

# On Hodge polynomials of Singular Character Varieties

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## Abstract

Let  $\mathcal{X}_\Gamma G := \text{Hom}(\Gamma, G)//G$  be the  $G$ -character variety of  $\Gamma$ , where  $G$  is a complex reductive group and  $\Gamma$  a finitely presented group. We introduce new techniques for computing Hodge-Deligne and Serre polynomials of  $\mathcal{X}_\Gamma G$ , and present some applications, focusing on the cases when  $\Gamma$  is a free or free abelian group. Detailed constructions and proofs of the main results will appear elsewhere.

## 1 Introduction

Let  $G$  be a connected reductive complex algebraic group, and  $\Gamma$  be a finitely presented group. The  $G$ -character variety of  $\Gamma$  is defined to be the (affine) geometric invariant theory (GIT) quotient

$$\mathcal{X}_\Gamma G = \text{Hom}(\Gamma, G)//G.$$

The most well studied families of character varieties include the cases when the group  $\Gamma$  is the fundamental group of a Riemann surface  $\Sigma$ , and its “twisted” variants. In these cases, the non-abelian Hodge correspondence (see, for example [Si]) shows that (components of)  $\mathcal{X}_\Gamma G$  are homeomorphic to certain moduli spaces of  $G$ -Higgs bundles which appear in connection to important problems in Mathematical-Physics: for example, these spaces play an important role in the quantum field theory interpretation of the geometric Langlands correspondence, in the context of mirror symmetry ([KW]).

The study of geometric and topological properties of character varieties is an active topic and there are many recent advances in the computation of their Poincaré polynomials and other invariants. For the surface group case ( $\Gamma = \pi_1(\Sigma)$  and related groups) the calculations of Poincaré polynomials started with Hitchin and Gothen, and have been pursued more recently by Hausel, Lettelier, Melilit, Rodriguez-Villegas, Schiffmann and others, who also considered the parabolic version of these character varieties (see [HRV, Me, Sc]). Those recent results use arithmetic methods: it is shown that the number of points of the corresponding moduli space over finite fields is given by a polynomial,

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which turns out to coincide with the  $E$ -polynomial of  $\mathcal{X}_\Gamma G$  ([HRV, Appendix]). Then, in the smooth case, the pure nature of the cohomology of Higgs bundles moduli spaces allows the derivation of the Poincaré polynomial from the  $E$ -polynomial.

On the other hand, for many important classes of *singular* character varieties, explicitly computable formulas for the  $E$ -polynomials (also called Serre, or Hodge-Euler polynomials) are very hard to obtain. In the articles of Logares, Muñoz, Newstead and Lawton [LMN], [LM] (using geometric methods) and of Baraglia and Hekmati [BH] (using arithmetic methods), the  $E$ -polynomials are computed for several character varieties, with  $G = GL(n, \mathbb{C})$ ,  $SL(n, \mathbb{C})$  and  $PGL(n, \mathbb{C})$  for small values of  $n$ , but the computations quickly become intractable for  $n$  higher than 3.

In this short article, we describe some of the techniques and constructions that we have recently developed for computations of  $E$ -polynomials of singular character varieties, and present some of their main applications.

The outline of the article is as follows. Section 2 covers notations and preliminaries on mixed Hodge and  $E$ -polynomials and on character varieties in the context of GIT. In section 3, we explain how to use equivariant mixed Hodge structures to study (the identity component of)  $\mathcal{X}_\Gamma G$  when  $\Gamma$  is a free abelian group and  $G$  a classical group. These character varieties have orbifold singularities and we can obtain their full mixed Hodge polynomials. In section 4, for arbitrary  $\Gamma$ , we define a stratification of  $GL(n, \mathbb{C})$ -character varieties (which also exists for  $G = SL(n, \mathbb{C})$  or  $PGL(n, \mathbb{C})$ ) which allows writing down an explicit plethystic exponential relation between generating functions of the  $E$ -polynomials of  $\mathcal{X}_\Gamma GL(n, \mathbb{C})$  and of its locus of irreducible representations  $\mathcal{X}_\Gamma^{\text{irr}} GL(n, \mathbb{C})$ . Finally, in Section 5, we consider the free group  $\Gamma = F_r$  of rank  $r$ , and announce the solution of a conjecture of Lawton and Muñoz: the  $E$ -polynomials of  $\mathcal{X}_{F_r} SL(n, \mathbb{C})$  and of  $\mathcal{X}_{F_r} PGL(n, \mathbb{C})$  coincide, for every  $n \in \mathbb{N}$ . For lack of space, the proofs are omitted and will be published elsewhere.

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## 2 Preliminaries on Hodge-Deligne polynomials, Affine GIT and Character Varieties

In this article, all algebraic varieties are defined over  $\mathbb{C}$ ,  $G$  is a connected reductive algebraic group, and  $\Gamma$  is a finitely presented group.

Let  $X$  be a quasi-projective variety (not necessarily irreducible), of complex dimension  $\leq d$ . Deligne showed that the compactly supported cohomology  $H_c^*(X) := H_c^*(X, \mathbb{C})$  can be endowed with a mixed Hodge structure whose mixed Hodge numbers are given by

$$h^{k,p,q}(X) := \dim_{\mathbb{C}} H_c^{k,p,q}(X) \in \mathbb{N}_0,$$

for  $k, p, q \in \{0, \dots, 2d\}$ , and we call  $(p, q)$  the  $k$ -weights of  $X$ , if  $h^{k,p,q} \neq 0$  (c.f. [De], [PS]).

Mixed Hodge numbers are symmetric in the weights,  $h^{k,p,q} = h^{k,q,p}$ , and  $\dim_{\mathbb{C}} H_c^k(X) = \sum_{p,q} h^{k,p,q}$ . Therefore, they provide the (compactly supported) Betti numbers, yielding the usual Betti numbers, by Poincaré duality, in the non-singular case. They are also the coefficients of the mixed Hodge polynomial of  $X$  on three variables,

$$\mu(X; t, u, v) := \sum_{k,p,q} h^{k,p,q}(X) t^k u^p v^q \in \mathbb{N}_0[t, u, v], \quad (1)$$

which specializes to the (compactly supported) Poincaré polynomial by setting  $u = v = 1$ ,  $P_t^c(X) := \mu(X; t, 1, 1)$  (and provides the usual Poincaré polynomial in the smooth situation). Plugging  $t = -1$ , mixed Hodge polynomials convert into the  $E$ -polynomial of  $X$ , or the *Serre polynomial* of  $X$ , given by

$$E(X; u, v) = \sum_{k,p,q} (-1)^k h^{k,p,q}(X) u^p v^q \in \mathbb{Z}[u, v].$$

From the  $E$ -polynomial we can compute the (compactly supported) Euler characteristic of  $X$  as  $\chi^c(X) = E(X; 1, 1) = \mu(X; -1, 1, 1)$ .

Serre polynomials satisfy an *additive property with respect to stratifications* by locally closed (in the Zariski topology) strata: if  $X$  has a closed subvariety  $Z \subset X$  we have (see, eg. [PS]),

$$E(X) = E(Z) + E(X \setminus Z).$$

The  $E$ -polynomial also satisfies (c.f. [DL, LMN]) a *multiplicative property for fibrations*. Namely, for a given algebraic fibration  $F \hookrightarrow X \rightarrow B$ , we have

$$E(X) = E(F) \cdot E(B)$$

in any of the following three situations:

- (i) the fibration is locally trivial in the Zariski topology of  $B$ ,
- (ii)  $F$ ,  $X$  and  $B$  are smooth, the fibration is locally trivial in the complex analytic topology, and  $\pi_1(B)$  acts trivially on  $H_c^*(F)$ , or
- (iii)  $X$ ,  $B$  are smooth and  $F$  is a complex connected Lie group.

We say  $X$  is of *Hodge-Tate type* (also called *balanced type*) if all the  $k$ -weights are of the form  $(p, p)$  with  $p \in \{0, \dots, k\}$ , in which case the sum in  $\mu(X)$  reduces to a one-variable sum. In particular, the  $E$ -polynomials of Hodge-Tate type varieties depend only the product  $uv$ , so we write  $x = uv$  and use the notation  $E(X; x) := E(X; \sqrt{x}, \sqrt{x}) \in \mathbb{Z}[x]$ .

Now let  $X$  be an affine algebraic variety, and let the reductive group  $G$  act algebraically on  $X$ . The induced action of  $G$  on the ring  $\mathbb{C}[X]$  of regular functions on  $X$  defines the (affine) GIT quotient

$$X//G := \text{Spec}(\mathbb{C}[X]^G),$$

where  $\mathbb{C}[X]^G$  is the subring of  $G$ -invariants in  $\mathbb{C}[X]$ . This quotient identifies  $G$ -orbits whose closures intersect, such that each point in the quotient classifies an equivalence class of orbits, leading to a stability condition. Let  $G_x \subset G$  be the stabilizer of  $x \in X$  and consider the orbit map through  $x$ ,  $\psi_x : G \rightarrow X; g \mapsto g \cdot x$ . We define  $x \in X$  to be *stable* if  $\psi_x$  is a proper map and *polystable* if the orbit  $G \cdot x$  is closed in  $X$ . Stability implies polystability, but not conversely.

GIT shows that the stable locus  $X^s \subset X$  is a Zariski open set (hence dense, when non-empty) and that the restriction of the affine quotient map  $\Phi : X \rightarrow X//G$  to the stable locus,  $X^s \rightarrow X^s/G$ , is a geometric quotient (or an orbit space), where  $\Phi(X^s)$  is Zariski open in  $X//G$ .

Now, consider a finitely presented group  $\Gamma$ . The (generally singular) algebraic variety of representations of  $\Gamma$  in  $G$  is

$$\mathcal{R}_{\Gamma}G = \text{Hom}(\Gamma, G).$$

Each  $\rho \in \mathcal{R}_\Gamma G$  is determined by  $\rho(\gamma)$ , for each generator  $\gamma \in \Gamma$ , and satisfying the relations of the group  $\Gamma$ . There is an algebraic action of  $G$  on the variety  $\mathcal{R}_\Gamma G$  by conjugation of representations,  $g^{-1}\rho g$ , yielding the  $G$ -character variety of  $\Gamma$ ,

$$\mathcal{X}_\Gamma G := \text{Hom}(\Gamma, G) // G,$$

as the GIT quotient.

By definition, polystable representations are representations  $\rho : \Gamma \rightarrow G$  whose orbits  $G \cdot \rho := \{g\rho g^{-1} : g \in G\}$  are Zariski closed in  $\mathcal{R}_\Gamma G$ . Alternatively, a representation  $\rho$  is polystable if and only if it is completely reducible (i.e, if  $\rho(\Gamma) \subset P \subset G$  for some proper parabolic  $P$  of  $G$ , then  $\rho(\Gamma)$  is contained in a Levi subgroup of  $P$ ). Denote the subset of polystable representations in  $\mathcal{R}_\Gamma G$  by  $\mathcal{R}_\Gamma^{ps} G \subset \mathcal{R}_\Gamma G$ , which is a Zariski locally-closed subvariety containing the stable locus  $\mathcal{R}_\Gamma^s G \subset \mathcal{R}_\Gamma G$ .

**Proposition 1.** [FL1] *There is a bijective correspondence:*

$$\mathcal{X}_\Gamma G = \mathcal{R}_\Gamma G // G \cong \mathcal{R}_\Gamma^{ps} G / G,$$

where the right hand side is called the polystable quotient.

We say that  $\rho$  is *irreducible* if  $\rho(\Gamma)$  is not contained in a proper parabolic subgroup of  $G$ . Alternatively,  $\rho$  is irreducible if it is polystable and  $Z_\rho$ , the centralizer of  $\rho(\Gamma)$  inside  $G$ , is a finite extension of the center  $ZG \subset G$ . Denote by  $\mathcal{R}_\Gamma^{irr} G \subset \mathcal{R}_\Gamma^{ps} G$  the subset of irreducible representations (being a Zariski open subset of  $\mathcal{R}_\Gamma G$ ,  $\mathcal{R}_\Gamma^{irr} G$  is a quasi-projective variety), and since irreducibility is well defined on  $G$ -orbits, denote by

$$\mathcal{X}_\Gamma^{irr} G := \mathcal{R}_\Gamma^{irr} G / G \tag{2}$$

the  $G$ -irreducible character variety of  $\Gamma$ , which is a geometric quotient, as it happens with the stable locus. In fact, it can be proved that irreducibility is equivalent to GIT stability for character varieties (see [CF, Thm. 1.3(1)]).

### 3 The Free Abelian Case

In this section, we are concerned with the determination of the mixed Hodge polynomials of character varieties  $\mathcal{X}_\Gamma G$  of the free abelian group of rank  $r$ ,  $\Gamma \cong \mathbb{Z}^r$ . As we always work over  $\mathbb{C}$ , we abbreviate the notation of the classical groups such as the linear group, special linear, special orthogonal and symplectic to  $GL_n$ ,  $SL_n$ ,  $SO_n$  and  $Sp_n$ , respectively (instead of  $GL(n, \mathbb{C})$ , etc).

The topology and geometry of the character varieties  $\mathcal{X}_{\mathbb{Z}^r} G$  was studied in [FL2, Sk], among others. Most important for us are the following facts:

- (i) there is only one irreducible component containing the trivial representation, that we denote by  $\mathcal{X}_{\mathbb{Z}^r}^0 G$  [Sk, Theorem 2.1],
- (ii) if the semisimple part of  $G$  is a classical group (ie, one of  $SL_n$ ,  $SO_n$  and  $Sp_n$ ), there exists an algebraic isomorphism

$$\mathcal{X}_{\mathbb{Z}^r}^0 G \cong (T_G)^r / W_G \tag{3}$$

where  $T_G$  is a maximal torus of  $G$ , and  $W_G$  its Weyl group [Sk, Theorem 2.1],

- (iii) the irreducibility of the free abelian character varieties  $\mathcal{X}_{\mathbb{Z}^r} G$  can be characterized, in terms of  $G$ : for example, if the semisimple part of  $G$  is a product of  $SL_n$ 's and  $Sp_n$ 's then  $\mathcal{X}_{\mathbb{Z}^r} G$  is irreducible, so that  $\mathcal{X}_{\mathbb{Z}^r} G \cong \mathcal{X}_{\mathbb{Z}^r}^0 G$  [FL2, Theorem 1.2].

We now focus on the determination of the mixed Hodge numbers of  $\mathcal{X}_{\mathbb{Z}^r} G$  when it is irreducible, or of  $\mathcal{X}_{\mathbb{Z}^r}^0 G$  when the algebraic isomorphism (3) applies. We start by explaining how mixed Hodge numbers transform under finite quotients.

Let  $X$  be a complex quasi-projective variety and  $F$  a finite group acting algebraically on it. The action of  $F$  on  $X$  induces an action on its cohomology. Since  $F$  acts by algebraic isomorphisms, it also induces an action on the mixed Hodge components. Then we can regard  $H^{k,p,q}(X)$  as  $F$ -modules, that we denote by  $[H^{k,p,q}(X)]_F$ . As in equation (1) for the mixed Hodge polynomial, we codify these in the *equivariant mixed Hodge polynomial*, defined by

$$\mu_F(X; t, u, v) := \sum_{k,p,q} [H^{k,p,q}(X)]_F t^k u^p v^q \in R(F)[t, u, v]$$

whose coefficients belong to  $R(F)$ , the representation ring of  $F$ . The polynomial  $\mu_F(X; t, u, v)$  may also be seen as a polynomial-weighted representation. For instance, one can consider equivariant cohomology to obtain an isomorphism

$$H^*(X/F) \cong H^*(X)^F \quad (4)$$

that respects mixed Hodge structures. In particular, this isomorphism allows us to identify the mixed Hodge polynomial of the quotient  $X/F$  as the coefficient of the trivial representation of  $\mu_F(X; t, u, v)$  when written on a basis of irreducible representations of  $F$ . Another important consequence for us is the inequality  $h^{k,p,q}(X) \geq h^{k,p,q}(X/F)$ , which holds since  $H^{k,p,q}(X/F)$  is given by the  $F$ -invariant part of  $H^{k,p,q}(X)$ . We conclude that if  $X$  is, for instance, a balanced variety, or if its mixed Hodge structure is actually *pure* (that is, if  $h^{k,p,q} \neq 0$  then  $k = p + q$ ), then the same holds for  $X/F$ .

We now summarize our strategy to obtain the mixed Hodge polynomials of  $\mathcal{X}_{\mathbb{Z}^r}^0 G$ , in the cases when the isomorphism (3) holds (so, these character varieties are isomorphic to finite quotients of algebraic tori). The only non-zero Hodge numbers of the maximal torus  $T_G \cong (\mathbb{C}^*)^n$  are  $h^{k,k,k}(T_G)$ . Moreover, its natural mixed Hodge structure satisfies:

$$H^{k,k,k}(T_G) \cong \bigwedge^k H^{1,1,1}(T_G).$$

So, the action of  $W_G$  on the cohomology ring can be understood from the one on the mixed Hodge component  $H^{1,1,1}(T_G)$ . The next three theorems are proved in [FS].

**Theorem 1.** *For a reductive group  $G$  satisfying (3), we have*

$$\mu(\mathcal{X}_{\mathbb{Z}^r}^0 G; t, u, v) = \frac{1}{|W_G|} \sum_{g \in W_G} [\det(I + tuv A_g)]^r$$

where  $A_g$  is the automorphism of  $H^{1,1,1}(T_G)$  induced by the action of  $g \in W_G$ .

The proof starts by establishing the  $r = 1$  case, and using the diagonal action for higher  $r$  as well as the isomorphism (3), together with the multiplicative relation for the equivariant polynomials  $\mu_{W_G}(T_G^r) = \mu_{W_G}(T_G)^{\otimes r}$ . We remark that Theorem 1 generalizes a formula for the Poincaré polynomial of  $\mathcal{X}_{\mathbb{Z}^r}^0 G$ , recently obtained in [St].

To further work with Theorem 1, we examine the induced action of  $W_G$  on  $H^{1,1,1}(T_G)$  for some classical groups. In the case  $G = GL_n$ , the Weyl group is the symmetric group  $S_n$  on  $n$  letters, which acts on  $H^{1,1,1}(T_G) \cong \mathbb{C}^n$  by permutation of coordinates, and we obtain a general formula in terms of partitions of  $n$ .

A partition of  $n \in \mathbb{N}$  is denoted by  $[k] = [1^{k_1} \dots j^{k_j} \dots n^{k_n}]$  where the exponent  $k_j \geq 0$  is the number of parts of size  $j \in \{1, \dots, n\}$ , so that  $n = \sum_{j=1}^n j \cdot k_j$ . Let  $\mathcal{P}_n$  denote the finite set of partitions of  $n$ .

**Theorem 2.** *The mixed Hodge polynomials of  $\mathcal{X}_{\mathbb{Z}^r} GL_n$  and of  $\mathcal{X}_{\mathbb{Z}^r} SL_n$  satisfy*

$$\mu(\mathcal{X}_{\mathbb{Z}^r} GL_n; t, u, v) = \mu(\mathcal{X}_{\mathbb{Z}^r} SL_n; t, x)(1 + tuv)^r = \sum_{[k] \in \mathcal{P}_n} \prod_{j=1}^n \frac{(1 - (-tuv)^j)^{k_j r}}{k_j! j^{k_j}},$$

By using similar considerations as for the  $GL_n$  case, we can also deduce a concrete formula for  $Sp_n$  in terms of *bipartitions*. A bipartition of  $n$ , denoted  $[a, b] \in \mathcal{B}_n$  consists of two partitions  $[a] \in \mathcal{P}_k$  and  $[b] \in \mathcal{P}_l$ , such that  $0 \leq k, l \leq n$  with  $k + l = n$ . One can show that bipartitions of  $n$  are in one-to-one correspondence with conjugacy classes in  $W_{Sp_n}$ , the Weyl group of  $Sp_n$ .

**Theorem 3.** *The mixed Hodge polynomial of  $\mathcal{X}_{\mathbb{Z}^r} Sp_n \mathbb{C}$  is given by*

$$\mu(\mathcal{X}_{\mathbb{Z}^r} Sp_n; t, u, v) = \frac{1}{2^n n!} \sum_{[a, b] \in \mathcal{B}_n} c_{[a, b]} \prod_{i=1}^k (1 - (-tuv)^i)^{a_i r} \prod_{j=1}^l (1 + (-tuv)^j)^{b_j r}$$

where  $c_{[a, b]}$  is the size of the conjugacy class in  $W_{Sp_n}$ , corresponding to  $[a, b] \in \mathcal{B}_n$ .

The same method allows to obtain explicit expressions for  $\mu(\mathcal{X}_{\mathbb{Z}^r}^0 G)$  in the case of other reductive  $G$ ; the special orthogonal groups  $SO_n$  will be addressed in a future work.

## 4 Generating functions for $E$ -polynomials

In this section we consider character varieties with arbitrarily bad singularities. In this case, there are formidable difficulties in computing the corresponding Poincaré polynomials in general, and previous explicit methods have dealt with the  $E$ -polynomials for low dimensional groups such as  $SL_2$  and  $SL_3$  ([LMN, LM, BH]).

By using the additive and multiplicative properties of  $E$ -polynomial, for  $G = GL_n$  we now address our new approach on  $E$ -polynomial computations based on a stratification of  $\mathcal{X}_{\Gamma} G$  that we term *by partition type*, and which works for arbitrary  $\Gamma$ .

Using standard arguments in GIT, any character variety admits a stratification by the dimension of the stabilizer of a given representation. When  $G$  is the general linear group  $GL_n$  (as well as the related groups  $SL_n$  and  $PGL_n$ ), there is a more convenient refined stratification that gives a lot of information on the corresponding character varieties  $\mathcal{X}_{\Gamma} G$  which we call *stratification by partition type*.

**Definition 1.** Let  $G = GL_n$  and  $[k] \in \mathcal{P}_n$ . We say that  $\rho \in \mathcal{R}_{\Gamma} G = \text{Hom}(\Gamma, G)$  is  $[k]$ -polystable if  $\rho$  is conjugated to  $\bigoplus_{j=1}^n \rho_j$  where each  $\rho_j$  is, in turn, a direct sum of  $k_j > 0$  irreducible representations of  $\mathcal{R}_{\Gamma}(GL_j)$ , for  $j = 1, \dots, n$  (by convention, if some  $k_j = 0$ , then  $\rho_j$  is not present in the direct sum).

We denote  $[k]$ -polystable representations by  $\mathcal{R}_{\Gamma}^{[k]} G$  and use similar terminology/notation for equivalence classes under conjugation  $\mathcal{X}_{\Gamma}^{[k]} G \subset \mathcal{X}_{\Gamma} G$ . It is to be noted that the trivial partition  $[n] = [n^1] \in \mathcal{P}_n$  corresponds exactly to the irreducible (or stable) locus:  $\mathcal{X}_{\Gamma}^{[n]} G = \mathcal{X}_{\Gamma}^{\text{irr}} G$ .

**Proposition 2.** *Fix  $n \in \mathbb{N}$ , and let  $G = GL_n$ . Then  $\mathcal{X}_{\Gamma} G = \bigsqcup_{[k] \in \mathcal{P}_n} \mathcal{X}_{\Gamma}^{[k]} G$ , as a disjoint union of locally closed quasi-projective varieties.*

The next result relates, by the plethystic exponential, the generating functions of the  $E$ -polynomials  $E(\mathcal{X}_{\Gamma} GL_n)$  to the corresponding generating functions of the  $E$ -polynomials of the irreducible character varieties  $E(\mathcal{X}_{\Gamma}^{\text{irr}} GL_n)$ .

The *plethystic exponential* of a formal power series  $f(x, y, z) = \sum_{n \geq 0} f_n(x, y) z^n \in \mathbb{Q}[x, y][[z]]$  is denoted by  $\text{PExp}(f)$ , and defined formally (in terms of the usual exponential) as  $\text{PExp}(f) := e^{\Psi(f)} \in \mathbb{Q}[x, y][[z]]$ , where  $\Psi$  acts on monomials as:  $\Psi(x^i y^j z^k) = \sum_{l \geq 1} \frac{x^i y^j z^{lk}}{l}$ , where  $(i, j, k) \in \mathbb{N}_0^3 \setminus \{(0, 0, 0)\}$ ,

and is  $\mathbb{Q}$ -linear on  $\mathbb{Q}[x, y][[z]]$ . This exponential plays a prominent role in the combinatorics of symmetric functions, and has applications in counting of gauge invariant operators in supersymmetric quantum field theories (see eg. [FHH]).

**Theorem 4.** *Let  $\Gamma$  be any finitely presented group. Then:*

$$\sum_{n \geq 0} E(\mathcal{X}_\Gamma GL_n; u, v) t^n = \text{PExp} \left( \sum_{n \geq 1} E(\mathcal{X}_\Gamma^{\text{irr}} GL_n; u, v) t^n \right).$$

The proofs of Theorem 4 and Proposition 2 are detailed in [FNZ]; they allow to write explicit expressions for  $E(\mathcal{X}_\Gamma GL_n)$ , for any group  $\Gamma$ , for which we have a formula for  $E(\mathcal{X}_\Gamma^{\text{irr}} GL_m)$ , for all  $m \leq n$ , by a simple finite algorithm (and vice-versa). The formula of Theorem 4 generalizes a formula of [MR] to an arbitrary group  $\Gamma$ , even if the corresponding  $GL_n$ -character variety is not of polynomial type.

## 5 The Free Group Case

In this last section, we describe applications of the above methods to the case of the free group of rank  $r$ ,  $\Gamma = F_r$ ; for simplicity we adopt the notations  $\mathcal{X}_r GL_n$ ,  $\mathcal{X}_r SL_n$ , etc, for the corresponding character varieties. In [MR], it was shown that  $\mathcal{X}_r^{\text{irr}} GL_n$  and  $\mathcal{X}_r GL_n$  are of polynomial type. Moreover, by counting points over finite fields and using a theorem of Katz ([HRV, Appendix]), Mozgovoy and Reineke found a formula for the  $E$ -polynomial of  $\mathcal{X}_r^{\text{irr}} GL_n$  that can be written as follows (dropping the  $x$  variable in  $E(X; x)$ , and using  $||[k]|| := k_1 + \dots + k_d$  for the length of a partition  $[k] \in \mathcal{P}_d$ ).

**Proposition 3.** [MR, FNZ] *For  $r, n \geq 2$ , we have:*

$$E(\mathcal{X}_r^{\text{irr}} GL_n) = (x-1) \sum_{d|n} \frac{\mu(n/d)}{n/d} \sum_{[k] \in \mathcal{P}_d} \frac{(-1)^{||[k]||}}{||[k]||} \binom{||[k]||}{k_1, \dots, k_d} \prod_{j=1}^d b_j(x^{n/d})^{k_j} x^{\frac{n(r-1)k_j}{d} \binom{j}{2}},$$

where  $\mu$  is the Möbius function, and the  $b_j(x)$  are polynomials defined by:

$$\left(1 + \sum_{n \geq 1} b_n(x) t^n\right) \left(1 + \sum_{n \geq 1} ((x-1)(x^2-1) \dots (x^n-1))^{r-1} t^n\right) = 1. \quad (5)$$

Using Propositions 2 and 3 and Theorem 4, we are able to write down very explicit expressions for  $E(\mathcal{X}_r^{[k]} GL_n)$ , the  $E$ -polynomials of all polystable strata of  $\mathcal{X}_r GL_n$  (see [FNZ, Secs. 5 and 6], where we also compute  $E(\mathcal{X}_\Gamma^{\text{irr}} GL_n)$  for other  $\Gamma$  and low  $n$ ).

We now provide a few lines on a forthcoming proof of the equality between the  $E$ -polynomials of  $\mathcal{X}_r SL_n$  and of  $\mathcal{X}_r PGL_n$  for all  $n \in \mathbb{N}$ . This has been conjectured in Lawton-Muñoz in [LM], who proved by explicit computation the cases  $n = 2$  and  $3$ .

In an analogous way as for  $GL_n$  (see Section 4), we can define the  $[k]$ -polystable loci  $\mathcal{X}_r^{[k]} SL_n$  and  $\mathcal{X}_r^{[k]} PGL_n$  as follows. For a partition  $[k] \in \mathcal{P}_n$ , the  $[k]$ -stratum of  $\mathcal{X}_r SL_n$  is defined by restriction of the corresponding one for  $GL_n$ :

$$\mathcal{X}_r^{[k]} SL_n := \{\rho \in \mathcal{X}_r^{[k]} GL_n \mid \det \rho = 1\},$$

where the determinant of a representation is an element of  $\mathcal{R}_r \mathbb{C}^*$ . By considering the action  $\mathcal{R}_r \mathbb{C}^* \times \mathcal{X}_r GL_n \rightarrow \mathcal{X}_r GL_n$  given by multiplication of (conjugacy classes of) representations, which is well defined on the GIT quotients and preserves the stratification of  $GL_n$ , we can define

$$\mathcal{X}_r^{[k]} PGL_n := \mathcal{X}_r^{[k]} GL_n / \mathcal{R}_r \mathbb{C}^* = \mathcal{X}_r^{[k]} GL_n / (\mathbb{C}^*)^r. \quad (6)$$

**Theorem 5.** [FNZ2] *For the free group  $F_r$ , we have the equalities:*

$$\begin{aligned} E(\mathcal{X}_r SL_n) &= E(\mathcal{X}_r PGL_n) = E(\mathcal{X}_r GL_n)(x-1)^{-r} \\ E(\mathcal{X}_r^{[k]} SL_n) &= E(\mathcal{X}_r^{[k]} PGL_n) = E(\mathcal{X}_r^{[k]} GL_n)(x-1)^{-r}, \end{aligned}$$

for every  $r$ ,  $n$  and partition  $[k] \in \mathcal{P}_n$ .

The proof of Theorem 5 uses geometric methods and has two parts. The easy part is the relation between the  $E$ -polynomials of  $\mathcal{X}_r^{[k]} PGL_n$  and of  $\mathcal{X}_r^{[k]} GL_n$ , which follows from the locally trivial (in the Zariski topology) fibration corresponding to the quotient (6). The difficult part is the relation between the strata  $\mathcal{X}_r^{[k]} PGL_n$  and  $\mathcal{X}_r^{[k]} SL_n$  which involves finite quotients: it requires the proof of the triviality of the action of the center  $\mathbb{Z}_n \subset SL_n$  on the cohomology (with compact support) of all the strata  $\mathcal{X}_r^{[k]} SL_n$ ; for this we use equivariant cohomology and a deformation retraction between  $\mathcal{X}_r^{irr} SL_n$  and the smooth part of the semialgebraic set  $\text{Hom}(F_r, SU(n))/SU(n)$  (see [FL]).

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