{n} \frac{B_l^r(x^h)^{k_{l,h}}}{k_{l,h}! \, h^{k_{l,h}}},$$

and the E-polynomial of the stratum corresponding to a partition $[m] \in \mathcal{P}_n$ is

$$e(\mathcal{X}_r^{[m]} GL_n) = \sum_{[[k]] \in \pi^{-1}[m]} \prod_{l,h=1}^{n} \frac{B_l^r(x^h)^{k_{l,h}}}{k_{l,h}! \, h^{k_{l,h}}} \ .$$

6.2. Serre polynomials for SL_n and PGL_n-character varieties of the free group. Recall that, by Proposition 4.4 and Theorem 4.9, E-polynomials of all strata for the SL_n and PGL_n-character varieties of the free group of rank r can be derived from the corresponding ones for GL_n, by dividing out by $(x-1)^r$:

$$e(\mathcal{X}_r^{[k]} SL_n) = e(\mathcal{X}_r^{[k]} PGL_n) = \frac{e(\mathcal{X}_r^{[k]} GL_n)}{(x-1)^r} \ ,$$

for every $[k] \in \mathcal{P}_n$, as for the whole character variety. So, Theorem 6.6 allows the computation of explicit formulae for all E-polynomials of $\mathcal{X}_r G$, with $G = GL_n$, SL_n or PGL_n.

As examples, this method recovers the polynomial $e(\mathcal{X}_r SL_2) = e(\mathcal{X}_r PGL_2)$ first obtained in [CL], and the polynomial $e(\mathcal{X}_r SL_3) = e(\mathcal{X}_r PGL_3)$ in [LM, Theorem 1] (compare also [BH]). We illustrate the method for $n = 3$.

Theorem 6.7. *For $s \geq 0$, we have:*

$$e(\mathcal{X}_{s+1} SL_3) = \tfrac{1}{2}(x-1)^{s+1}(x+1)^s x + \tfrac{1}{3}(x^2 + x + 1)^s x(x+1) +$$

$$+ (x-1)^{2s} \Big((x+1)^s [x^{3s}(x^2+x+1)^s + x^{s+1} - 2x^{3s}] + x^{3s} - x^{s+1} + \tfrac{x}{6}(x+1) \Big)$$

Proof. By Theorem 6.6, we get for GL_3

$$e(\mathcal{X}_r GL_3) = B_3^r(x) + B_2^r(x)B_1^r(x) + \frac{B_1^r(x^3)}{3} + \frac{B_1^r(x^2)B_1^r(x)}{2} + \frac{B_1^r(x)^3}{6}$$

where the first term corresponds to $e(\mathcal{X}_r^{irr} GL_3)$, the second to $e(\mathcal{X}_r^{[1\,2]} GL_3)$ and remaining 3 terms to $e(\mathcal{X}_r^{[1^3]} GL_3)$ (see [FNZ, Figure 4.1] showing rectangular partitions for $n = 3$). Replacing $B_j^r(x)$, $j = 1, 2, 3$, by the expressions in Lemma 6.5 we obtain the result for $\mathcal{X}_r GL_3$, and the case of SL_3 follows immediately. $\qquad\square$

Our method allows explicit expressions for $n = 4$ and beyond. In fact, Theorem 6.6 provides the decomposition

$$e(\mathcal{X}_r GL_4) = e(\mathcal{X}_r^{[4]} GL_4) + e(\mathcal{X}_r^{[1\ 3]} GL_4) + e(\mathcal{X}_r^{[2^2]} GL_4) + e(\mathcal{X}_r^{[1^2\ 2]} GL_4) + e(\mathcal{X}_r^{[1^4]} GL_4)$$

$$= B_4(x) + B_3(x)B_1(x) + \frac{B_2(x)^2}{2} + \frac{B_2(x^2)}{2} + \frac{B_2(x)B_1(x^2)}{2} + \frac{B_2(x)B_1(x)^2}{2}$$

(6.4) $$+ \frac{B_1(x^4)}{4} + \frac{B_1(x^3)B_1(x)}{3} + \frac{B_1(x^2)^2}{8} + \frac{B_1(x^2)B_1(x)^2}{4} + \frac{B_1(x)^4}{24},$$

as the sum of the 5 strata (which comprise the 11 terms coming from the rectangular partitions in [FNZ, Figure 4.2]). Combining (6.4) with formulae in Lemma 6.5 yields the computation of the E-polynomial for SL_4 (and then also PGL_4). This formula is new.

Theorem 6.8. *The E-polynomial of the SL_4-character variety of F_{s+1} is:*

$$
\begin{aligned}
e(\mathcal{X}_{s+1} SL_4) &= (x-1)^{3s+1}\Big[(x+1)^{2s}\frac{x^{2s}}{2} + (x+1)^s\big(x^{3s}(x^2+x+1)^s - 2x^{3s} - x^{2s} + \frac{3x^s}{2}\big)\Big] \\
&+ (x-1)^{3s+1}\big[x^{3s} + \frac{x^{2s}}{2} - \frac{3x^s}{2} + \frac{11}{24}\big] + \frac{1}{24}(x-1)^{3s+3} \\
&+ (x-1)^{3s}(x+1)^{2s}(-x^{6s} + \frac{x^{2s}}{2}) \\
&+ (x-1)^{3s}(x+1)^s x^{6s}\big[(x^2+x+1)^s(x^3+x^2+x+1)^s - 2(x^2+x+1)^s + 3\big] \\
&+ (x-1)^{3s}(x+1)^s\big[x^{3s}\big((x^2+x+1)^s - 2\big) - x^{2s} + \frac{x^s}{2}\big] \\
&+ (x-1)^{3s}(-x^{6s} + x^{3s} + \frac{x^{2s}}{2} - \frac{x^s}{2} + \frac{1}{2}) \\
&+ (x-1)^{2s+2}\frac{(x-1)^{s+1}}{4} \\
&+ (x-1)^{2s+1}\frac{(x+1)^s}{2}(-(x+1)^s x^s + x^s - \frac{1}{2}) \\
&+ (x-1)^{2s}(x+1)^s\frac{x^s}{2}(1 - (x+1)^s) \\
&+ (x-1)^{s+1}\Big[(x+1)\frac{x}{3}(x^2+x+1)^s + \frac{(x+1)^{2s}}{8}(x^2+2x+2)\Big] \\
&+ (x-1)^s(x+1)^{2s}\big[\frac{x^{2s+1}}{2}((x^2+1)^s - 1) + \frac{x-1}{4}\big] \\
&- \frac{1}{4}(x+1)^{s+1}(x^2+1)^s + \frac{1}{4}(x^3+x^2+x+1)^{s+1}.
\end{aligned}
$$

Formulas for all n can be obtained in exactly the same way, from the combinatorics of rectangular partitions, and the formula for B_n in Proposition (6.2).

6.3. Irreducibility and Euler characteristics.
It is clear that the above method, combining Proposition 6.2, Lemma 6.5 and Theorem 6.6, provides the same kind of expressions for $e(\mathcal{X}_r SL_n) = e(\mathcal{X}_r PGL_n)$, and for every $n \in \mathbb{N}$. Additionally, we can prove irreducibility and compute all Euler characteristics of $\mathcal{X}_r^{[k]} G$ for $G = GL_n$, SL_n and PGL_n, and all $[k] \in \mathcal{P}_n$. We start with the GL_n case.

Lemma 6.9. *The degree of the polynomial $B_n^r(x)$ is*

$$\deg(B_n^r(x)) = n^2(r-1) + 1.$$

Proof. From Proposition 6.2, $\deg B_n^r(x)$ will be

$$(6.5) \qquad 1 + \max_{d|n} \max_{[k] \in \mathcal{P}_d} \deg \left(\prod_{j=1}^{d} b_j(x^{n/d})^{k_j} x^{\frac{n(r-1)k_j}{d} \binom{j}{2}} \right),$$

where $b_n(x)$ are defined by $(1 + \sum_{n \geq 1} a_n(x) t^n)(1 + \sum_{n \geq 1} b_n(x) t^n) = 1$, with $a_n(x) = \left((x-1)(x^2-1) \ldots (x^n-1) \right)^{r-1}$, so that $\deg a_i(x) = \frac{i(i+1)}{2}(r-1)$. By the recursive definition of $b_j(x)$ in terms of the $a_i(x)$ (see Equations (6.3)), it is easy to see that $b_j(x) = -a_j(x) +$ lower degree terms, hence $\deg b_j(x) = \deg a_j(x) = \binom{j+1}{2}(r-1)$. Now, the maximum in (6.5) is achieved for $d = n$ because partitions of smaller n have lower degree. Hence:

$$\begin{aligned}
\deg B_n^r(x) &= 1 + \max_{[k] \in \mathcal{P}_n} \deg \prod_{j=1}^{n} b_j(x)^{k_j} x^{(r-1)k_j \binom{j}{2}} \\
&= 1 + \max_{[k] \in \mathcal{P}_n} \sum_{j=1}^{n} \left(\binom{j+1}{2}(r-1)k_j + (r-1)k_j \binom{j}{2} \right) \\
&= 1 + \max_{[k] \in \mathcal{P}_n} \sum_{j=1}^{n} k_j j^2 (r-1) .
\end{aligned}$$

Using the restriction $n = \sum_{j=1}^{n} j k_j$, it is clear that the maximum is achieved for the partition $[n]$, i.e. all $k_j = 0$ except $k_n = 1$, so that $\deg B_n^r(x) = n^2(r-1)+1$ as wanted. $\square$

Corollary 6.10. *Every $[k]$-polystable stratum $\mathcal{X}_r^{[k]} GL_n$ is an irreducible algebraic variety, and has zero Euler characteristic.*

Proof. The irreducible stratum corresponds to the partition $[n]$ and its polynomial is given by $e(\mathcal{X}_r^{irr} GL_n) = B_n^r(x)$ (see Theorem 6.6 and Remark 4.2). The monomial of top degree of $B_n^r(x)$ is, by the proof of Lemma 6.9,

$$(x-1)\frac{\mu(1)}{1} \frac{(-1)^1}{1} \binom{1}{1} b_n(x)^1 x^{(r-1)\cdot 1 \cdot \binom{n}{2}} = (x-1) a_n(x) x^{(r-1)\binom{n}{2}} ,$$

whose leading coefficient equals the leading coefficient of $a_n(x)$, which is 1. Therefore, $\mathcal{X}_r^{irr} GL_n$ is an irreducible variety. All polystable strata can be expressed as symmetric products of irreducible strata of lower dimension (see [FNZ, Proposition 4.5]), hence all of them are irreducible. The statement about Euler characteristics follows by substituting $x = 1$ in Theorem 6.6, since $B_n^r(1) = 0$, for all $n, r \in \mathbb{N}$ (see Proposition 6.2 and Lemma 6.5). $\square$

The next Corollary is immediate from our constructions.

Corollary 6.11. *For every $[k] \in \mathcal{P}_n$, the strata $\mathcal{X}_r^{[k]} SL_n$ and $\mathcal{X}_r^{[k]} PGL_n$ are irreducible algebraic varieties.*

Finally, we compute the Euler characteristics of the SL_n and PGL_n-character varieties of the free group. It turns out that the only strata contributing are of the form $[d^{n/d}] \in \mathcal{P}_n$, indexed by the divisors d of n.

Proposition 6.12. *The Euler characteristics of the SL_n and PGL_n character varieties of the free group are*

$$\chi(\mathcal{X}_r SL_n) = \chi(\mathcal{X}_r PGL_n) = \varphi(n)n^{r-2}$$

where $\varphi(n)$ is Euler's totient function. The Euler characteristics for the strata of the form $[d^{n/d}] \in \mathcal{P}_n$ are

$$\chi(\mathcal{X}_r^{[d^{n/d}]} SL_n) = \chi(\mathcal{X}_r^{[d^{n/d}]} PGL_n) = \frac{\mu(d)}{d} n^{r-1} \ ,$$

otherwise $\chi(\mathcal{X}_r^{[k]} SL_n) = 0$, where $\mu(n)$ is the arithmetic Möbius function.

Proof. By Theorems 4.6 and 5.11, the Euler characteristics for SL_n and PGL_n-character varieties are equal and can be computed strata by strata, by setting $x = 1$ in the GL_n polynomials of Proposition 6.2 and Theorem 6.6, and using Proposition 4.4:

(6.6) $$\chi(\mathcal{X}_r^{[m]} SL_n) = \chi(\mathcal{X}_r^{[m]} PGL_n) = \left[\frac{e(\mathcal{X}_r^{[m]} GL_n)}{(x-1)^r}\right]_{x=1} .$$

To begin, we show the formula:

(6.7) $$\left[\frac{B_n^r(x)}{(x-1)^r}\right]_{x=1} = \left[\frac{-1}{(x-1)^{r-1}} \frac{\mu(n)}{n} b_1(x^n)\right]_{x=1} = \mu(n)n^{r-2} .$$

Indeed, by the recursive definition (6.3), the term $(x-1)^{n(r-1)}$ can be factored out in $a_n(x)$ and, hence, the same applies to every $b_n(x)$. Then, for $n = 1$ we get $b_1(x) = -(x-1)^{r-1}$. However, for $m > 1$, we have $\left[\frac{b_m(x)}{(x-1)^{r-1}}\right]_{x=1} = 0$, so all terms with $b_m(x)$ for $m > 1$, in the formula of Proposition 6.2, disappear from the calculation of the Euler characteristic. For $b_1(x)$ we have the following expression, for every $t \in \mathbb{N}$:

$$b_1(x^s)^t = -(x^s-1)^{t(r-1)} = (-1)^t (x-1)^{t(r-1)} (1 + x + \cdots + x^{s-1})^{t(r-1)} .$$

Therefore, if $t > 1$, $\left[\frac{b_1(x^s)^t}{(x-1)^{r-1}}\right]_{x=1} = 0$, while, for $t = 1$, we have $\left[\frac{b_1(x^s)}{(x-1)^{r-1}}\right]_{x=1} = -s^{r-1}$.

Note that products $b_i(x) \cdot b_j(x)$, $i \neq j$, necessarily have a zero of order greater than r in $x = 1$. Hence, in Lemma 6.5, after setting $\frac{B_n^r(x)}{(x-1)^r}$, the only terms which do not vanish are those with $j = 1$ and $k_1 = 1$ and $k_j = 0$ for $j = 2, \ldots, d$. Therefore $d = 1$ and the only partition contributing is $[1] \in \mathcal{P}_1$, yielding the equality in Equation (6.7). Similarly, we have $\left[\frac{B_n^r(x^s)}{(x-1)^r}\right]_{x=1} = \mu(n)s^r n^{r-2}$, for every $s \in \mathbb{N}$.

To finish, we look at Theorem 6.6. For rectangular partitions with two or more blocks of different sizes (i.e. giving terms of the form $B_{n_1}^r(x^{s_1}) \cdot B_{n_2}^r(x^{s_2})$) $x = 1$ is a zero of order greater than r, then we get rid of them in the final calculation. This leaves us to just consider rectangular partitions $[[k]]$ with a single block $l \times h$ and $k_{l,h} = 1$ and the rest $k_{l,h} = 0$: they are indexed by divisors of n. Let $[k] \in \mathcal{P}(n)$ be a partition by blocks of just one size $[k] = [d^{n/d}]$. In the expression of Theorem 6.6, put $l = d$ and $h = n/d$ and have

$$\chi(\mathcal{X}_r^{[d^{n/d}]} SL_n) = \left[\frac{e(\mathcal{X}_r^{[d^{n/d}]} GL_n)}{(x-1)^r}\right]_{x=1} = \left[\frac{1}{(x-1)^r} \frac{d}{n} B_d(x^{n/d})\right]_{x=1} = \frac{\mu(d)}{d} n^{r-1} \ ,$$

while for other strata $[k] \in \mathcal{P}(n)$ with two or more blocks of different sizes, the Euler characteristic is zero. To get the Euler characteristic of the full character variety, we sum

up over all strata, this is, over all divisors of n:

$$\chi(\mathcal{X}_r SL_n) = \sum_{d|n} \left[\frac{1}{(x-1)^r} \frac{d}{n} B_d(x^{n/d}) \right]_{x=1} = \sum_{d|n} \frac{\mu(d)}{d} n^{r-1} = \varphi(n) n^{r-2},$$

by using Möbius inversion formula with the Euler function, $\varphi(n) = \sum_{d|n} \frac{n}{d} \mu(d)$. $\qquad \square$

Remark 6.13. This result extends [MR, Theorem 1.4] to the SL_n case and, moreover, calculates the Euler characteristic of all strata, providing a geometrical meaning to the calculation in terms of the Euler function. Also, note that the stratum $[1^n]$ corresponds to an abelian character variety, and this computation agrees with the one in [FS, Corollary 5.16].

Example 6.14. (1) As an application of Proposition 6.12, for $n = 4$, we get

$$\chi(\mathcal{X}_r SL_4) = \chi(\mathcal{X}_r PGL_4) = 2 \cdot 4^{r-2},$$

where the only strata that contribute to the Euler characteristic are:

$$\chi(\mathcal{X}_r^{[1^4]} SL_4) = \chi(\mathcal{X}_r^{[1^4]} PGL_4) = 4^{r-1}, \qquad \chi(\mathcal{X}_r^{[2^2]} SL_4) = \chi(\mathcal{X}_r^{[2^2]} PGL_4) = -2 \cdot 4^{r-2}.$$

(2) In the case when n is a prime number p, we similarly get:

$$\chi(\mathcal{X}_r SL_p) = (p-1) \cdot p^{r-2},$$

where the only strata contributing to the Euler characteristic are:

$$\chi(\mathcal{X}_r^{[1^p]} SL_p) = p^{r-1}, \qquad \chi(\mathcal{X}_r^{[p]} SL_p) = -p^{r-2},$$

and similarly for $\mathcal{X}_r PGL_p$.

(3) More generally, if $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_s^{a_s}$ is the prime decomposition of n, the only strata contributing to the Euler characteristic of the SL_n and the PGL_n character variety of F_r are those of the form $[d^{n/d}]$, where $d = p_{i_1} \cdot p_{i_2} \cdots p_{i_t}$ is square-free since then $\mu(d) \neq 0$. These strata are the same for $n^* := p_1 \cdot p_2 \cdots p_s$ and for all n with the same prime divisors. Indeed,

$$\chi(\mathcal{X}_r^{[1^n]} SL_n) = n^{r-1},$$

$$\chi(\mathcal{X}_r^{[p_i^{n/p_i}]} SL_n) = -\frac{1}{p_i} n^{r-1},$$

$$\chi(\mathcal{X}_r^{[(p_i p_j)^{n/p_i p_j}]} SL_n) = \frac{1}{p_i p_j} n^{r-1}, \qquad \cdots$$

$$\chi(\mathcal{X}_r^{[(n^*)^{n/n^*}]} SL_n) = (-1)^s \frac{1}{n^*} n^{r-1},$$

with sum

$$\prod_{p|n,\, p\ prime} (1 - \frac{1}{p}) n^{r-1} = \varphi(n) n^{r-2}.$$

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