ROOTS OF GENERALISED HERMITE POLYNOMIALS WHEN BOTH PARAMETERS ARE LARGE

DAVIDE MASOERO AND PIETER ROFFELSEN

Abstract. We study the roots of the generalised Hermite polynomials $H_{m,n}$ when both m and n are large. We prove that the roots, when appropriately rescaled, densely fill a bounded quadrilateral region, called the elliptic region, and organise themselves on a deformed rectangular lattice, as was numerically observed by Clarkson. We describe the elliptic region and the deformed lattice in terms of elliptic integrals and their degenerations.

Keywords: Generalised Hermite polynomials; roots asymptotics; Painleve IV; Boutroux Curves; Tritronquee solution.

Contents

1. Introduction

For $m, n \in \mathbb{N}$, the generalised Hermite polynomial $H_{m,n}$ is the polynomial of degree $m \times n$ defined by the determinantal formula

$$H_{m,n}(z) = \gamma_{m,n} \begin{vmatrix} h_m(z) & h_{m+1}(z) & \dots & h_{m+n-1}(z) \\ h_m^{(1)}(z) & h_{m+1}^{(1)}(z) & \dots & h_{m+n-1}^{(1)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ h_m^{(n-1)}(z) & h_{m+1}^{(n-1)}(z) & \dots & h_{m+n-1}^{(n-1)}(z) \end{vmatrix}$$

where $h_k^{(l)}(z)$ denotes the l-th derivative of the k-th Hermite polynomial

$$h_k(z) = (-1)^k e^{z^2} \frac{\partial^k}{\partial z^k} \left[e^{-z^2} \right]$$

and $\gamma_{m,n} \in \mathbb{C}^*$ is a, for the purpose of this paper, irrelevant constant multiplier.

The goal of the present paper is to study the asymptotic distribution of the roots of $H_{m,n}$ as $m+n \to \infty$. We call the asymptotics unrestricted since, in contrast to our previous paper [21], we do not require any of the two parameters to remain bounded.

The most striking property of the generalised Hermite polynomials is that they yield families of rational solutions to the fourth Painlevé equation

$$\omega_{zz} = \frac{1}{2\omega}\omega_z^2 + \frac{3}{2}\omega^3 + 4z\omega^2 + 2(z^2+1-2\theta_\infty)\omega - \frac{8\theta_0^2}{\omega}, \quad \theta := (\theta_0, \theta_\infty) \in \mathbb{C}^2. \tag{1}$$

For example, the functions $\omega^{(\mathrm{I})}_{m,n} = \frac{d}{dz}\log\frac{H_{m+1,n}}{H_{m,n}}$ solve the above equation with parameters $\theta_0 = \frac{1}{2}n, \theta_\infty = m + \frac{1}{2}n + 1$ [23]. This establishes an explicit relation among poles of rational solutions of Painleve IV and roots of generalised Hermite polynomials; hence the problem of our interest can be restated as the study of the asymptotic distribution of singularities of the Hermite-family of rational solutions of the Painleve IV equation [21].

We mentioned in our previous paper [21] that the asymptotic analysis of rational solutions of Painlevé equations (and more generally special solutions) has recently been the object of intense study, see e.g. [11, 19]. This is even more the case at the time of writing since in the meanwhile many new results have appeared in the literature [4, 6, 25]. There is a clear reason for this: most often when a problem in applied or pure mathematics is solved by means of a solution of a Painlevé equation (see e.g. [12]), this is singled-out by some special (i.e. non generic) asymptotic expansion, which reflects itself in a special associated Riemann-Hilbert problem amenable to a thorough analysis to a degree not attainable for generic solutions [1].

The above is clearly the case for generalised Hermite polynomials. In fact in [21] we showed that in the case of the generalised Hermite polynomials the isomonodromic deformation method for Painleve IV simplifies dramatically. To be more precise, we proved the following theorem, which characterises the roots of $H_{m,n}$ by means of an eigenvalue problem for a specific class of anharmonic oscillators.

Theorem 1 (Theorem 2.2 in [21])**.** *For $m, n \in \mathbb{N}$, the point $a \in \mathbb{C}$ is a root of $H_{m,n}$ if and only if there exists $b \in \mathbb{C}$ such that the anharmonic oscillator*

$$\psi''(\lambda) = (\lambda^2 + 2a\lambda + a^2 - (2m+n) - \frac{b}{\lambda} + \frac{n^2-1}{4\lambda^2})\psi(\lambda), \tag{2}$$

satisfies the following two properties:

(1) ***Apparent Singularity Condition.*** *The resonant singularity at $\lambda = 0$ is apparent or equivalently the monodromy around the singularity is scalar. In a formula,*

$$\psi(e^{2\pi i}\lambda) = (-1)^{n+1}\psi(\lambda), \quad \forall \psi \text{ solution of (2)}.$$

(2) ***Quantisation Condition.*** *There exists a non-zero solution of (2) which solves the following boundary value problem*

$$\lim_{\lambda \to +\infty} \psi(\lambda) = \lim_{\lambda \to 0^+} \psi(\lambda) = 0 .$$

Studying the asymptotic solution of the above inverse problem, we obtain our main result, which we name the *bulk asymptotics*:

(1) We determine the region of the complex plane which asymptotically gets filled densely by the roots. This is a quadrilateral domain that we call the elliptic region following Buckingham [6], who had already described it.

[1] On the other side, the structure of the Painlevé equations can be properly understood only when studying the general solution, see [15, 16, 17].

(2) We describe the bulk asymptotics of the roots, that is, we obtain an asymptotic description of the roots uniformly on any fixed compact subset of the interior of the elliptic region. In particular we show that the roots asymptotically organise themselves on a deformed lattice, thus confirming Clarkson's numerical findings [8, 9].

The paper is organised as follows. In Section 2 we state our main results, and we announce our results concerning the critical asymptotics, that is, the asymptotics of roots approaching the four corners of the elliptic region (we postpone a detailed analysis of this to a forthcoming publication, in preparation). In Section 3 we develop a complex WKB method for equation (2), Section 4 is devoted to the study of the elliptic region and in Section 5 we prove the main theorems concerning the bulk asymptotics. Finally in the appendix we collect some results from the theory of elliptic integrals and of Stokes complexes which are used in the main body.

Before we begin our paper, in Figure 1 we show the reader a pictorial description of our results, which is worth a thousand lemmas.

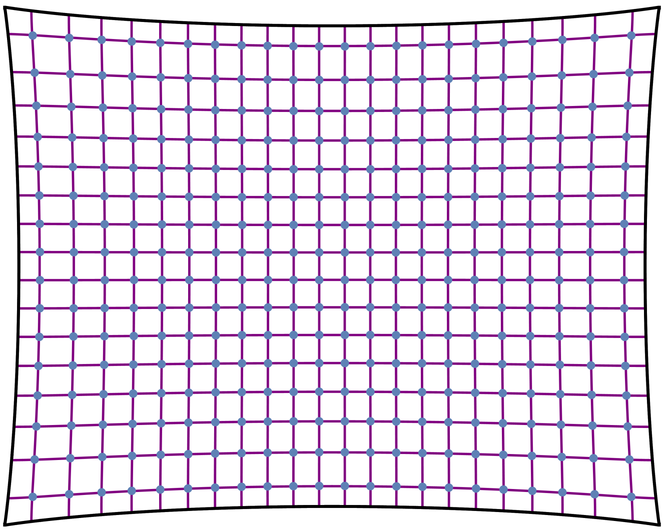

FIGURE 1. Asymptotically the roots lie on the vertices of a deformed lattice. The asymptotic prediction is stunningly precise even for moderate m, n. In the picture, the elliptic region with in purple the deformed lattice, and in blue the roots of $H_{m,n}(z)$ with $(m, n) = (22, 16)$.

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2. Results

We introduce a large parameter $E = 2m + n$, and the scaled parameters α, β, ν,

$$E = 2m + n, \quad \alpha = E^{-\frac{1}{2}} a, \quad \beta = E^{-\frac{3}{2}} b, \quad \nu = \frac{n}{E}. \tag{3}$$

Without loss in generality we assume that $m \geq n$, since $H_{n,m}(z) = H_{m,n}(iz)$, hence $\nu \in [0, \frac{1}{3}]$. We further assume that $\nu \in (0, \frac{1}{3}]$. The case $\nu = 0$ requires a different approach and was dealt with in our previous paper [21].

By scaling the independent variable $\lambda \to E^{\frac{1}{2}}\lambda$, the problem laying before us is the rigorous study of the no-logarithm and quantisation condition on the anharmonic oscillator

$$\psi''(\lambda) = \left(E^2 V(\lambda; \alpha, \beta, \nu) - \frac{1}{4\lambda^2}\right)\psi(\lambda), \tag{4}$$

$$V(\lambda; \alpha, \beta, \nu) = \lambda^2 + 2\alpha\lambda + \alpha^2 - 1 - \beta\lambda^{-1} + \frac{\nu^2}{4}\lambda^{-2}, \tag{5}$$

in the $E \to +\infty$ limit. Our approach to the inverse monodromy problem is based on the complex WKB analysis of equation (4). The latter builds on the approximation of solutions by means of the (multivalued) WKB functions

$$\psi = V^{-\frac{1}{4}} e^{\pm E \int^{\lambda} \sqrt{V(\mu)} d\mu}, \tag{6}$$

where V is the function appearing in (5). Notice that in the above formula we have neglected the term $-\frac{1}{4\lambda^2}$. This is called the Langer modification and it is necessary to obtain a correct approximation when regular singularities are present; we will discussed it further when proving our results.

2.1. **The Elliptic Region.** In this subsection we introduce the compact region K_a in the complex α-plane which asymptotically gets filled with the roots of $H_{m,n}(E^{\frac{1}{2}}\alpha)$ as $E \to \infty$. This region is defined as the projection onto the α-plane of a compact set $K \subseteq \{(\alpha, \beta) \in \mathbb{C}^2\}$ which asymptotically gets filled with the solutions (α, β) of the no-logarithm and quantisation condition on (4).

Before defining K, we first introduce the elliptic curve underlying the elliptic integrals in the WKB functions (6). Consider the affine curve

$$\Gamma_{\alpha,\beta} = \{P = (\lambda, y) \in \mathbb{C}^2, y^2 = \lambda^4 + 2\alpha\lambda^3 + (\alpha^2 - 1)\lambda^2 - \beta\lambda + \frac{\nu^2}{4} = \lambda^2 V(\lambda)\}.$$

This curve can be compactified by adding two points at infinity, $\infty_\pm$, in accordance with the rule $\lim_{P \to \infty_\pm} \frac{y}{\lambda^2} = \pm 1$. In this way we obtain a -possibly singular- elliptic curve, which we denote by $\widehat{\Gamma} = \widehat{\Gamma}(\alpha, \beta)$, with corresponding projection $(\lambda, y) \mapsto \lambda$ onto the Riemann sphere. The pull-back of the multivalued differential $\sqrt{V(\lambda)}d\lambda$ is the meromorphic differential $\omega = \frac{y}{\lambda}d\lambda$ on $\hat{\Gamma}$.

In order to introduce K, we first define the Stokes complex associated with the potential $V(\lambda)$. In the WKB analysis of (4) the Stokes lines of $V(\lambda)$ play an important role. These are defined

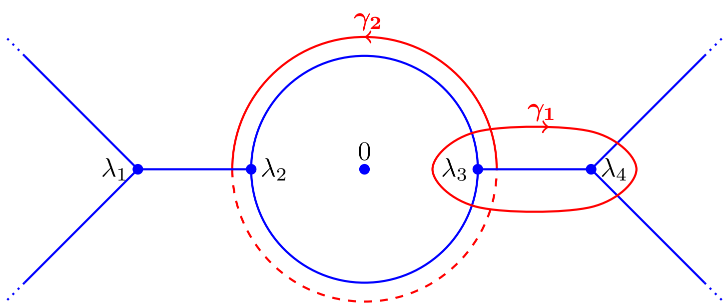


FIGURE 2. In blue the Stokes complex and in red the two cycles γ_1 and γ_2 used in equations (10), where λ_k, $1 \leq k \leq 4$, are the zeros of $V(\lambda)$.

as the levels sets $\Re \int_{\lambda^*}^{\lambda} \sqrt{V(\lambda)} d\lambda = 0$ in $\mathbb{P}^1$, where λ^* any zero of $V(\lambda)$. The Stokes complex $\mathcal{C} = \mathcal{C}(\alpha, \beta) \subseteq \mathbb{P}^1$ of $V(\lambda)$ is defined as the union of all its Stokes lines and zeros. For example, in Figure 2 the Stokes complex of $V(\lambda)$ is depicted with $(\alpha, \beta) = (0, 0)$.

Definition 1. *We define R as the set of $(\alpha, \beta) \in \mathbb{C}^2$ such that the Stokes complex $\mathcal{C}(\alpha, \beta)$ is homeomorphic to the Stokes complex at $(0, 0)$, denote its closure by $K = \overline{R}$ and define K_a as the projection of K onto the α-plane. We call K_a the elliptic region.*

The region K_a is a quadrilateral domain, invariant under complex conjugation and reflection in the origin: It is a simply connected region whose boundary is a Jordan curve composed of four analytic pieces, which we call edges, meeting at four corners, as in Figure 4. The interior of K_a corresponds to regular elliptic curves, while edges and corners are made up of singular elliptic curves: the edges correspond to the coalescence of a pair of branching points, and the corners correspond to the coalescence of three branching points.

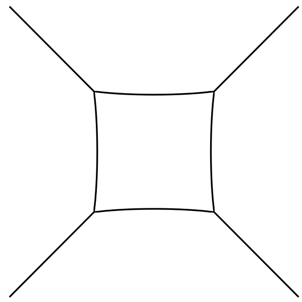


FIGURE 3. $\{\psi(\alpha) = 0\}$

Call $e_k, 1 \leq k \leq 4$, the edges of K_a and $c_k, 1 \leq k \leq 4$, the corners of K_a, see Figure 4, so that

$$\partial K_a = e_1 \sqcup e_2 \sqcup e_3 \sqcup e_4 \sqcup \{c_1, c_2, c_3, c_4\}.$$

For $1 \leq k \leq 4$, the corner c_k of K_a equals the unique root of

$$C(\alpha) := \alpha^8 - 6(3\nu^2 + 1)\alpha^4 + 8(1 - 9\nu^2)\alpha^2 - 3(9\nu^4 + 6\nu^2 + 1) \tag{7}$$

in the k-th quadrant of the complex α-plane. For $1 \leq k \leq 4$, the edge e_k is a smooth curve in the half-plane $\{\frac{1}{2}\pi(k-2) < \arg \alpha < \frac{1}{2}\pi k\}$ with end-points c_{k-1} and c_k, where $c_{-1} := c_4$. We have the following implicit parametrisation of ∂K_a: let $x = x(\alpha)$ be the unique algebraic function which solves the quartic

$$3x^4 + 4\alpha x^3 + (\alpha^2 - 1)x^2 - \frac{\nu^2}{4} = 0$$

analytically in the complex α-plane with $x(\alpha) \sim \frac{\nu}{2}\alpha^{-1}$ as $\alpha \to \infty$ and branch-cuts the diagonals $[c_1, c_3]$ and $[c_2, c_4]$. On the same cut plane there exists a unique algebraic function $y = y(\alpha)$ which solves

$$y^2 = \alpha^2 + 6x\alpha + 6x^3 - 1$$

with $y(\alpha) \sim \alpha$ as $\alpha \to \infty$. We set

$$\psi(\alpha) = \tfrac{1}{2}\Re\left[\alpha y + \tfrac{1}{2}(1-\nu)\log(p_1) - \log(p_2) + \nu \log(p_3)\right], \tag{8}$$

where

$$p_1 = 1 - 2x\alpha - 2x^2, \quad p_2 = 2x + \alpha + y, \quad p_3 = \frac{x(\alpha^2 + 5x\alpha + 4x^2 - 1) + \frac{1}{2}\nu y}{x^2}.$$

Then ψ is a univalued harmonic function on the cut plane and its level set $\{\phi(\alpha) = 0\}$ consists of the boundary ∂K_a plus four additional lines which emanate from the corners and go to infinity along the asymptotic directions $e^{\frac{\pi}{4}(2k-1)}\infty$, $1 \leq k \leq 4$, see Figure 3. Buckingham [6, Sections 1.1 and 3.2] gives a slightly different but equivalent parametrisation of the boundary, which also includes the four additional lines emanating from the corners.

To formulate our asymptotic results, we need to introduce a pair of functions on K and K_a. For any $(\alpha, \beta) \in K$ we define the basis of cycles $\{\gamma_1, \gamma_2\}$ as in Figure 2. By Definition 1 we have $\Re \oint_{\gamma_1} \omega = \Re \oint_{\gamma_2} \omega = 0$, thus yielding a real mapping from K (resp. K_a) to $\mathbb{R}^2$:

$$\mathcal{S} : K \to \mathbb{R}^2, (\alpha, \beta) \mapsto (-is_1(\alpha, \beta), -is_2(\alpha, \beta)), \qquad \mathcal{S}_a = \mathcal{S} \circ \Pi_a^{-1} \tag{9}$$

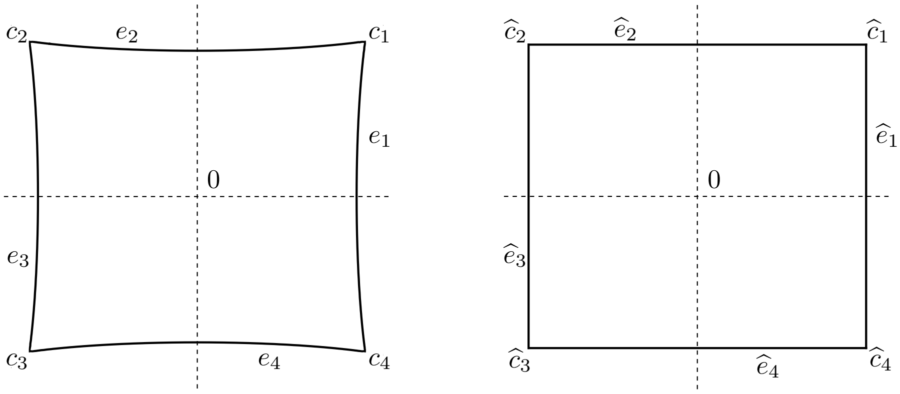


FIGURE 4. Edges and corners of respectively K_a and Q.

with

$$s_1(\alpha,\beta) = \int_{\gamma_1} \omega + \frac{i\pi(1-\nu)}{2}, \tag{10a}$$

$$s_2(\alpha,\beta) = \int_{\gamma_2} \omega \tag{10b}$$

and $\Pi_a : K \to K_a$ the projection onto K_a, which we prove to be invertible.

It turns out that K (resp. K_a) are homeomorphic under $\mathcal{S}$ (resp. $\mathcal{S}_a$) to the rectangle

$$Q := \left[-\tfrac{1}{2}(1-\nu)\pi, +\tfrac{1}{2}(1-\nu)\pi\right] \times [-\nu\pi, +\nu\pi], \tag{11}$$

and the above homeomorphisms map the interior of K (resp. K_a) C^∞-diffeomorphically onto the interior of Q. Moreover the image of the edges e_k are the edges $\hat{e}_k$ of the rectangle and the image of the corners c_k are the corners $\hat{c}_k$ of the rectangle, as shown in Figure 4.

Remark 1. *For $1 \le k \le 4$, the edges e_k and e_{k+1} meet at the corner c_k with interior angle equal to $\frac{2}{5}\pi$. In particular $\mathcal{S}_a$ is not conformal.*

2.2. **Bulk Asymptotics.** For sake of simplicity we state our result when $m, n \to \infty$ with the ratio $\frac{m}{n}$ fixed. We thus choose $p \ge q$, p, q either equal or co-prime, and fix the ratio $\frac{m}{n} = \frac{p}{q}$. Hence the numbers m, n take values in the sequences $m = tq, n = tp, t \in \mathbb{N}^*$. Correspondingly $\nu = \frac{p}{2q+p} \in (0, \frac{1}{3}]$ is fixed and the large parameter E belongs to the sequence $(2q+p)t, t \in \mathbb{N}^*$.

Definition 2. *Given an integer number $m \in \mathbb{N}^*$ we denote $I_m = \{-m+1, -m+3, \dots, m-1\} \subseteq \mathbb{Z}$. $\forall (j,k) \in I_m \times I_n$ we let $(\alpha_{j,k}, \beta_{j,k}) \in K$ be the unique solution of $\mathcal{S}(\alpha,\beta) = (\frac{\pi j}{E}, \frac{\pi k}{E})$.*

Definition 3. *A filling fraction is a real number $\sigma \in (0,1)$. We let $I_m^\sigma = I_m \cap [\sigma(-m+1), \sigma(m-1)]$ and define $Q^\sigma \subset Q$ as the closed rectangle $[-\frac{\pi\lfloor\sigma(m-1)\rfloor}{E}, \frac{\pi\lfloor\sigma(m-1)\rfloor}{E}] \times [-\frac{\pi\lfloor\sigma(n-1)\rfloor}{E}, \frac{\pi\lfloor\sigma(n-1)\rfloor}{E}]$, which in the large E limit converges to $\sigma \cdot Q$.*

Finally we define $K^\sigma = \mathcal{S}^{-1}(Q^\sigma)$ and $K_a^\sigma = \Pi_a(K^\sigma)$ as the projection of K^σ on the α-plane.

Our main result is the following theorem which provides an asymptotic formula for the roots of the generalised Hermite polynomials as well as an estimate of the error.

Theorem 2. *Fix a filling fraction $\sigma \in (0,1)$. Then there exists an $R_\sigma > 0$ such that for E large enough the following hold true:*

(1) Each disc with center $E^{\frac{1}{2}}\alpha_{j,k}$ and radius $R_\sigma E^{-\frac{3}{2}}$, $(j,k) \in I_m^\sigma \times I_n^\sigma$, contains a unique root of the generalised Hermite polynomial $H_{m,n}$.

(2) Let $\mathcal{K}^\sigma = \{a \in \mathbb{C}, E^{-\frac{1}{2}}a \in K_a^\sigma\}$, then the ϵ-neighbourhood of $\mathcal{K}^\sigma$ with radius $R_\sigma E^{-\frac{3}{2}}$ contains exactly $\lfloor \sigma m \rfloor \times \lfloor \sigma n \rfloor$ roots of the generalised Hermite polynomials.

Clarkson [8, 10] observed numerically that the zeros of generalised Hermite polynomial $H_{m,n}$ seem to organise themselves on the intersection points of a deformed grid of m vertical and n horizontal lines, see the top pictures in Figure 5. The mapping $\mathcal{S}_a$ rectifies this deformed grid. Namely, after Theorem 2, the images under $\mathcal{S}_a$ of the rescaled roots of $H_{m,n}$ asymptotically organise themselves along the intersection points of the true orthogonal grid made of the respective equally spaced m vertical and n horizontal lines

$$l_v^{(j)} = \{(x,y) \in Q, x = \frac{\pi j}{E}\} \quad (j \in I_m), \quad l_h^{(k)} = \{(x,y) \in Q, y = \frac{\pi k}{E}\} \quad (k \in I_n), \tag{12}$$

see Figure 5.

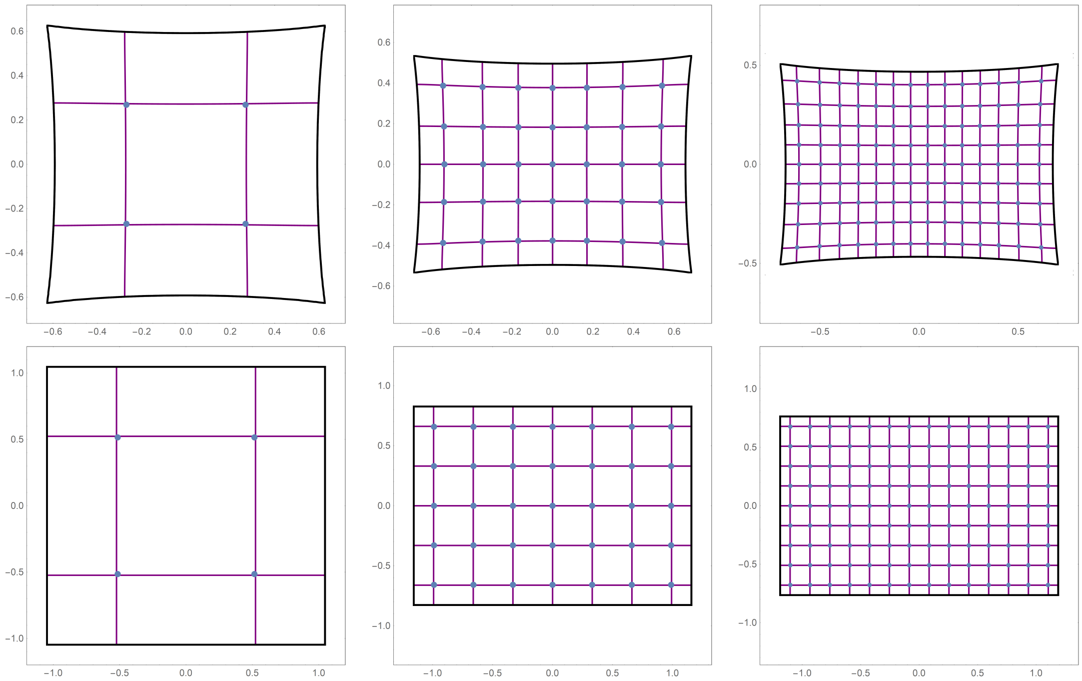


FIGURE 5. On the top row the elliptic region K_a with in purple the inverse images under $\mathcal{S}_a$ of the grid lines (12) and the roots of $H_{m,n}(E^{\frac{1}{2}}\alpha)$ in blue, and on the bottom row the rectangle Q with in purple the grid lines (12) and the images of the roots of $H_{m,n}(E^{\frac{1}{2}}\alpha)$ under $\mathcal{S}_a$ in blue for the respective values $(m,n) = (2,2)$, $(7,5)$, $(14,9)$.

The deformed lattice is actually a regular lattice for small α's. Indeed we have the following immediate corollary of Theorem 2.

Corollary 1. *Fix N_0 and suppose $|j|, |k| \leq N_0$. The roots lie on the regular lattice generated by the vectors $\frac{\mathcal{K}(\frac{1-\nu}{1+\nu})}{E\sqrt{1+\nu}}$ and $i\frac{\mathcal{K}(\frac{2\nu}{1+\nu})}{E\sqrt{1+\nu}}$, where $\mathcal{K}(m)$ denotes the standard complete elliptic integral of the first kind with respect to the parameter $m = k^2$.*

More precisely, the following asymptotic formula holds

$$\alpha_{j,k} = \frac{1}{\sqrt{1+\nu}} \left(\mathcal{K}\left(\frac{1-\nu}{1+\nu}\right) \frac{j}{E} + \mathcal{K}\left(\frac{2\nu}{1+\nu}\right) \frac{k}{E} i \right) + O(E^{-2}) \tag{13}$$

Note that the above equation describes a square lattice when $\nu = \frac{1}{3}$.

Announcement of a result concerning the critical behaviour. We announce here the formula describing the critical asymptotics of roots of generalised Hermite polynomials. Global error estimates, and full proofs will be provided in a forthcoming publication [22].

Theorem 2 does not cover the asymptotics of roots approaching the corner of the elliptic region. These are called critical asymptotics, and, according to our result, are described by means of the Tritronquee solution y_T of the Painlevé I equation

$$y''(z) = 6y^2(z) - z \, . \tag{14}$$

Remarkably, also the critical asymptotics for the poles of rational solutions of the Painlevé II equation are described in terms of the Tritronquee solution, as proven in [7].

Let us be more precise. The Tritronquee solution, which was discovered by Boutroux [5], can be defined as the unique solution of equation (14) which does not have poles in the closed sector $|\arg z| \leq \frac{4\pi}{5}$ [11]. It has however an infinite number of poles [19], which a fortiori lie in the sector $|\arg z| > \frac{4\pi}{5}$. Let c_1 be the upper right corner of the elliptic region, and p be a pole of the Tritronquee solution y_T. For E large enough, there is a unique root α_p of $H_{m,n}$ with the following asymptotic behaviour

$$\alpha_p = c_1 - 4\kappa p E^{-\frac{2}{5}} + \mathcal{O}(E^{-\frac{3}{5}}) \text{ as } E \to \infty \, . \tag{15}$$

Here κ is a constant, independent of p, defined by the equation

$$\kappa^5 = \frac{2c_1^3(2+c_1^2)^3}{(c_1^4 - 3\nu^2 - 1)(c_1^4 + 4c_1^2 + 3\nu^2 + 1)}, \quad -\frac{3}{4}\pi - \frac{\pi}{10} \leq \arg\kappa < -\frac{3}{4}\pi + \frac{\pi}{10}.$$

The critical asymptotics for the other corners are given by the above formulas, upon substituting c_1 for the corresponding $c_k, k \neq 1$, and by choosing the appropriate solution of the quintic equation.

2.3. **Note on the Literature.** The problem of describing the asymptotic location of roots of generalised Hermite polynomials when both parameters are large were raised by Peter Clarkson in [9]. The slightly more general problem of describing the asymptotics of generalised Hermite polynomials was the object of much recent interest, see e.g. [21, 6, 25]. In our previous paper [21] we dealt with the asymptotic location of roots in case when one of the two parameters stays bounded. Buckingham studied the asymptotics of generalised Hermite polynomials when both parameters are large in the complement of the elliptic region, that is, in the region where no roots are present [6]. We present here the first correct (a previous attempt is carefully analysed in [6, 25], and shown to be erroneous) asymptotic description of roots of generalised Hermite polynomials when both parameters are large.

3. WKB asymptotics

In the present section we study the inverse monodromy problem of equation (4), which characterises roots of generalised Hermite polynomials in the rescaled variables α, β, in the large E limit. We do this by developing a suitable complex WKB method. We remark that, in the present section and unless otherwise stated, $m, n, E = 2m+n$ are arbitrary positive real numbers, since most results hold true irrespective of their integer nature.

We consider the complex WKB method for an equation of the kind

$$\psi''(x) = (E^2 V(\lambda) + r(\lambda))\psi(\lambda) \, . \tag{16}$$

Specifying to $V = V(\lambda;\alpha,\beta,\nu)$ and $r = -\frac{1}{4\lambda^2}$, we obtain the main object of our analysis, namely equation (4).

The WKB asymptotic is based on the approximation of solutions by means of the multivalued functions

$$J(\lambda,\lambda') = e^{-ES(\lambda,\lambda')-\frac{1}{4}L(\lambda,\lambda')}, \tag{17}$$
$$S(\lambda,\lambda') = \int_{\lambda'}^{\lambda}\sqrt{V(\mu)}d\mu, \quad L(\lambda,\lambda') = \int_{\lambda'}^{\lambda}\frac{V'(\mu)}{V(\mu)}d\mu$$

In order to measure the difference between a WKB approximation $J(\lambda,\lambda')$ and a solution $\psi(\lambda,\lambda_0,\lambda')$, we introduce the ratio $z(\lambda,\lambda_0) := z(\lambda,\lambda_0,\lambda') = \frac{\psi(\lambda,\lambda_0,\lambda')}{J(\lambda,\lambda')}$ and impose the boundary condition $z(\lambda_0,\lambda_0) = 1$ (which implies that z does not depend on λ'). The ratio (after a few algebraic steps, see [13, §Section 4] for the details) is shown to satisfy the following Volterra integral equation

$$z(\lambda,\lambda_0) = 1 - \frac{1}{E}\int_{\lambda_0}^{\lambda}K(\lambda,\lambda')F(\lambda')d\lambda'\,, \quad B(\lambda,\lambda') = \frac{e^{2ES(\lambda,\lambda')}-1}{2}$$
$$F(\lambda) = \frac{\phi(\lambda)}{V^{\frac{1}{2}}(\lambda)}\,, \quad \phi(\lambda) = -r(\lambda) + \frac{-4V''(\lambda)V(\lambda)+5V'^2(\lambda)}{16V(\lambda)^2} \tag{18}$$

We begin the analysis of the integral equation by studying the singularity of the forcing form $F(\lambda)d\lambda$.

Lemma 1. *The only singularities, on the compact Riemann surface $\widehat{\Gamma}$, of the form $F(\lambda)d\lambda$ are the zeros of V, namely the branching points.*

Proof. The form is manifestly regular outside $\lambda = 0,\infty$ and $V(\lambda) = 0$: in a neighbourhood of ∞, $\rho(\lambda) = O(\lambda^{-3})$; in a neighbourhood of 0, $\rho(\lambda) = O(1)$; while at a zero of V, ρ is is not integrable. □

Remark 2. *We notice that formula (17) is the so-called Langer modified WKB approximation. In fact, the standard WKB approximation is obtained by studying approximate solutions of the form $\widetilde{J}(\lambda,\lambda') = e^{-\widetilde{S}(\lambda,\lambda')}$ with $\widetilde{S}(\lambda,\lambda') = \int_{\lambda'}^{\lambda}\sqrt{Q(\mu)}d\mu$, $Q = E^2V + r$. This choice leads to an integral equation with the forcing term*

$$\widetilde{F}(\lambda)d\lambda = \frac{-4Q''(\lambda)Q(\lambda)+5Q'^2(\lambda)}{16Q^{\frac{5}{2}}(\lambda)}d\lambda\,.$$

The latter 1-form coincides asymptotically with $E^{-1}F(\lambda)d\lambda$ in all of $\widehat{\Gamma}$ but for a neighbourhood of the Fuchsian singularity $\lambda = 0$, where $F(\lambda)d\lambda$ is integrable while $\widetilde{F}(\lambda)d\lambda$ is not. For such a reason, the standard WKB approximation fails at that point, while the Langer modified WKB provides the correct result.

Definition 4. *Let λ_0 be a regular point of $F(\lambda)d\lambda$. We say that $\gamma : [0,1] \to \overline{\mathbb{C}}, \gamma(0) = \lambda_0$ is an admissible curve if γ is a rectifiable curve such that $\Re S(\gamma(t),\lambda_0)$ is monotone not increasing on $[0,1]$.*

Proposition 1 (Analytic continuation and WKB approximation)**.** *If γ is admissible, then for large enough E there exists a unique solution $\psi(\lambda,\lambda_0,\lambda')$ of (16) such that*

$$\left|\frac{\psi(\gamma(t),\lambda_0,\lambda')}{J(\gamma(t),\lambda')} - 1\right| \le e^{\frac{\rho(\gamma(t))}{E}} - 1 \tag{19}$$

with

$$\rho(\gamma(t)) = \int_0^t |F(s)\dot{\gamma}(s)|ds \ . \tag{20}$$

Proof. Details can be found in [19]. The integral equation (18) is of the form $z = 1 + E^{-1}B[1]$, where B is a continuous linear (integral) operator on the space $L^\infty([0,1])$, and 1 the constant function 1. We construct its unique solution by means of the Neumann series $z = \sum_{l=0}^{\infty} E^{-l}B^l[1]$ where $B^l[1]$ is the l-th power (or iterate) of B applied to the constant function 1.

We want to estimate the operator norm $\|B^l\|_\infty$ of K^l. By definition of admissible curve we have that $|B(\lambda,\lambda')| \leq 1$, from which it follows that $\|B\|_\infty \leq \rho(\gamma(1))$. Since by iterating B l-times we are integrating on a simplex of dimension l, which has volume $\frac{1}{l!}$ and thus $\|B^l\|_\infty \leq \frac{\|B\|_\infty^l}{l!} = \frac{\rho^l(\gamma(1))}{l!}$. The thesis follows from the latter estimate. □

In the case of the potential $V = V(\lambda,\alpha,\beta,\nu)$, we have that $S(\lambda,\lambda') = \pm E\frac{\lambda^2}{2} + O(\lambda), E > 0$ as $\lambda \to \infty$. It follows that the lines of steepest descent (or ascent) for the function $S(\lambda,\lambda')$ are asymptotic to one of the following four rays, the rays with argument $0, \frac{\pi}{2}, -\frac{\pi}{2}$, or π. This corresponds to the fact that there are four Stokes sectors, each with a unique subdominant solution. Alternatively, we can think of the complex plane with four adjoined points at infinity, namely $\pm\infty, \pm i\infty$, and of the unique solution subdominant at one of each adjoined points. In order to solve the inverse monodromy problem characterising the roots of the generalised Hermite polynomials we have to understand the linear relations among these four solutions, and among them and the solution subdominant at $\lambda = 0$. Such linear relations can be computed in WKB approximation and the error can be estimated by means of a number of those error terms ρ's described in the previous proposition.

Definition 5. *Let us consider the complex plane cut along the negative real semi-axis. Denote by $\chi_+, \psi_0, \psi_{\pm 1}$ the unique (up to a normalising constant) solutions of the anharmonic oscillator such that $\lim_{\lambda\to 0}\chi_+(\lambda) = \lim_{\lambda\to+\infty}\psi_0(\lambda) = \lim_{\lambda\to\pm i\infty}\psi_{\pm 1}(\lambda) = 0$.*

We define $\Gamma_\pm$ as the set of admissible paths such that $\lim_{t\to 0^+}\gamma(t) = 0, \lim_{t\to 1^-}\gamma(t) = \pm i\infty$, and Γ_L, Γ_R the set of admissible paths such that $\lim_{t\to 0^+}\gamma(t) = -\infty, \lim_{t\to 1^-}\gamma(t) = i\infty$ passing $\lambda = 0$ respectively from the left or from the right (namely $\int_0^1 d\arg\gamma(t) = \pm\pi$ provided that γ belongs to γ_R or γ_L). Finally, we let

$$\rho_\pm = \inf_{\gamma\in\Gamma_\pm} \rho(\gamma(1)) \ ,$$

and

$$\rho_{L,R} = \inf_{\gamma\in\Gamma_{L,R}} \rho(\gamma(1)) \ .$$

Remark 3. *In principle one can define the minimal errors $\rho(\infty,\pm i\infty)$ and $\rho(0,\infty)$ of all admissible paths joining ∞ and $\pm i\infty$, and of all admissible paths joining 0 and $+\infty$. However this is unnecessary for two opposite reasons: The error $\rho(\infty,\pm i\infty) = 0$ because these asymptotic directions belong to neighbouring Stokes sectors, see [19]; The error $\rho(\infty,\pm i\infty) := \infty$ because, due to the topology of the Stokes complex, there are no admissible paths connecting 0 and ∞ if $(\alpha,\beta) \in K$.*

Definition 6. *Fixed ν, we notice that the errors $\rho_\pm$, and ρ_L, ρ_R are functions of α,β. We say that a compact subset $D \subseteq \mathbb{C}^2 \ni (\alpha,\beta)$ has the W property, if the above functions (i.e. the ρ's) are bounded. We often denote such domains by D_W.*

By construction, any compact subset K' of R has the W property. By lower semi-continuity, any such K' admits an epsilon neighbourhood with the W property.

We now prove the WKB estimate for the eigenvalue conditions and for the trivial-monodromy condition. For sake of clarity, we analyse them in three distinct theorems (the first about the quantisation condition, the other two about the trivial monodromy condition).

Theorem 3. *Fix $\nu \in (0, \frac{1}{3}]$, not necessarily rational, let α, β belong to a domain D on which $\rho_R, \rho_\pm$ are uniformly bounded.*

On the cut plane $\mathbb{C} \setminus \mathbb{R}_-$, let the solutions χ_+, ψ_0 of equation (4) be uniquely determined up to multiplicative pre-factors as the solutions subdominant at $0, +\infty$. Denote their Wronskian by $Wr[\chi_+, \psi_0]$.

Then there exist $C_0, E_0 > 0$ such that, for all the $E \geq E_0$ the following estimate holds (after a suitable normalisation of χ_+, ψ_0):

$$\left|(Wr[\chi_+, \psi_0] + 1)e^{E \oint_{\gamma_1} \sqrt{V}} + 1\right| \leq \frac{C_0}{E} \qquad \forall(\alpha, \beta) \in D. \tag{21}$$

Theorem 4. *Let D_W a domain with the W property. Fix $\nu \in (0, \frac{1}{3}]$ rational and restrict E to integer values such that $E\nu = n \in \mathbb{N}$.*

On the plane cut along the positive imaginary semi-axis, let ψ_1^L, ψ_1^R be the solutions of equation (4) uniquely determined up to multiplicative pre-factors as the solutions subdominant at $+i\infty$ respectively to the left and to the right of the branch cut. Denote their Wronskian by $Wr[\psi_1^R, \psi_1^L]$.

Then there exist $C_0, E_0 > 0$ such that, for all $E \geq E_0$ the following hold true:

(1) the monodromy of equation (4) is trivial if and only if $Wr[\psi_1^R, \psi_1^L] = 0$;

(2) we have the estimate (after suitable normalisation of $\psi_1^{R,L}$)

$$\left|(Wr[\psi_1^R, \psi_1^L] + 1)e^{-E \oint_{\gamma_2} \sqrt{V} + i\pi n} + 1\right| \leq \frac{C_0}{E} \qquad \forall(\alpha, \beta) \in D_W. \tag{22}$$

Below we compute the full monodromy matrix in WKB approximation, even though the above theorem suffices to compute asymptotically the roots of the generalised Hermite polynomials. We do provide the complete monodromy matrix because, to our knowledge, this has not appeared in the scientific literature at the time of writing.

Theorem 5. *Fix $\nu \in (0, \frac{1}{3}]$, not necessarily rational, and a domain D_W with the W property.*

On the plane cut along the negative real semi-axis, let $\psi_{\pm 1}$ be the solutions of equation (4) uniquely determined up to multiplicative pre-factors as the solutions subdominant at $\pm i\infty$.

Then there exist $C_0, E_0 > 0$ such that for all $E \geq E_0$ the following hold true:

(1) the pair $\{\psi_1, \psi_{-1}\}$ is a basis of solutions;

(2) letting $\mathcal{M}$ denote the monodromy of equation (4) with respect to the basis $\{\psi_1, \psi_{-1}\}$, we have

$$\|\mathcal{M} \cdot \mathcal{M}_W^{-1} - \mathbb{I}_2\| \leq \frac{C_0}{E} \qquad \forall(\alpha, \beta) \in D_W, \text{ with}$$

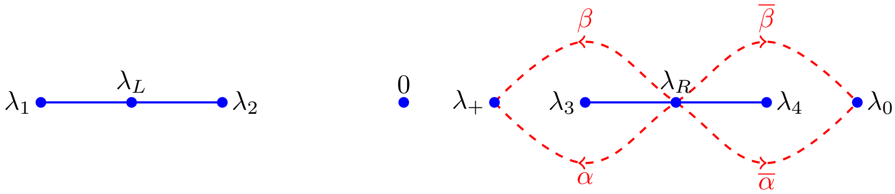
$$\mathcal{M}_W = \begin{pmatrix} -2\cos\pi n - e^{E \oint_{\gamma_2} \sqrt{V}} & \kappa(\alpha, \beta)\left(1 + e^{i\pi n} e^{E \oint_{\gamma_2} \sqrt{V}}\right) \\ -\kappa^{-1}(\alpha, \beta)\left(1 + e^{-i\pi n} e^{E \oint_{\gamma_2} \sqrt{V}}\right) & e^{E \oint_{\gamma_2} \sqrt{V}} \end{pmatrix}, \quad \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{23}$$

where $k(\alpha, \beta)$ is a non-zero (immaterial) normalising constant that can be chosen in such a way that the above estimate hold.

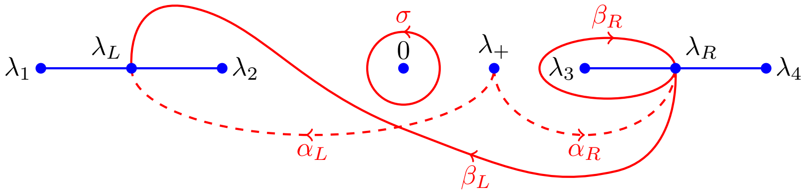
Remark 4. *After the estimates (21,22), in WKB approximation the quantisation and monodromy conditions read:*

$$E \oint_{\gamma_1} \sqrt{V} = (2k + 1)\pi, k \in \mathbb{Z} \tag{24}$$

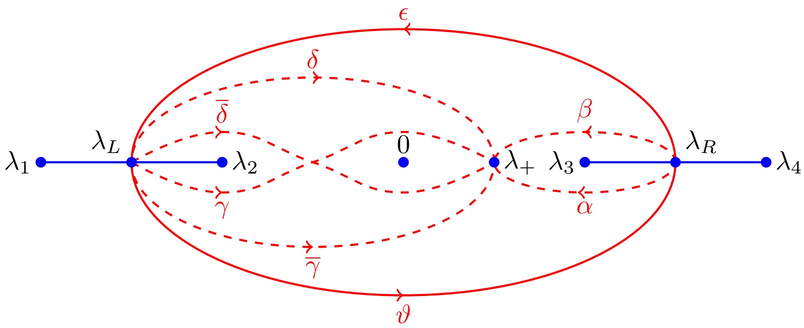
$$E \oint_{\gamma_2} \sqrt{V} = 2l\pi, l \in \mathbb{Z} \text{ if } n \text{ odd}, \ l \in \mathbb{Z} + \frac{1}{2} \text{ if } n \text{ even} \tag{25}$$



(A) The paths used in the proof of Theorem 3.



(B) The paths used in the proof of Theorem 4.



(C) The paths used in the proof of Theorem 5.

FIGURE 6. The paths used to prove the main theorems in this section.

Notice that in this approximation the quantisation and monodromy conditions are on an equal footing despite of their different definition. This is however consistent with the following fact: that there is a dual inverse monodromy problem which interchanges the roles of m, n, see [21].

Finally notice that the trace and determinant of the monodromy matrix $\mathcal{M}$, are exact in the WKB approximation, as indeed the trace and the determinant of $\mathcal{M}_W$ are, respectively, $-2\cos\pi n$ and 1.

Proof of Theorem 3, Theorem 4, and Theorem 5. We choose points $\lambda_+, \lambda_L, \lambda_R, \lambda_0$ as in Figures 6.

Proof of Theorem 3. We normalise the solutions $\chi_+, \psi_{\pm 1}, \psi_0$, subdominant at $\lambda = 0, \lambda = \pm i\infty, \lambda = \infty$, by the requirements $\chi_+(\lambda_+) = \psi_{\pm 1}(\lambda_R) = \psi_0(\lambda_0) = 1$.

Since $\rho_R < \infty$, the solutions $\{\psi_1, \psi_{-1}\}$ form a basis. In fact, by Proposition 1,

$$\lim_{\lambda \to -i\infty} \left| \psi_1(\lambda) e^{\int_{\lambda_R}^{\lambda} \sqrt{V(\mu)} + \frac{V'(\mu)}{V(\mu)}} - 1 \right| \leq E^{-1} \rho_R \to 0\,,$$

where $\sqrt{V}$ is chosen in such a way that $\lim_{\lambda \to \pm i\infty} \Re \int_{\lambda_R}^{\lambda} \sqrt{V(\mu)} = \pm\infty$. Hence ψ_1 is eventually dominant at $-i\infty$, while ψ_{-1} is by definition subdominant at $-i\infty$.

We write

$$\chi_+ = A\psi_1 + B\psi_{-1}\,, \quad \psi_0 = \overline{A}\psi_1 + \overline{B}\psi_{-1}\,, \tag{26}$$

so that

$$Wr[\chi_+,\psi_0] = Wr[\psi_1,\psi_{-1}]B\overline{A}\big(\frac{A\overline{B}}{B\overline{A}} - 1\big)\,. \tag{27}$$

To compute $A, B, \overline{A}, \overline{B}$ in the WKB approximation, we first notice that

$$A = \lim_{\lambda\to -i\infty}\frac{\chi_+(\lambda)}{\psi_1(\lambda)},\ \overline{A} = \lim_{\lambda\to -i\infty}\frac{\psi_0(\lambda)}{\psi_1(\lambda)},\ B = \lim_{\lambda\to i\infty}\frac{\chi_+(\lambda)}{\psi_{-1}(\lambda)},\ \overline{B} = \lim_{\lambda\to i\infty}\frac{\psi_0(\lambda)}{\psi_{-1}(\lambda)}.$$

The above limits can be computed using the WKB approximations of $\chi_+, \psi_{\pm 1}, \psi_0$ via Proposition 1 - see [19] for a guided introduction to these kind of computations:

$$\begin{aligned}
&|Ae^{-\int_\alpha\left(\sqrt{V}+\frac{V'}{4V}\right)} - 1| \leq E^{-1}(\rho_- + \rho_R),\\
&|\overline{A}e^{-\int_{\overline{\alpha}}\left(\sqrt{V}+\frac{V'}{4V}\right)} - 1| \leq E^{-1}\rho_R,\\
&|Be^{-\int_\beta\left(\sqrt{V}+\frac{V'}{4V}\right)} - 1| \leq E^{-1}(\rho_+ + \rho_R),\\
&|\overline{B}e^{-\int_{\overline{\beta}}\left(\sqrt{V}+\frac{V'}{4V}\right)} - 1| \leq E^{-1}\rho_R.
\end{aligned} \tag{28}$$

Here the paths $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ are chosen as in the Figure 6a. We notice that $\oint_{\alpha-\beta+\overline{\beta}-\overline{\alpha}}\sqrt{V} = -\oint_{\gamma_1}\sqrt{V}$ and $\oint_{\alpha-\beta+\overline{\beta}-\overline{\alpha}}\frac{V'}{V} = \oint_{\gamma_1}\frac{V'}{V}$. Combining formula (27) with the estimates (28), and latter homological computations, we obtain the thesis.

We now prove Theorem 4. We let $\chi_+, \psi_{-1}, \psi_1^R, \psi_1^L$ be the solutions subdominant at $0, -i\infty, +i\infty$ to the right (left) of the cut, normalised by the requirements $\chi_+(\lambda+) = \psi_{-1}(\lambda_R) = \psi_1^R(\lambda_R) = \psi_1^L(\lambda_L) = 1$. We begin by proving part (1). It follows from $\rho(0, i\infty) < \infty$ that $\psi_1^{R,L}$ and χ_+ are linearly independent if E is big enough (by reasoning similarly as in the beginning of the proof of Theorem 3). It follows that $\psi_1^R(e^{2i\pi}\lambda) = (-1)^{n+1}\psi_1^R(e^{2i\pi}\lambda)$ if and only if $\lambda = 0$ is an apparent singularity. Moreover $\psi_1^R(e^{2i\pi}\lambda) = (-1)^{n+1}\psi_1^R(e^{2i\pi}\lambda)$ if and only if ψ_1^R is subdominant at $+i\infty$ to the left of the branch-cut, which is equivalent to $Wr[\psi_1^R, \psi_1^L] = 0$.

We now prove part (2). We write

$$\psi_1^{R,L} = A^{R,L}\chi_+ + B^{R,L}\psi_{-1} \tag{29}$$

so that

$$Wr[\psi_+^R, \psi_1^L] = Wr[\chi_+, \psi_{-1}]A^L B^R\big(\frac{A^R B^L}{A^L B^R} - 1\big)\,. \tag{30}$$

To compute $A^{R,L}, B^{R,L}$ in the WKB approximation, we reason as above. Noticing that

$$A^{R,L} = \lim_{\lambda\to -i\infty}\frac{\psi^{R,L}(\lambda)}{\chi_+(\lambda)},\ B^{R,L} = \lim_{\lambda\to 0}\frac{\psi^{R,L}(\lambda)}{\psi_{-1}(\lambda)},$$

we obtain, via Proposition 1, the WKB estimates

$$\begin{aligned}
&|A^{R,L}e^{-\int_{\alpha_{R,L}}\left(\sqrt{V}+\frac{V'}{4V}\right)} - 1| \leq E^{-1}(\rho_- + \rho_{R,L}),\\
&|B^{R,L}e^{-\int_{\beta_{R,L}}\left(\sqrt{V}+\frac{V'}{4V}\right)} - 1| \leq E^{-1}(\rho_+ + \rho_-).
\end{aligned} \tag{31}$$

Here the paths $\alpha_R, \alpha_L, \beta_L, \beta_R, \sigma$ are as in Figure 6b. Combining equations (30) and (31), and noticing that $\alpha_R - \alpha_L + \beta_L - \beta_R = \gamma_2 - \sigma$, we obtain the thesis.

We now prove Theorem 5. To this aim we fix the following paths $\alpha, \overline{\alpha}, \gamma, \overline{\gamma}, \delta, \overline{\delta}, \varepsilon, \vartheta$ on $\widehat{\Gamma}$ as in Figure 6c. In the plane cut along the *positive* real semi axis, we define $\psi_{\pm 1}^L$ as the solution subdominant at $\pm i\infty$ with the normalisation $\psi_{\pm 1}^L(\lambda_L) = 1$. We use this alternative pair of solutions to find a convenient LU factorisation of the monodromy matrix:

(1) We denote by $\mathcal{M}_U$ the matrix of change of basis among the solutions $\psi^R_{\pm 1}$ and $\psi^L_{\pm 1}$, $(\psi^L_1, \psi^L_{-1})^T = \mathcal{M}_U(\psi^R_1, \psi^R_{-1})^T$, when $\psi^R_{\pm 1}$ are analytically continued to λ_L along an arc passing above $\lambda = 0$.
(2) We denote by $\mathcal{M}_D$ the matrix of change of basis, $(\psi^R_1, \psi^R_{-1})^T = \mathcal{M}_D(\psi^L_1, \psi^L_{-1})^T$, when $\psi^L_{\pm 1}$ are analytically continued to λ_R along an arc passing below $\lambda = 0$.
(3) We thus have $\mathcal{M} = \mathcal{M}_D \cdot \mathcal{M}_U$, where, as it turns out, $\mathcal{M}_U$ is lower triangular while $\mathcal{M}_D$ is upper triangular.

We analyse M_U first. We notice that $\mathcal{M}_U$ is equivalently defined as the matrix of change of basis among $\psi^R_{\pm 1}$ and $\psi^L_{\pm 1}$, when we work on the complex plane cut along the negative imaginary semi axis, and we define ψ^R_{-1} as the solution subdominant at $-i\infty$ right to the cut (and such that $\psi^R_{-1}(\lambda_R) = 1)$), and ψ^L_{-1} as the solution subdominant at $-i\infty$ left to the cut (and such that $\psi^L_{-1}(\lambda_L) = 1)$). With such a geometry, the solutions ψ^L_1 and ψ^R_1 coincide but for a normalising factor, which can be easily computed by WKB approximation:

$$\psi^L_1 = F\psi^R_1 \,, \quad |Fe^{-\int_\varepsilon \sqrt{V} + \frac{V'}{4V}} - 1| \leq \frac{\rho_L + \rho_R}{E} \,. \tag{32}$$

On the contrary, ψ^L_{-1}, ψ^R_{-1} are not in general proportional. To overcome such a difficulty, we utilise the solution χ_+. Writing

$$\chi_+ = A\psi^R_1 + B\psi^R_{-1} = C\psi^L_1 + D\psi^L_{-1} \tag{33}$$

we have that

$$\mathcal{M}_U = \begin{pmatrix} F & 0 \\ D^{-1}(A - CF) & D^{-1}B \end{pmatrix} \tag{34}$$

The WKB approximation of the terms A, B was computed in (28) above. The same sort of computations show that

$$\begin{aligned} &|Ce^{-\int_\gamma \left(\sqrt{V} + \frac{V'}{4V}\right)} - 1| \leq E^{-1}\big(\rho_- + \rho_L\big), \\ &|De^{-\int_\delta \left(\sqrt{V} + \frac{V'}{4V}\right)} - 1| \leq E^{-1}\big(\rho_+ + \rho_L\big). \end{aligned} \tag{35}$$

We can repeat the very same arguments to compute $\mathcal{M}_D$, after having introduced a branch-cut on the positive imaginary semi-axis. We have that

$$\psi^R_{-1} = T\psi^L_{-1} \,, \quad |Te^{-\int_\theta \sqrt{V} + \frac{V'}{4V}} - 1| \leq \frac{\rho_L + \rho_R}{E} \,. \tag{36}$$

Defining

$$\chi_+ = \overline{C}\psi^L_1 + \overline{D}\psi^L_{-1} \,, \tag{37}$$

we have

$$\mathcal{M}_D = \begin{pmatrix} A^{-1}\overline{C} & A^{-1}(\overline{D} - BT) \\ 0 & T \end{pmatrix} \tag{38}$$

where WKB approximations of $\overline{C}, \overline{D}$ are given by

$$\begin{aligned} &|\overline{C}e^{-\oint_{\overline{\gamma}} \left(\sqrt{V} + \frac{V'}{4V}\right)} - 1| \leq E^{-1}\big(\rho_- + \rho_L\big), \\ &|\overline{D}e^{-\oint_{\overline{\delta}} \left(\sqrt{V} + \frac{V'}{4V}\right)} - 1| \leq E^{-1}\big(\rho_+ + \rho_L\big). \end{aligned} \tag{39}$$

Using (34,38) we obtain

$$\mathcal{M} = \mathcal{M}_D \cdot \mathcal{M}_U = \begin{pmatrix} A^{-1}\overline{C}F + D^{-1}(A - CF) & A^{-1}(\overline{D} - BT)D^{-1}B \\ D^{-1}(A - CF)T & D^{-1}BT \end{pmatrix} \tag{40}$$

Substituting $A, B, C, D, \overline{C}, \overline{D}, F, T$ with their WKB approximations (28,35,39,32,36), and using some elementary homological calculations (e.g. $\epsilon + \bar{\gamma} - \alpha = \gamma_2$), we obtain that the WKB approximation $\mathcal{M}_W$ of $\mathcal{M}$ is

$$\mathcal{M}_W = \begin{pmatrix} -2\cos \pi n - \exp E \oint_{\gamma_2} \sqrt{V} & \kappa(1 + e^{i\pi n} \exp E \oint_{\gamma_2} \sqrt{V}) \\ -\kappa^{-1}(1 + e^{-i\pi n} \exp E \oint_{\gamma_2} \sqrt{V}) & \exp E \oint_{\gamma_2} \sqrt{V} \end{pmatrix} \tag{41}$$

where $\kappa = -e^{\int_{\alpha-\beta} \sqrt{V} + \frac{V'}{4V}} e^{-i\pi n}$.

Finally the estimate $\|\mathcal{M} \cdot \mathcal{M}_W^{-1} - \mathbb{I}_2\| \leq \frac{C_0}{E}$ can be proven by using the factorisation $\mathcal{M} = \mathcal{M}_D \mathcal{M}_U$. Indeed, if we let $\mathcal{M}_{U(W)}, \mathcal{M}_{D(W)}$ denote the WKB approximation of $\mathcal{M}_{U,D}$ when which are obtained substituting $A, B, C, D, \overline{C}, \overline{D}, F, T$ with their WKB approximations, then it is straightforward to show that $\|\mathcal{M}_D \cdot \mathcal{M}_{D(W)}^{-1} - \mathbb{I}_2\| \leq \frac{C_0}{E}$ and $\|\mathcal{M}_U \cdot \mathcal{M}_{U(W)}^{-1} - \mathbb{I}_2\| \leq \frac{C_0}{E}$. □

4. The Elliptic Region

This section is devoted to the study of the regions R, K and K_a, defined in Definition 1, as well as the projection $\Pi_a : K \to K_a$ and mapping $\mathcal{S} : K \to Q$ introduced in equation (9), which we reintroduce here. For convenience of the reader we have collected some explicit formulas (that we will need in the sequel) for complete elliptic integrals and their Jacobian in the Appendix A.

The regions R, K and K_a are defined entirely in terms of the Stokes geometry of the potential

$$V(\lambda; \alpha, \beta) = \lambda^2 + 2\alpha\lambda + \alpha^2 - 1 - \beta\lambda^{-1} + \frac{\nu^2}{4}\lambda^{-2}. \tag{42}$$

We therefore quickly review some basic notions of WKB theory concerning Stokes complexes, see [14, 24, 18] for more details.

Firstly we recall the notion of turning points.

Definition 7. *A zero of multiplicity n of* (42) *is called a turning point of degree n of the potential, for $n \geq 1$. The turning points and $\lambda = 0$ are called the critical points of the potential. Other points are called generic.*

Let λ_0 be a generic point and consider, for any choice of sign, the so called action integral

$$S(\lambda_0, \lambda) = \int_{\lambda_0}^{\lambda} \sqrt{V(\mu; \alpha, \beta)} d\mu$$

defined on the universal covering of the λ-plane minus critical points. Let i_{λ_0} be the level curve $\Re S(\lambda_0, \lambda) = 0$ through λ_0 on the universal covering and consider its projection $\widetilde{i}_{\lambda_0}$ onto the λ plane minus critical points. Clearly $\widetilde{i}_{\lambda_0}$ cannot self-intersect and is hence either diffeomorphic to a circle or a line. We call $\widetilde{i}_{\lambda_0}$ the level curve through λ_0.

Note that different level curves cannot intersect and thus the set of level curves forms a complete partition of the λ-plane minus critical points. We have the following important dichotomy.

Lemma 2. *Let λ_0 be a generic point of the potential* (42) *and $\widetilde{i}_{\lambda_0}$ its corresponding level curve. If $\widetilde{i}_{\lambda_0}$ is diffeomorphic to a circle, then it is homotopic to a simple encircling of $\lambda = 0$ in the λ-plane minus critical points. Otherwise $\widetilde{i}_{\lambda_0}$ is diffeomorphic to a line. Let $x \mapsto \gamma_{\lambda_0}(x)$ be a diffeomorphism of $\mathbb{R}$ onto $\widetilde{i}_{\lambda_0}$, then, for $\epsilon \in \{\pm 1\}$, we have the following dichotomy: either*

- *(i)* $\lim_{x \to \epsilon\infty} = \infty$ *and the curve is asymptotic to one of the four rays $e^{\frac{1}{4}(2k-1)\pi i}\mathbb{R}_+$, $k \in \mathbb{Z}_4$, in the λ-plane.*
- *(ii) or* $\lim_{x \to \epsilon\infty} = \lambda_*$ *with λ_* a turning point of the potential.*

Proof. See Strebel [24]. □

In alignment with the above lemma, we make the following definition.

Definition 8. *Considering the dichotomy in Lemma 2, when $\widetilde{i}_{\lambda_0}$ is diffeomorphic to a line, we call, in case*

(i) the asymptotic direction $\infty_k := e^{\frac{1}{4}(2k-1)\pi i}\infty$ an endpoint of $\widetilde{i}_{\lambda_0}$,

(ii) the corresponding turning point λ_* an endpoint of $\widetilde{i}_{\lambda_0}$.

We call $\widetilde{i}_{\lambda_0}$ a Stokes line if at least one endpoint is a turning point.

We define $\mathbb{C}_\infty$ as the complex plane with the addition of four marked points at infinity $\infty_k := e^{\frac{1}{4}(2k-1)\pi i}\infty$, $1 \leq k \leq 4$, with the unique topology making the following map a homeomorphism,

$$\begin{aligned} &L : \mathbb{C}_\infty \to \mathbb{D} \cup \{e^{\frac{1}{4}(2k-1)\pi i} : 1 \leq k \leq 4\}, \\ &L(\rho e^{i\phi}) = \frac{2}{\pi}\arctan(\rho)e^{i\phi} \quad (\rho \in \mathbb{R}_{\geq 0}, \phi \in \mathbb{R}), \\ &L(\infty_k) = e^{\frac{1}{4}(2k-1)\pi i} \quad (1 \leq k \leq 4), \end{aligned}$$

where $\mathbb{D}$ denotes the open unit disc. We denote $\mathbb{C}_\infty^* = \mathbb{C}_\infty \setminus \{0\}$.

Definition 9. *The Stokes complex $\mathcal{C} = \mathcal{C}(\alpha, \beta) \subseteq \mathbb{C}_\infty^*$ of the potential (42) is the union of the marked points at infinity and the Stokes lines and turning points of the potential. The corresponding internal Stokes complex is defined as the union of the turning points and those Stokes lines with only turning points as endpoints.*
We call two Stokes complexes isomorphic if there exists a topological homeomorphism between them which respects the markings. Such an isomorphism always has an extension to an automorphism of $\mathbb{C}_\infty^*$.

From a topological point of view the Stokes complex $\mathcal{C}$ is an embedded graph into $\mathbb{C}_\infty^*$ with vertices equal to the turning points and the four points at infinity, with edges given by the Stokes lines.

In the following proposition we summarise some basic facts concerning the Stokes complexes under consideration.

Proposition 2. *Let $\mathcal{C}$ be the Stokes complex of the potential (42), then*

- *For any turning point λ^*, say of degree n, there are precisely $n+2$ Stokes lines emanating from it, counting Stokes lines with all end points equal to λ_* double.*
- *$\mathcal{C}$ is connected;*
- *The internal Stokes complex contains precisely one Jordan curve, the interior of which contains $\lambda = 0$;*
- *If two different Stokes lines have the same endpoints, then they are homotopically inequivalent in $\mathbb{C}_\infty^*$.*

Proof. See [14, 24]. □

As an example, let us consider the Stokes complex corresponding to the potential (42) with $(\alpha, \beta) = (0, 0)$. In Figure 7 its depicted as an embedded graph in $\mathbb{C}_\infty^*$, and in Figure 2 it is depicted in $\mathbb{C}$, with real turning points

$$\begin{aligned} \lambda_1 &= -\sqrt{\tfrac{1}{2}(1 + \sqrt{1 - \nu^2})}, \quad \lambda_2 = -\sqrt{\tfrac{1}{2}(1 - \sqrt{1 - \nu^2})}, \\ \lambda_3 &= +\sqrt{\tfrac{1}{2}(1 - \sqrt{1 - \nu^2})}, \quad \lambda_4 = +\sqrt{\tfrac{1}{2}(1 + \sqrt{1 - \nu^2})}. \end{aligned} \tag{43}$$

For convenience of the reader, we reintroduce the regions R, K and K_a in the following definition.

Definition 10. *We denote by $R \subseteq \mathbb{C}^2$ the set of all $(\alpha, \beta) \in \mathbb{C}^2$ for which the corresponding Stokes complex $\mathcal{C}(\alpha, \beta)$ is isomorphic to $\mathcal{C}(0, 0)$. Furthermore, we define*

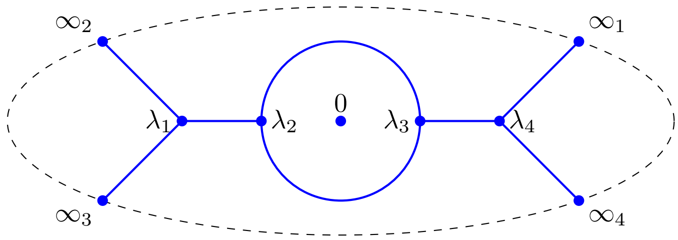


FIGURE 7. The Stokes complex $\mathcal{C}(0,0)$ as an embedded graph in $\mathbb{C}^*_\infty$.

- *K as the closure of R;*
- *R_a and R_b as the projection of R onto respectively the α-plane and β-plane, with corresponding projections Π_a and Π_b;*
- *K_a and K_b as the projection of K onto respectively the α-plane and β-plane.*

We call K_a the elliptic region.

We proceed in discussing the mapping $\mathcal{S}$ introduced in (9). Strictly speaking, the formulas defining $\mathcal{S}$ are only unambiguously defined on the region R and it will require a bit work to show that the formulae have a well-defined continuous extension to the closure $K = \overline{R}$. Firstly, note that s_1 and s_2 are uniquely defined by (10) for $(\alpha, \beta) \in R$ and indeed, due the the Stokes geometry of the potential on R,

$$\Re s_1 = 0, \quad \Re s_2 = 0. \tag{44}$$

We define the mapping

$$\mathcal{S} : R \to \mathbb{R}^2, (\alpha, \beta) \mapsto (-is_1(\alpha, \beta), -is_2(\alpha, \beta)), \tag{45}$$

and notice that set R is invariant under negation and complex conjugation, and that $(0,0) \in R$.

Lemma 3. *The set R is invariant under complex conjugation $(\alpha, \beta) \mapsto (\overline{\alpha}, \overline{\beta})$ and reflection $(\alpha, \beta) \mapsto (-\alpha, -\beta)$. Furthermore, for $(\alpha, \beta) \in R$,*

$$\mathcal{S}(\overline{\alpha}, \overline{\beta}) = (\mathcal{S}_1(\alpha, \beta), -\mathcal{S}_2(\alpha, \beta)),$$
$$\mathcal{S}(-\alpha, -\beta) = (-\mathcal{S}_1(\alpha, \beta), -\mathcal{S}_2(\alpha, \beta)),$$

and in particular $\mathcal{S}(0,0) = (0,0)$.

Proof. This follows from the symmetries

$$V(\lambda; \overline{\alpha}, \overline{\beta}) = \overline{V(\overline{\lambda}; \alpha, \beta)}, \quad V(\lambda; -\alpha, -\beta) = V(-\lambda; \alpha, \beta). \qquad \square$$

The remainder of the present section consists of two subsections. In Subsection 4.1 we study the sets R, K and the map $\mathcal{S}$. In particular we prove the following

Proposition 3. *The set R is a smooth 2-dimensional regular submanifold of $\{(\Re\alpha, \Im\alpha, \Re\beta, \Im\beta) \in \mathbb{R}^4\}$. The mapping $\mathcal{S}$ has a unique continuous extension $\mathcal{S} : K \to Q$ which is a homeomorphism and maps R diffeomorphically onto Q°.*

In Subsection 4.2 we prove the following

Proposition 4. *The projection $\Pi_a : K \to K_a$ is a homeomorphism which maps R diffeomorphically onto R_a.*

Moreover we study the domain $\Pi_a(K) = K_a \subset \mathbb{C}$: we prove that it is a quadrilateral domain and we prove the explicit description of its boundary as given in Section 2.

4.1. K **and the map** $\mathcal{S}$**.** We start this subsection by estimating of the range of $\mathcal{S}$.

Lemma 4. *The image of R under $\mathcal{S}$ is contained in the open rectangle*

$$Q^{\circ} := \left(-\tfrac{1}{2}(1-\nu)\pi, +\tfrac{1}{2}(1-\nu)\pi\right) \times (-\nu\pi, +\nu\pi)\,.$$

Proof. The meromorphic differential ω has four poles, $0_{\pm} := (\lambda = 0, y = \pm\frac{\nu}{2})$ and $\infty_{\pm}$, and the corresponding residues are easily computed,

$$\operatorname*{Res}_{P=0_{\pm}} \omega = \pm\frac{\nu}{2}, \quad \operatorname*{Res}_{P=\infty_{\pm}} \omega = \mp\frac{1}{2}\,. \tag{46}$$

Consider the oriented contours δ_i, $1 \le i \le 6$, defined as in Figure 8, where the two blue lines are the Stokes line between λ_1 and λ_2 and the Stokes line between λ_3 and λ_4 acting as branch cuts, such that all the contours lie in the same sheet of the elliptic curve $\widehat{\Gamma}$ where $\omega \sim \frac{1}{2}\nu\lambda^{-1}d\lambda$. We define

$$r_i = -i\int_{\delta_i} \omega \quad (1 \le i \le 6), \tag{47}$$

all of which are real and positive due to the Stokes geometry of the potential. Furthermore, $r_1 = r_6$ and $r_3 = r_4$.

Since 0_+ and ∞_- lie in the same sheet as the above contours, the residue theorem yields

$$r_2 + r_5 = 2\pi \operatorname*{Res}_{\lambda=0_+} \omega = \pi\nu,$$

$$r_1 + r_2 + r_3 + r_4 + r_5 + r_6 = 2\pi \operatorname*{Res}_{\lambda=\infty_-} \omega = \pi.$$

It follows in particular that

$$r_4 + r_6 = \tfrac{1}{2}(1-\nu)\pi. \tag{48}$$

Note that $s_2(\alpha, \beta) = (r_5 - r_2)i$ and since $r_2 + r_5 = \pi\nu$ with $r_2, r_5 > 0$, we have

$$-\pi\nu < -is_2(\alpha, \beta) < +\pi\nu.$$

Similarly $s_1(\alpha, \beta) = -2r_4 i + \frac{1}{2}(1-\nu)\pi i$ and it follows from equation (48) that $0 < r_4 < \frac{1}{2}(1-\nu)\pi$, so

$$-\tfrac{1}{2}(1-\nu)\pi < -is_1(\alpha, \beta) < +\tfrac{1}{2}(1-\nu)\pi,$$

which finishes the proof of the lemma. □

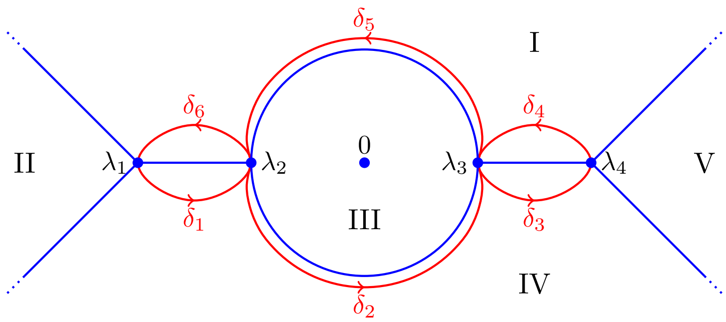


FIGURE 8. Schematic representation of contours used in the proof of Lemma 4 in red, with the Stokes complex in blue, and the regions $I, \ldots, V$ used in the proof of Lemma 6.

To proceed further with our analysis we introduce the notion of Boutroux curves [5, 19, 2, 3] tailored to our setting.

Definition 11. *We call the elliptic curve $\widehat{\Gamma}$ a Boutroux curve if $\Re \oint_\gamma \omega = 0$ for any closed cycle γ in $\widehat{\Gamma}$. We denote by Ω the set of all $(\alpha, \beta) \in \mathbb{C}^2$ such that $\widehat{\Gamma}(\alpha, \beta)$ is a Boutroux curve.*

By definition of the region R, we have $R \subseteq \Omega$. In proving Lemma 5 and Proposition 3, we have to deal with deformations of Boutroux curves for which we have collected the necessary results in Appendix B.

Firstly we focus our attention on Lemma 5. We make use of the factorisation

$$\lambda^2 V = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4),$$

where the turning points are defined, up to permutations, by the following algebraic constraints

$$\lambda_1\lambda_2\lambda_3\lambda_4 = \tfrac{1}{4}\nu^2, \tag{49a}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = \tfrac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) - 2, \tag{49b}$$

$$\tfrac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = -\alpha, \tag{49c}$$

$$\tfrac{1}{4}\nu^2(\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} + \lambda_4^{-1}) = \beta. \tag{49d}$$

When $(\alpha, \beta) \in R$, we may unambiguously, i.e. not just up to permutation, define the turning points via the Stokes complex and we will always do this in alignment with Figure 7.

Having introduced the notion of Boutroux curves we proceed with the proof of Proposition 3 - actually with a stronger version of it which we call Theorem 6 - together with an explicit description of $K \setminus R$, see Corollary 3.

Our analysis is divided in several preparatory lemmas. In Lemma 5 we show that R is a smooth submanifold of $\mathbb{C}^2$, in Lemma 6 we show that $\mathcal{S}$ is injective, in Lemma 7 we show that K is compact, in Lemmas 8,9 we show that $K \setminus R$ is made of singular elliptic curves, and eventually in Lemma 10 we prove that $\mathcal{S}$, as defined in R, admits a unique extension to K, which maps K to Q.

Lemma 5. *The set R is a smooth 2-dimensional regular submanifold of $\{(\Re\alpha, \Im\alpha, \Re\beta, \Im\beta) \in \mathbb{R}^4\}$ and the mapping $\mathcal{S}$ is locally diffeomorphic[2] on R. Furthermore, for any given $(\alpha^*, \beta^*) \in R$, there exist simply connected open sets $U \subseteq \mathbb{C}$ and $V \subseteq \mathbb{C}$, containing respectively α^* and β^*, and a diffeomorphism $B : U \to V$, such that*

$$R \cap (U \times V) = \{(\alpha, B(\alpha)) : \alpha \in U\}.$$

Proof. As $(\alpha^*, \beta^*) \in R$, we know that the turning points of $V(\lambda; \alpha^*, \beta^*)$ are all simple. Hence we can take small open discs $U_0 \subseteq \mathbb{C}$ and $V_0 \subseteq \mathbb{C}$, centred respectively at α^* and β^*, such that on $U_0 \times V_0$ the turning points of $V(\lambda; \alpha, \beta)$ do not coalesce. On this set the turning points $\lambda_i = \lambda_i(\alpha, \beta)$ are analytic in (α, β), $1 \leq i \leq 4$, and the defining equations (10) of $s_1(\alpha, \beta)$ and $s_2(\alpha, \beta)$ have unique analytic continuation to $U_0 \times V_0$, the result of which we denote by $s_1(\alpha, \beta)$ and $s_2(\alpha, \beta)$ as well. The idea of the proof is to make use of the fact that R is locally defined by equations (44).

Using the standard decomposition into real and imaginary part,

$$\alpha = \alpha_R + i\alpha_I, \quad \beta = \beta_R + i\beta_I, \quad s_i = s_i^R + i s_i^I \quad (i = 1, 2),$$

the analytic mapping

$$(s_1, s_2) : U_0 \times V_0 \to \mathbb{C}^2, (\alpha, \beta) \mapsto (s_1(\alpha, \beta), s_2(\alpha, \beta))$$

can be rewritten as a real smooth mapping

$$H : (\alpha_R, \alpha_I, \beta_R, \beta_I) \mapsto (s_1^R, s_1^I, s_2^R, s_2^I)$$

on an open environment of $(\alpha_R^*, \alpha_I^*, \beta_R^*, \beta_I^*)$.

[2] In this paper, diffeomorphic always means C^∞-diffeomorphic.

Due to equation (86), we know that (s_1, s_2) is locally biholomorphic near (α^*, β^*) and thus H is locally diffeomorphic near $(\alpha_R^*, \alpha_I^*, \beta_R^*, \beta_I^*)$. To prove the thesis, we proceed in showing that the mapping $(\beta_R, \beta_I) \mapsto (s_1^R, s_2^R)$ is a local diffeomorphism near (β_R^*, β_I^*), for $(\alpha_R, \alpha_I) = (\alpha_R^*, \alpha_I^*)$ fixed.

To this end we apply the implicit function theorem to show that $\{\Re\alpha, \Im\alpha\}$ are good local coordinates on the zero set

$$\Re s_1(\alpha, \beta) = 0, \quad \Re s_2(\alpha, \beta) = 0.$$

This requires that the Jacobian determinant

$$\delta = \delta(\alpha, \beta) = \begin{vmatrix} \frac{\partial s_1^R}{\partial \beta_R} & \frac{\partial s_1^R}{\partial \beta_I} \\ \frac{\partial s_2^R}{\partial \beta_R} & \frac{\partial s_2^R}{\partial \beta_I} \end{vmatrix}$$

does not vanish at $(\alpha, \beta) = (\alpha^*, \beta^*)$. By the Cauchy-Riemann equations, we have $\delta = \Im\left[\frac{\partial s_1}{\partial \beta} \cdot \overline{\frac{\partial s_2}{\partial \beta}}\right]$. Now, note that

$$\frac{\partial s_1}{\partial \beta} = -\frac{1}{2}\int_{\gamma_1} \widetilde{\omega}, \quad \frac{\partial s_2}{\partial \beta} = -\frac{1}{2}\int_{\gamma_2} \widetilde{\omega},$$

where $\widetilde{\omega}$ is the holomorphic differential form $\widetilde{\omega} = \frac{d\lambda}{y}$. As γ_1 and γ_2 are homologically independent, $(p_1, p_2) := (-2\frac{\partial s_1}{\partial \beta}, -2\frac{\partial s_2}{\partial \beta})$ form a pair of $\mathbb{R}$-linearly independent periods of the elliptic curve $\widehat{\Gamma}(\alpha, \beta)$. Therefore $p_1\overline{p}_2 \notin \mathbb{R}$ and thus $\delta(\alpha, \beta) \neq 0$ on $U_0 \times V_0$ and in particular at $(\alpha, \beta) = (\alpha^*, \beta^*)$. We note that this can also be proven using the explicit formulae in Appendix A.

By the implicit function theorem, and the fact that (s_1, s_2) is a local biholomorphism, there exist simply connected open sets $U \subseteq U_0$ and $V \subseteq V_0$ with $\alpha^* \in U$ and $\beta^* \in V$, and a diffeomorphism $B : U \to V$ such that

$$\{(\alpha, B(\alpha)) : \alpha \in U\} = \{(\alpha, \beta) \in U \times V : \Re s_1(\alpha, \beta) = \Re s_2(\alpha, \beta) = 0\}$$

and hence in particular $B(\alpha^*) = \beta^*$. Furthermore, note that

$$\{(\alpha, B(\alpha)) : \alpha \in U\} = \Omega \cap (U \times V).$$

Applying Lemma 15, it follows that the Stokes complex $\mathcal{C}(\alpha, B(\alpha))$ is homeomorphic to $\mathcal{C}(\alpha^*, \beta^*)$ and hence $(\alpha, B(\alpha)) \in R$ for all $\alpha \in U$. Therefore

$$R \cap (U \times V) = \Omega \cap (U \times V) = \{(\alpha, B(\alpha)) : \alpha \in U\}. \tag{50}$$

By (i), we may choose a simply connected open environment $W \subseteq \{(\Re\alpha, \Im\alpha, \Re\beta, \Im\beta) \in \mathbb{R}^4\}$ of (α^*, β^*), with $W \subseteq U \times V$, such that $H|_W : W \to \mathbb{R}^4$ is a diffeomorphism onto its open image $H(W) \subseteq \mathbb{R}^4$. So $(W, H|_W)$ is a local chart of $\{(\Re\alpha, \Im\alpha, \Re\beta, \Im\beta) \in \mathbb{R}^4\}$ and

$$H|_W(R \cap W) = H(W) \cap \{(\sigma_1^R, \sigma_1^I, \sigma_2^R, \sigma_2^I) \in \mathbb{R}^4 : \sigma_1^R = \sigma_2^R = 0\}.$$

We conclude that R is a smooth 2-dimensional regular submanifold of $\{(\Re\alpha, \Im\alpha, \Re\beta, \Im\beta) \in \mathbb{R}^4\}$. Furthermore, $(R \cap W, (s_1^I, s_2^I))$ is a local chart of R, and as $\mathcal{S} = (s_1^I, s_2^I)$, we find that $\mathcal{S}$ is locally diffeomorphic on R. □

As a corollary of the previous lemma we obtain that $R_a \subseteq \mathbb{C}$ and $R_b \subseteq \mathbb{C}$ are both open. Furthermore, the turning points $\lambda_i = \lambda_i(\alpha, \beta)$, $1 \leq i \leq 4$, are smooth on R.

Next we turn our attention to Proposition 3. To prove it, we need a number of preparatory lemmas.

Lemma 6. *The mapping $\mathcal{S}$ is injective.*

Proof. We just give a rough sketch of the proof and refer the interested reader to [3, 18] for a complete treatment of similar arguments. Suppose $\mathcal{S}(\alpha,\beta)=\mathcal{S}(\alpha',\beta')$ and let respectively $\lambda_1,\ldots,\lambda_4$ and $\lambda_1',\ldots,\lambda_4'$ be corresponding turning points. Furthermore, let us denote corresponding differentials by ω and ω'.

Note that the Stokes complex of the potential $V(\lambda,\alpha,\beta)$ naturally cuts the complex λ-plane into five disjoint open connected regions $I,\ldots,V$ as in Figure 8. By choosing the sign correctly, the action integral

$$S(\lambda)=\int_{\lambda_1}^{\lambda}\omega$$

defines a uniformisation of I (resp. homeomorphism from $\overline{I}$) onto the open (resp. closed) left-half plane, mapping the turning points $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ to the respective marked points 0, $r_6 i$, $(r_5+r_6)i$ and $(r_4+r_5+r_6)i$, where r_4,r_5,r_6 as defined in Lemma 4.

Similarly, the Stokes complex of the potential $V(\lambda,\alpha',\beta')$ cuts the complex plane into five pieces $I',\ldots,V'$ and

$$S'(\lambda)=\int_{\lambda_1'}^{\lambda}\omega'$$

defines a uniformisation of I' (resp. homeomorphism from $\overline{I'}$) onto the open (resp. closed) left-half plane, which maps the turning points $\lambda_1',\lambda_2',\lambda_3',\lambda_4'$ to the same respective marked points 0, $r_6 i$, $(r_5+r_6)i$ and $(r_4+r_5+r_6)i$, as $\mathcal{S}(\alpha,\beta)=\mathcal{S}(\alpha',\beta')$.

Thus $\chi:\overline{I}\to\overline{I'}$, defined by

$$\chi:=S'|_{\overline{I'}}^{-1}\circ S|_{\overline{I}},$$

is a homeomorphism which maps I conformally and bijectively onto I', mapping turning point λ_k to λ_k' for $1\leq k\leq 4$.

Without going into further detail, one may prove that χ can be uniquely extended to an automorphism of $\mathbb{P}^1$, mapping each region $J\in\{I,\ldots V\}$ to its corresponding region J' and in particular

$$\chi(0)=0,\quad \chi(\infty)=\infty,\quad \chi(\lambda_i)=\lambda_i'\quad (1\leq i\leq 4).$$

It follows that χ is a dilation. There are only two dilations which preserve equations (49a) and (49b), namely $\chi(\lambda)\equiv+\lambda$ and $\chi(\lambda)\equiv-\lambda$. Since by construction χ leaves the asymptotic direction $e^{-\frac{3}{4}\pi i}\infty$ invariant, we must have $\chi(\lambda)\equiv+\lambda$. In particular $(\alpha',\beta')=(\alpha,\beta)$ by equations (49c) and (49d), and we conclude that $\mathcal{S}$ is indeed injective. ◻

Lemma 7. *The region $K=\overline{R}$ is compact.*

Proof. Suppose K is not compact, then R is unbounded. Take a sequence $(\alpha_n,\beta_n)_{n>1}$ in R such that $|\alpha_n|+|\beta_n|\to\infty$ as $n\to\infty$. Let us write $\widetilde{\beta}_n=\frac{\beta_n}{\alpha_n^3}$ for $n\geq 1$. By replacing $(\alpha_n,\beta_n)_{n>1}$ by an appropriate subsequence if necessary, we may assume that we are in one of the following four scenarios:

(i) $\alpha_n\to\alpha^*\in\mathbb{C}$ and $|\beta_n|\to\infty$,
(ii) $|\alpha_n|\to\infty$ and $|\widetilde{\beta}_n|\to\infty$,
(iii) $|\alpha_n|\to\infty$ and $\widetilde{\beta}_n\to\widetilde{\beta}^*\in\mathbb{C}^*$, or
(iv) $|\alpha_n|\to\infty$ and $\widetilde{\beta}_n\to 0$

as $n\to\infty$.

Each of the four cases leads to a contradiction in a similar fashion and we therefore limit our discussion to one of them, case (iii). Setting $\lambda=\alpha_n\mu$, we have

$$\sqrt{V(\lambda;\alpha_n,\beta_n)}d\lambda=\alpha_n^2\sqrt{\mu^2+2\mu+1-\widetilde{\beta}_n\mu^{-1}-\alpha_n^{-2}+\alpha_n^{-4}\frac{\nu^2}{4}\mu^{-2}}d\mu. \qquad (51)$$

Let us write the turning points of $V(\lambda;\alpha_n,\beta_n)$ by $\lambda_j^n = \lambda_j(\alpha_n,\beta_n)$ and define $\mu_j^n := \alpha_n^{-1}\lambda_j^n$ for $1 \leq j \leq 4$ and $n \geq 1$. Then, by replacing $(\alpha_n,\beta_n)_{n>1}$ by an appropriate subsequence if necessary, we may assume that there exists a permutation $\sigma \in S_4$ such that

$$\mu^n_{\sigma(4)} \to 0, \quad \mu^n_{\sigma(j)} \to u_j \in \mathbb{C}^* \quad (1 \leq j \leq 3)$$

as $n \to \infty$, where $\{u_1, u_2, u_3\}$ are the roots of $\mu^3 + 2\mu^2 + \mu - \widetilde{\beta}^*$.

However, $\mu = 0$ is a simple pole of the differential (51). Recalling definition (47) of $r_4, r_6 \in \mathbb{R}_+$ in the proof of Lemma 4, we hence obtain, as $n \to \infty$ and thus $\mu^n_{\sigma(4)} \to 0$, that either $r_4 \to +\infty$ or $r_6 \to +\infty$, contradicting equation (48). $\square$

We define the border of R by $\delta R = \overline{R} \setminus R$ in $\mathbb{R}^4$, not to be confused with the topological boundary ∂R of R, so that $K = R \sqcup \delta R$. In order to extend the mapping $\mathcal{S}$ continuously to K, we have understand what happens with the turning points and cycles γ_1, γ_2 upon approaching the border. We thus proceed with studying the border of R.

Let us denote the discriminant of the polynomial

$$\lambda^4 + 2\alpha\lambda^3 + (\alpha^2 - 1)\lambda^2 - \beta\lambda + \tfrac{1}{4}\nu^2$$

by

$$\begin{aligned}\Delta(\alpha,\beta) = &-27\beta^4 + 4\alpha(3-\alpha^2)\beta^3 + 2\left(3\nu^2(5\alpha^2-6) + 2(\alpha^2-1)^2\right)\beta^2 \\ &+ 4\nu^2\alpha\left(6\nu^2 - (1-\alpha^2)(10-\alpha^2)\right)\beta + \nu^2\left(4\nu^4 + \nu^2(\alpha^4 - 20\alpha^2 - 8) + 4(1-\alpha^2)^3\right).\end{aligned}$$

As a first step, in the following lemma we prove that the border of R is characterised by the merging of turning points.

Lemma 8. *We have $\delta R \subseteq \Omega \cap \{\Delta(\alpha,\beta) = 0\}$.*

Proof. Firstly we show that $\delta R \subseteq \Omega$. Let $(\alpha^*,\beta^*) \in \delta R$, then there exists a sequence $(\alpha_n,\beta_n)_{n\geq 1}$ in R such that $(\alpha_n,\beta_n) \to (\alpha^*,\beta^*)$ as $n \to \infty$. Take any Jordan curve γ in $\mathbb{C}^*$ minus turning points of $V(\lambda;\alpha^*,\beta^*)$, which encircles an even number of turning points counting multiplicity, then $\sqrt{V(\lambda)}$ can be chosen single-valued along γ and

$$\Re \oint_\gamma \sqrt{V(\lambda;\alpha^*,\beta^*)}d\lambda = \lim_{n\to\infty} \Re \oint_\gamma \sqrt{V(\lambda;\alpha_n,\beta_n)}d\lambda = 0.$$

It follows that $(\alpha^*,\beta^*) \in \Omega$ and hence $\delta R \subseteq \Omega$.

Next, let $(\alpha^*,\beta^*) \in \delta R$ and suppose that $\Delta(\alpha^*,\beta^*) \neq 0$. The proof of equation (50) can be used line for line to show that there exists an open environment $W \subseteq \{(\Re\alpha, \Im\alpha, \Re\beta, \Im\beta) \in \mathbb{R}^4\}$ of (α^*,β^*) such that

$$W \cap \Omega = \{(\alpha,\beta) \in W : \mathcal{C}(\alpha,\beta) \sim \mathcal{C}(\alpha^*,\beta^*)\}.$$

As $W \cap \Omega \cap R = W \cap R \neq \emptyset$, it follows that $\mathcal{C}(\alpha^*,\beta^*) \sim \mathcal{C}(0,0)$ and hence $(\alpha^*,\beta^*) \in R$. But $R \cap \delta R = \emptyset$ and we have arrived at a contradiction. We conclude that $\Delta(\alpha^*,\beta^*) = 0$. $\square$

By the above lemma, at points on the border δR two or more of the turning points of $V(\lambda)$ have merged. We classify points on the border by the isomorphism class of their Stokes complex. To this end, we introduce the isomorphism classes $E_1, \ldots, E_4$ and $C_1, \ldots, C_4$ in Figure 9. For convenience, we have included the isomorphism class G_0 of the Stokes complex at $(\alpha,\beta) = (0,0)$ in the figure.

The possible mergers of turning points near the border are heavily constrained, since two turning points can only merge if they are connected by a Stokes line, due to Lemma 16, and Lemma 17 gives a similar constraint for the merging of three turning points. This in turn constraints the possible Stokes complexes attainable at the border and the following lemma shows that the isomorphism classes introduced in Figure 9 indeed suffice to cover the entire border.

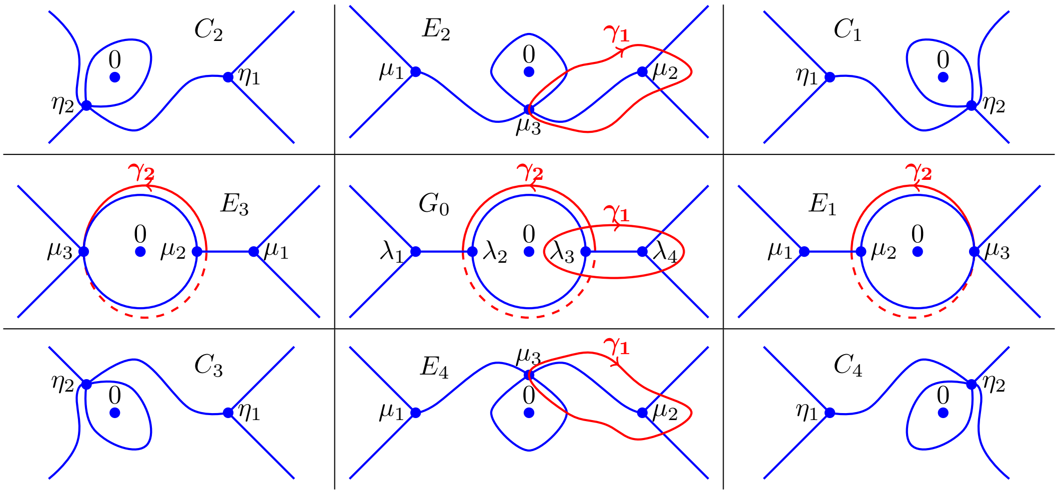


FIGURE 9. Schematic representation of isomorphism classes $E_1, \ldots, E_4$ and $C_1, \ldots, C_4$ and G_0 of Stokes complexes in blue with corresponding contours used in the proof of Lemma 10 in red. Note that G_0 is the isomorphism class of $\mathcal{C}(0,0)$.

Lemma 9. *For any point $(\alpha, \beta) \in \delta R$ on the border the associated Stokes complex $\mathcal{C}(\alpha, \beta)$ falls in one of the classes $E_k, 1 \leq k \leq 4$ or $C_k, 1 \leq k \leq 4$, introduced in Figure 9.*

Proof. Let $(\alpha^*, \beta^*) \in \delta R$, then there exists a sequence $(\alpha_n, \beta_n)_{n \geq 1}$ in R such that $(\alpha_n, \beta_n) \to (\alpha^*, \beta^*)$ as $n \to \infty$. Let $\{\mu_1, \ldots, \mu_m\}$ be the turning points of $V(\lambda; \alpha^*, \beta^*)$, discarding multiplicity, so that $2 \leq m \leq 4$. By Lemma 8, we know that $m = 2$ or $m = 3$. Let us denote the turning points of $V(\lambda; \alpha_n, \beta_n)$ by $\lambda_j^n = \lambda_j(\alpha_n, \beta_n)$ for $1 \leq j \leq 4$ and $n \geq 1$, then we may assume, by replacing $(\alpha_n, \beta_n)_{n \geq 1}$ by an appropriate subsequence if necessary, that there exists a surjective mapping $\sigma : \{1, 2, 3, 4\} \to \{1, 2, \ldots, m\}$ such that $\lambda_j^n \to \mu_{\sigma(j)}$ as $n \to \infty$ for $1 \leq j \leq 4$.

Let us first consider the case $m = 3$ and label the turning points of $V(\lambda; \alpha^*, \beta^*)$ such that μ_1 and μ_2 are simple and μ_3 is the double turning point. Applying Lemma 16, either

(i) $\lambda_1^n, \lambda_2^n \to \mu_3$ as $n \to \infty$,
(ii) $\lambda_2^n, \lambda_3^n \to \mu_3$ as $n \to \infty$, or
(iii) $\lambda_3^n, \lambda_4^n \to \mu_3$ as $n \to \infty$.

In case (i), it follows from Lemma 14 that

- μ_3 has one emanating Stokes line asymptotic to $e^{+\frac{3}{4}\pi i}\mathbb{R}_+$ and another asymptotic to $e^{-\frac{3}{4}\pi i}\mathbb{R}_+$,
- μ_1 or μ_2 has one emanating Stokes line asymptotic to $e^{+\frac{1}{4}\pi i}\mathbb{R}_+$ and another asymptotic to $e^{-\frac{1}{4}\pi i}\mathbb{R}_+$.

It is now easy to see that $\mathcal{C}(\alpha^*, \beta^*) \in E_3$, using Proposition 2, Lemma 13 and the fact that $(\alpha^*, \beta^*) \in \Omega$ by Lemma 8.

In case (iii) it follows analogously that $\mathcal{C}(\alpha^*, \beta^*) \in E_1$.

In case (ii), it follows from Lemma 14 that we may, if necessary, renumber $\{\mu_1, \mu_2\}$ such that

- μ_1 has one emanating Stokes line asymptotic to $e^{+\frac{3}{4}\pi i}\mathbb{R}_+$ and another asymptotic to $e^{-\frac{3}{4}\pi i}\mathbb{R}_+$,
- μ_2 has one emanating Stokes line asymptotic to $e^{+\frac{1}{4}\pi i}\mathbb{R}_+$ and another asymptotic to $e^{-\frac{1}{4}\pi i}\mathbb{R}_+$.

It is now easy to see that either $\mathcal{C}(\alpha^*,\beta^*)\in E_2$ or $\mathcal{C}(\alpha^*,\beta^*)\in E_4$, using Proposition 2 and Lemma 13.

Next, let us consider $m=2$ and label the turning points of $V(\lambda;\alpha^*,\beta^*)$ so that μ_1 is simple and μ_2 is the triple turning point. By Lemma 17, either

(1) $\lambda_1^n,\lambda_2^n,\lambda_3^n\to\mu_2$ as $n\to\infty$, or
(2) $\lambda_2^n,\lambda_3^n,\lambda_4^n\to\mu_2$ as $n\to\infty$.

Similarly as above, it follows that in case (1) that either $\mathcal{C}(\alpha^*,\beta^*)\in C_2$ or $\mathcal{C}(\alpha^*,\beta^*)\in C_3$, and in case (2) that either $\mathcal{C}(\alpha^*,\beta^*)\in C_1$ or $\mathcal{C}(\alpha^*,\beta^*)\in C_4$. □

We define

$$\widetilde{e}_k=\{(\alpha,\beta)\in\delta R:\mathcal{C}(\alpha,\beta)\in E_k\},\quad \widetilde{c}_k=\{(\alpha,\beta)\in\delta R:\mathcal{C}(\alpha,\beta)\in C_k\}\quad(1\leq k\leq 4),$$

then Lemma 9 implies the following partition of the border,

$$\delta R=\widetilde{e}_1\sqcup\widetilde{e}_2\sqcup\widetilde{e}_3\sqcup\widetilde{e}_4\sqcup\widetilde{c}_1\sqcup\widetilde{c}_2\sqcup\widetilde{c}_3\sqcup\widetilde{c}_4. \tag{52}$$

Let us denote by l_+ and l_- respectively the upper and lower Stokes line with respect to 0, as depicted in Figure 7, with endpoints λ_2 and λ_3. Then we have the following corollary of Lemma 9 which characterises the separate parts of the border in the disjoint union (52).

Corollary 2. *Near the edges $\widetilde{e}_k$, $1\leq k\leq 4$, two of the turning points of $V(\lambda)$ merge,*

$\widetilde{e}_1$: here λ_3 and λ_4 merge,
$\widetilde{e}_2$: here λ_2 and λ_3 merge via the vanishing of l_-,
$\widetilde{e}_3$: here λ_2 and λ_3 merge via the vanishing of l_+,
$\widetilde{e}_4$: here λ_1 and λ_2 merge.

Similarly, the corners $\widetilde{c}_k$ are characterised by the merging of three turning points,

$\widetilde{c}_1$: here λ_2, λ_3 and λ_4 merge via the vanishing of l_-,
$\widetilde{c}_2$: here λ_1, λ_2 and λ_3 merge via the vanishing of l_-,
$\widetilde{c}_3$: here λ_1, λ_2 and λ_3 merge via the vanishing of l_+,
$\widetilde{c}_4$: here λ_2, λ_3 and λ_4 merge via the vanishing of l_+.

Proof. Let $(\alpha^*,\beta^*)\in\widetilde{e}_1$ and let $(\alpha_n,\beta_n)_{n\geq 1}$ be a sequence in R which converges to (α^*,β^*). Let $\lambda_k^n=\lambda_k(\alpha_n,\beta_n)$ denote the turning points of $V(\lambda;\alpha_n,\beta_n)$ for $1\leq k\leq 4$ and $n\geq 1$, and μ_1,μ_2,μ_3 denote the turning points of $V(\lambda;\alpha^*,\beta^*)$ as in Figure 9 for the class E_1.

We have to show that $\lambda_k^n\to\mu_{\sigma_0(k)}$ for $1\leq k\leq 4$, where $\sigma_0(k)=k$ for $1\leq k\leq 3$ and $\sigma_0(4)=3$. Suppose this is not the case, then there exists a subsequence $(\alpha_{n_m},\beta_{n_m})_{m\geq 1}$ together with a surjective mapping $\sigma:\{1,2,3,4\}\to\{1,2,3\}$ not equal to σ_0, such that $\lambda_k^{m_n}\to\mu_{\sigma(k)}$ as $n\to\infty$ for $1\leq k\leq 4$. But, by applying the argument in Lemma 9 to the subsequence $(\alpha_{n_m},\beta_{n_m})_{m\geq 1}$, it then follows that the Stokes complex $\mathcal{C}(\alpha^*,\beta^*)$ must fall in one of the classes E_2,E_3 or E_4, which contradicts that $(\alpha^*,\beta^*)\in\widetilde{e}_1$. We conclude that $\lambda_k^n\to\mu_{\sigma_0(k)}$ for $1\leq k\leq 4$ so that $\widetilde{e}_1$ is indeed characterised by the merging of the turning points λ_3 and λ_4. The other characterisations are shown analogously. □

Now that we know which turning points merge on the different parts of the border, it is clear how to extend $\mathcal{S}$ continuously to K, as detailed in the proof of the following lemma.

Lemma 10. *The mapping $\mathcal{S}$ on R has a unique continuous extension to K,*

$$\mathcal{S} : K \to Q, \quad Q = \left[-\tfrac{1}{2}(1-\nu)\pi, +\tfrac{1}{2}(1-\nu)\pi\right] \times [-\nu\pi, +\nu\pi],$$

with

$$\mathcal{S}(\widetilde{e}_k) \subseteq \widehat{e}_k, \quad \mathcal{S}(\widetilde{c}_k) \subseteq \{\widehat{c}_k\} \quad (1 \le k \le 4),$$

where the open edges and corners $\widehat{e}_k, \widehat{c}_k$, $1 \le k \le 4$ of Q are defined as in Figure 4.
Furthermore, for $1 \le k \le 4$, the 'edge' $\widetilde{e}_k$ is a smooth 1-dimensional regular submanifold of $\{(\Re\alpha, \Im\alpha, \Re\beta, \Im\beta) \in \mathbb{R}^4\}$ and the restricted mapping

$$\mathcal{S}|_{\widetilde{e}_k} : \widetilde{e}_k \to \widehat{e}_k \tag{53}$$

is locally diffeomorphic.

Proof. We first discuss the extension of $\mathcal{S}$ to $R \cup \widetilde{e}_1$. Let $(\alpha, \beta) \in \widetilde{e}_1$ and take any sequence $(\alpha_n, \beta_n)_{n \ge 1}$ in R with $(\alpha_n, \beta_n) \to (\alpha^*, \beta^*)$ as $n \to \infty$. We label the turning points $\{\mu_1, \mu_2, \mu_3\}$ of $V(\lambda; \alpha, \beta)$ such that μ_3 is the double turning point, μ_2 is the unique simple turning point adjacent to μ_3 and μ_1 is the remaining simple turning point, in accordance with Figure 9.

Let us denote the turning points of $V(\lambda; \alpha_n, \beta_n)$ by $\lambda_j^n = \lambda_j(\alpha_n, \beta_n)$ for $1 \le j \le 4$ and $n \ge 1$. Due to Corollary 2,

$$\lambda_1^n \to \mu_1, \quad \lambda_2^n \to \mu_2, \quad \lambda_3^n \to \mu_3, \quad \lambda_4^n \to \mu_3$$

as $n \to \infty$ and thus

$$s_1(\alpha_n, \beta_n) \to +\tfrac{1}{2}(1-\nu)\pi i, \qquad s_2(\alpha_n, \beta_n) \to s_2^{e_1}(\alpha, \beta) \quad (n \to \infty),$$

where

$$s_2^{e_1}(\alpha, \beta) = \int_{\gamma_2} \omega \tag{54}$$

with γ_2 as defined in Figure 9 on the class E_1. In conclusion,

$$\mathcal{S}(\alpha_n, \beta_n) \to (+\tfrac{1}{2}(1-\nu)\pi, -is_2^{e_1}(\alpha, \beta)) \quad (n \to \infty) \tag{55}$$

and we therefore set

$$\mathcal{S}(\alpha, \beta) = (+\tfrac{1}{2}(1-\nu)\pi, -is_2^{e_1}(\alpha, \beta)).$$

It is easy to see that $\mathcal{S}(\alpha, \beta) \in \widehat{e}_1$, following the lines in the proof of Lemma 4.

To prove continuity of $\mathcal{S}$ on $R \cup \widetilde{e}_1$, it remains to be shown that $s_2^{e_1}$ is continuous on $\widetilde{e}_1$. We proceed in showing that $\widetilde{e}_1$ is a regular smooth submanifold of $\mathbb{R}^4$ and that $s_2^{e_1}$ is locally diffeomorphic on $\widetilde{e}_1$, which in particular implies the required continuity.

Let $(\alpha^*, \beta^*) \in \widetilde{e}_1$, then $\Delta(\alpha^*, \beta^*) = 0$. Take a simply connected open environment U of α^*, a simply connected open environment V of β^* and a biholomorphism $B_\Delta : U \to V$ such that

$$W := \{\Delta(\alpha, \beta) = 0\} \cap (U \times V) = \{(\alpha, B_\Delta(\alpha)) : \alpha \in U\}.$$

Note that W is a regular complex submanifold of $\mathbb{C}^2$ and $\widetilde{e}_1 \cap (U \times V) \subseteq W$, which allows us to study $\widetilde{e}_1$ locally within W. To this end, we extend $s_2^{e_1}$ to a analytic mapping on W and show that it is biholomorphic at (α^*, β^*).

Let $\{\mu_1^*, \mu_2^*, \mu_3^*\}$ be the turning points of $V(\lambda; \alpha^*, \beta^*)$, labelled in accordance with Figure 9 as above. Let $D_j \subseteq \mathbb{C}^*$ with $\mu_j^* \in D_j$, $1 \le j \le 3$, be mutually disjoint open discs. We define the following analytic function on $D_1 \times D_2 \times D_3$,

$$\widetilde{S}_2(\mu_1, \mu_2, \mu_3) = \oint_{\gamma_2} \frac{\sqrt{(\mu - \mu_1)(\mu - \mu_2)}(\mu - \mu_3)}{\mu} d\mu,$$

with branch and contour chosen consistently with $s_2^{e_1}$ in equation (54).
For $1 \le j \le 3$, there exists a unique analytic function $\mu_j : W \to \mathbb{C}^*$ with $\mu_j(\alpha^*, \beta^*) = \mu_j^*$,

$1 \le j \le 3$, such that $\mu_j(\alpha,\beta)$ is a turning point of $V(\lambda;\alpha,\beta)$ for $(\alpha,\beta) \in W$. We extend $s_2^{e_1}$, defined in (54), analytically to W, by setting

$$s_2^{e_1}(\alpha,\beta) = s_2^{e_1}(\alpha, B_\Delta(\alpha)) = \widetilde{S}_2(\mu_1(\alpha,B_\Delta(\alpha)),\mu_2(\alpha,B_\Delta(\alpha)),\mu_3(\alpha,B_\Delta(\alpha))),$$

for $(\alpha,\beta) \in W$.

Now note that

$$\Omega \cap W = \{(\alpha,\beta) \in W : \Re s_2^{e_1}(\alpha,\beta) = 0\}.$$

We wish to show that $s_2^{e_1} : W \to \mathbb{C}$ is locally biholomorphic at (α^*,β^*), i.e.

$$\frac{\partial}{\partial\alpha} s_2^{e_1}(\alpha, B_\Delta(\alpha)) \tag{56}$$

does not vanish at $\alpha = \alpha^*$.

In order to compute (56), we introduce, inspired by (49a) and (49b),

$$\widetilde{S}_3(\mu_1,\mu_2,\mu_3) = \mu_1\mu_2\mu_3^2 - \tfrac{1}{4}\nu^2,$$
$$\widetilde{S}_4(\mu_1,\mu_2,\mu_3) = \mu_1\mu_2 + 2\mu_1\mu_3 + 2\mu_2\mu_3 - \tfrac{1}{2}(\mu_1^2+\mu_2^2) + 2,$$

and compute the Jacobian

$$|J_{(\widetilde{S}_2,\widetilde{S}_3,\widetilde{S}_4)}(\mu)| = \pm 2\mu_3(\mu_2-\mu_1)(\mu_3-\mu_1)^{\frac{3}{2}}(\mu_3-\mu_2)^{\frac{3}{2}}, \quad J_{(\widetilde{S}_2,\widetilde{S}_3,\widetilde{S}_4)}(\mu) := \left(\frac{\partial \widetilde{S}_{m+1}}{\partial \mu_n}\right)_{1\le m,n\le 3}.$$

It follows that $M := J_{(\widetilde{S}_2,\widetilde{S}_3,\widetilde{S}_4)}(\mu^*)$ is invertible. By the chain rule, we have

$$\begin{pmatrix} \frac{\partial \widetilde{S}_2}{\partial\alpha} \\ \frac{\partial \widetilde{S}_3}{\partial\alpha} \\ \frac{\partial \widetilde{S}_4}{\partial\alpha} \end{pmatrix} = J_{(\widetilde{S}_2,\widetilde{S}_3,\widetilde{S}_4)}(\mu) \cdot \begin{pmatrix} \frac{\partial \mu_1}{\partial\alpha} \\ \frac{\partial \mu_2}{\partial\alpha} \\ \frac{\partial \mu_3}{\partial\alpha} \end{pmatrix}$$

which, when specialised to $(\alpha,\beta) = (\alpha^*,\beta^*)$, gives

$$\begin{pmatrix} \frac{\partial}{\partial\alpha} s_2^{e_1}(\alpha, B(\alpha))|_{\alpha=\alpha^*} \\ 0 \\ 0 \end{pmatrix} = M \cdot \begin{pmatrix} \frac{\mu_1^*}{\mu_3^*-\mu_1^*} \\ \frac{\mu_2^*}{\mu_3^*-\mu_2^*} \\ \frac{\mu_3^*(\mu_1^*+\mu_2^*-2\mu_3^*)}{2(\mu_3^*-\mu_1^*)(\mu_3^*-\mu_2^*)} \end{pmatrix}$$

Since M is invertible, it thus follows that (56) does not vanishes at $\alpha = \alpha^*$ and hence $s_2^{e_1}(\alpha,\beta)$ is a local biholomorphism at $(\alpha,\beta) = (\alpha^*,\beta^*)$. Therefore, there exists a simply connected open environment $W_0 \subseteq W$ of (α^*,β^*) such that

$$(F_1,F_2) : W_0 \to \mathbb{R}^2, (\alpha,\beta) \mapsto (\Re\mathcal{S}_2(\alpha,\beta), \Im\mathcal{S}_2(\alpha,\beta))$$

is a diffeomorphism onto its open image $F(W_0) \subseteq \mathbb{R}^2$ with

$$F(W_0) \cap (\{0\}\times\mathbb{R}) = \{0\} \times I,$$

where $I \subseteq \mathbb{R}$ is an open interval. We apply Lemma 15 with

$$T := \{(\alpha,\beta) \in W_0 : \Re\mathcal{S}_2(\alpha,\beta) = 0\},$$

giving

$$T = W_0 \cap \Omega = W_0 \cap \widetilde{e}_1.$$

Let us in particular note that, by choosing an open environment $Z \subseteq \{(\Re\alpha,\Im\alpha,\Re\beta,\Im\beta) \in \mathbb{R}^4\}$ of (α^*,β^*) with $Z \cap \{\Delta(\alpha,\beta)=0\} \subseteq W_0$, we have

$$Z \cap \{\Delta(\alpha,\beta) = 0\} \cap \Omega = Z \cap \widetilde{e}_1. \tag{57}$$

We conclude that $\widetilde{e}_1$ is a smooth 1-dimensional regular submanifold of $\{(\Re\alpha,\Im\alpha,\Re\beta,\Im\beta) \in \mathbb{R}^4\}$. Furthermore $F_2|_T$ maps T diffeomorphically onto I. As $\mathcal{S}|_T = F_2|_T$, it follows that $\mathcal{S}|_{\widetilde{e}_1}$:

$\widetilde{e}_1 \to \widehat{e}_1$ is locally diffeomorphic at $(\alpha, \beta) = (\alpha^*, \beta^*)$ and in particular continuous. Combined with (55), this shows that the extension of $\mathcal{S}$ to $R \cup \widetilde{e}_1$ is indeed continuous.

Next, we will extend $\mathcal{S}$ to $R \cup \widetilde{e}_1 \cup \widetilde{e}_2$ in much the same way. For $(\alpha, \beta) \in \widetilde{e}_2$, we define

$$\mathcal{S}(\alpha, \beta) = (-is_1^{e_2}(\alpha, \beta), +\nu\pi),$$

where

$$s_1^{e_2}(\alpha, \beta) = \int_{\gamma_1} \omega + \frac{i\pi(1-\nu)}{2} \tag{58}$$

with γ_1 as defined in Figure 9 on the class E_2, and the branch chosen consistently with the definition of s_1. Analogously to the above, it is shown that $\widetilde{e}_2$ is a smooth 1-dimensional regular submanifold of $\{(\Re\alpha, \Im\alpha, \Re\beta, \Im\beta) \in \mathbb{R}^4\}$, and that $\mathcal{S}|_{\widetilde{e}_2}\widetilde{e}_2 \to \widehat{e}_2$ is locally diffeomorphic.

Continuity of the extension $\mathcal{S}$ to $R \cup \widetilde{e}_1 \cup \widetilde{e}_2$ on $\widetilde{e}_2$ is proven as it was on $\widetilde{e}_1$, noting that $\overline{\widetilde{e}_1} \cap \widetilde{e}_2 = \emptyset$ by equation (57).

Similarly, it is shown that $\widetilde{e}_3$ and $\widetilde{e}_4$ are smooth 1-dimensional regular submanifolds, we have $\overline{\widetilde{e}_i} \cap \widetilde{e}_j = \emptyset$ for $1 \le i, j \le 4$ with $i \ne j$, and we extend $\mathcal{S}$ to $R \cup \widetilde{e}_1 \cup \widetilde{e}_2 \cup \widetilde{e}_3 \cup \widetilde{e}_4$, such that $\mathcal{S}(\widetilde{e}_i) \subseteq \widehat{e}_i$ and $\mathcal{S}|_{\widetilde{e}_i}\widetilde{e}_i \to \widehat{e}_i$ is locally diffeomorphic for $1 \le i \le 4$.

Finally, we define $\mathcal{S}(\alpha, \beta) = \widehat{c}_i$ if $(\alpha, \beta) \in \widetilde{c}_i$ for $1 \le i \le 4$, so that $\mathcal{S} : K \to Q$, and it remains to be shown that $\mathcal{S}$ is continuous at such points.

Let $(\alpha, \beta) \in c_1$ and let us denote the simple and triple turning point of $V(\lambda; \alpha, \beta)$ by respectively η_1 and η_2, in accordance with Figure 9. Suppose $(\alpha_n, \beta_n)_{n\ge1}$ is a sequence in R such that $(\alpha_n, \beta_n) \to (\alpha, \beta)$ as $n \to \infty$ and let us denote the turning points of $V(\lambda; \alpha_n, \beta_n)$ by $\lambda_j^n = \lambda_j(\alpha_n, \beta_n)$ for $1 \le j \le 4$ and $n \ge 1$. It follows from Lemmas 14 and 17 that

$$\lambda_1^n \to \eta_1, \quad \lambda_j^n \to \eta_2 \quad (2 \le j \le 4)$$

and hence

$$\mathcal{S}(\alpha_n, \beta_n) = (-is_1(\alpha_n, \beta_n), -is_2(\alpha_n, \beta_n)) \to \widehat{c}_1$$

as $n \to \infty$.

Similarly, if $(\alpha_n, \beta_n)_{n\ge1}$ is a sequence in $\widetilde{e}_1$ which converges to (α, β), we denote the turning points of $V(\lambda; \alpha_n, \beta_n)$ by μ_j^n, $1 \le j \le 3$ in accordance with Figure 9, and Lemma 17 yields

$$\mu_1^n \to \eta_1, \quad \mu_j^n \to \eta_2 \quad (2 \le j \le 3)$$

and thus

$$\mathcal{S}(\alpha_n, \beta_n) = (+\tfrac{1}{2}(1-\nu)\pi, -is_2^{e_1}(\alpha_n, \beta_n)) \to \widehat{c}_1$$

as $n \to \infty$.

We handle sequences in $\widetilde{e}_2$ converging to (α^*, β^*) similarly and finally note that the sets $\widetilde{c}_i, 1, \le i \le 4$ are discrete and $(\alpha, \beta) \notin \overline{\widetilde{e}_i}$ for $3 \le i \le 4$. It follows that $\mathcal{S}$ is continuous at $(\alpha, \beta) \in \widetilde{c}_1$. Continuity at points in $\widetilde{c}_j$ for $2 \le j \le 4$ are proven analogously and it follows that $\mathcal{S}$ is globally continuous. □

We now have all the ingredients to prove the following theorem, which generalises Proposition 3.

Theorem 6. *The continuous extension $\mathcal{S} : K \to Q$ defined in Lemma 10 is a homeomorphism and maps the interior R of K diffeomorphically onto the open rectangle Q°.*
For $1 \le i \le 4$, the extension $\mathcal{S}$ maps the 'edge' $\widetilde{e}_i$ diffeomorphically [3] onto the open edge $\widehat{e}_i$ of Q. Furthermore, for $1 \le i \le 4$, the set $\widetilde{c}_i$ is a singleton and $\mathcal{S}$ maps $\widetilde{c}_i \equiv \{\widetilde{c}_i\}$ to the corner $\widehat{c}_i$ of Q.

[3] diffeomorphically with respect to the geometric structure on $\widetilde{e}_i$ as a smooth regular submanifold of $\{(\Re\alpha, \Im\alpha, \Re\beta, \Im\beta) \in \mathbb{R}^4\}$, see Lemma 10.

Proof. We know that the restriction of $\mathcal{S}$ to R is an injective smooth mapping from R to Q°, and hence, by the domain invariance theorem, $\widehat{R} = \mathcal{S}(R)$ is an open subset of Q° and $\mathcal{S}$ maps R diffeomorphically onto $\widehat{R}$. By construction of $\mathcal{S}$ in Lemma 10, we have $\mathcal{S}(\delta R) \subseteq \partial Q$. Since, by Lemma 7, K is compact, $\partial\widehat{R} \subseteq \mathcal{S}(\delta R)$ and hence $\partial\widehat{R} \subseteq \partial Q$.

There exists only one non-empty open subset of Q° whose boundary is contained in ∂Q°, namely Q° itself. We conclude that $\widehat{R} = Q^\circ$ and in particular $\mathcal{S}$ maps R diffeomorphically onto the open rectangle Q°.

It follows that $\mathcal{S}(\delta R) = \partial\widehat{R} = \partial Q$ and we have $\mathcal{S}(K) = Q$, i.e. $\mathcal{S}$ is surjective.

Let $1 \leq i \leq 4$, then, by construction of $\mathcal{S}$ in Lemma 10, we have $\mathcal{S}^{-1}(\widehat{e}_i) \subseteq \widetilde{e}_i$, so the restricted mapping $\mathcal{S}|_{\widetilde{e}_i} : \widetilde{e}_i \to \widehat{e}_i$ is surjective. Analogously to the proof of Lemma 6 it is shown that the restricted mapping $\mathcal{S}|_{e_i}$ is injective. Since, by Lemma 10, the restricted mapping $\mathcal{S}|_{e_i} : e_i \to \widehat{e}_i$ is locally diffeomorphic it follows that $\mathcal{S}$ maps $\widetilde{e}_i$ diffeomorphically onto $\widehat{e}_i$. We conclude that $\mathcal{S}$ maps $R \cup \widetilde{e}_1 \cup \widetilde{e}_2 \cup \widetilde{e}_3 \cup \widetilde{e}_4$ bijectively onto $Q^\circ \cup \widehat{e}_1 \cup \widehat{e}_2 \cup \widehat{e}_3 \cup \widehat{e}_4$.

Let $1 \leq i \leq 4$, then, by construction of $\mathcal{S}$ in Lemma 10, we have $\mathcal{S}^{-1}(\widehat{c}_i) \subseteq \widetilde{c}_i$. Hence the surjectivity of $\mathcal{S}$ implies that $\widetilde{c}_i$ is non-empty. On the other hand, it is easy to see that $\widetilde{c}_i$ has at most one element, applying the same line of argument as in the proof of Lemma 6. So $\widetilde{c}_i$ is a singleton and $\mathcal{S}$ maps $\widetilde{c}_i \equiv \{\widetilde{c}_i\}$ to the corner $\widehat{c}_i$ of Q.

All together we obtain that $\mathcal{S}$ is a continuous bijection. Since K is compact, $\mathcal{S}$ sends closed sets to closed sets and therefore $\mathcal{S}$ is a homeomorphism. □

Corollary 3. *Let $1 \leq k \leq 4$, then the corner $\widetilde{c}_k = \{\widetilde{c}_k\}$ of K is the unique point $(\alpha, \beta) \in \mathbb{C}^2$ such that the Stokes complex $\mathcal{C}(\alpha, \beta)$ of $V(\lambda)$ lies in the isomorphism class C_k, introduced in Figure 9. The edge $\widetilde{e}_k$ equals the set of all $(\alpha, \beta) \in \mathbb{C}^2$ such that the Stokes complex $\mathcal{C}(\alpha, \beta)$ lies in the isomorphism class E_k, also introduced in Figure 9.*

Proof. Let us consider the edge $\widetilde{e}_1$. Note that equation (54) defines a mapping

$$s_2^{e_1} : \{(\alpha, \beta) \in \mathbb{C}^2 : \mathcal{C}(\alpha, \beta) \in E_i\} \to i(-\nu\pi, +\nu\pi)$$

which is easily proven to be injective by the same line of argument as the proof of Lemma 6. But $\widetilde{e}_1$ is a subset of its domain and $s_2^{e_1} = i(-\nu\pi, +\nu\pi)$ by Theorem 6. It follows that $\widetilde{e}_1$ equals the domain of $s_2^{e_1}$. The other characterisations are proven similarly. □

4.2. **The Projection Π_a and Region K_a.** In this subsection we derive explicit parametrisations of the boundary ∂K_a and border δR, proving in particular the parametrisation of ∂K_a in Section 2, see equation (8). Along the way we establish the following

Lemma 11. *The projection Π_a maps the border δR homeomorphically onto ∂R_a.*

Proposition 4 follows from the above lemma using standard topological arguments.

Proof of Proposition 4. We already know that R_a is open, K_a is compact and $\overline{R}_a = K_a$. Due to Theorem 6, we know that δR is homeomorphic to the Jordan curve ∂Q. It thus follows from Lemma 11 that $\partial R_a = \partial K_a$ is a Jordan curve and R_a equals its interior since R_a is open and connected.

Consider the mapping $g = \Pi_a \circ \mathcal{S}^{-1} : Q \to K_a$. It follows from Lemma 5 and Theorem 6 that $g|_{Q^\circ} : Q^\circ \to R_a$ is a local diffeomorphism. Furthermore, g maps the jordan curve ∂Q homeomorphically onto the Jordan curve ∂R_a. It follows from the latter and compactness of K that $g|_{Q^\circ} : Q^\circ \to R_a$ is proper. Therefore $g^{-1}(\alpha)$ is finite for any $\alpha \in R_a$, and since $g|_{Q^\circ}$ is locally diffeomorphic, it follows that $g|_{Q^\circ} : Q^\circ \to R_a$ is a covering map. Since Q° and R_a are simply connected, $g|_{Q^\circ}$ must be injective and thus g is a continuous bijection which maps Q° diffeomorphically onto R_a. As K is compact, g maps closed sets to closed sets and thus g is a homeomorphism. The proposition now follows since $\Pi_a = g \circ \mathcal{S}$. □

We proceed with deriving explicit parametrisations for ∂K_a and δR. Let us denote

$$e_k = \Pi_a(\widetilde{e}_k), \quad c_k = \Pi_a(\widetilde{c}_k) \quad (1 \leq k \leq 4),$$

so that

$$\partial R_a \subseteq \Pi_a(\delta R) = e_1 \cup e_2 \cup e_3 \cup e_4 \cup \{c_1, c_2, c_3, c_4\},$$

since K is compact.

We know that the edges and corners of K parametrise singular Boutroux curves, that is

$$\delta R \subseteq \Omega \cap \{\Delta(\alpha, \beta) = 0\}.$$

Thus the algebraic surface $\{\Delta(\alpha, \beta) = 0\}$ plays an important role in our analysis of them. We first study the corners of K which correspond to branching points of this algebraic surface. Then we derive aformentioned explicit parametrisations.

Let us first consider the corners of K. They correspond the merging of three turning points and are thus branching points of the algebraic surface $\{\Delta(\alpha, \beta) = 0\}$. Therefore the c_k, $1 \leq k \leq 4$, are roots of the discriminant of $\Delta(\alpha, \beta)$ w.r.t. β, and thus zeros of

$$C(\alpha) := \alpha^8 - 6(3\nu^2 + 1)\alpha^4 + 8(1 - 9\nu^2)\alpha^2 - 3(9\nu^4 + 6\nu^2 + 1). \tag{59}$$

Direct computation gives that the (finite) branching locus of $\{\Delta(\alpha, \beta) = 0\}$ is given by the discrete set

$$\{(\alpha, \beta) \in \mathbb{C}^2 : C(\alpha) = 0, \Delta(\alpha, \beta) = 0\} = \{(\alpha, \beta) \in \mathbb{C}^2 : C(\alpha) = 0, \beta = f(\alpha)\}, \tag{60}$$

where

$$f(\alpha) := -\frac{\alpha^2(\alpha^2 - 2) + 3\nu^2 + 1}{6\alpha}.$$

We therefore obtain the following relation,

$$\widetilde{c}_k = (c_k, f(c_k)) \quad (1 \leq k \leq 4). \tag{61}$$

It remains to be determined to which root of $C(\alpha)$ the corner $\widetilde{c}_k$ corresponds, for $1 \leq k \leq 4$. We have the following lemma concerning the roots of $C(\alpha)$.

Lemma 12. *The polynomial $C(\alpha)$ has precisely two real roots and two purely imaginary roots.*

Proof. Application of Descartes' rule of signs on $C(\pm i\alpha)$ immediately gives that $C(\alpha)$ has precisely two purely imaginary roots. Note that

$$C'(\alpha) = 8\alpha(\alpha^2 + 2)(\alpha^2 - (1 - 3\nu^2))(\alpha^2 - (1 + 3\nu^2))$$

and direct computation gives that $C(\alpha) < 0$ for each of the five real roots of $C'(\alpha)$. Since $C(\alpha) \sim 8\alpha^8$ as $\alpha \to \infty$, it follows that $C(\alpha)$ has precisely two real roots. ☐

By Lemma 12 and the fact that $C(\alpha)$ is real and symmetric, we may define u_k as the unique root of $C(\alpha)$ in the k-th quadrant of the complex plane $\{\alpha \in \mathbb{C}\}$ for $1 \leq k \leq 4$, so that

$$u_2 = -\overline{u}_1, \quad u_3 = -u_1, \quad u_4 = \overline{u}_1. \tag{62}$$

We also define v_k as the unique root of $C(\alpha)$ within $i^{k-1}\mathbb{R}_+$ so that

$$v_3 = -v_1, \quad v_4 = -v_2,$$

see Figure 11. Due to the symmetries studied in Lemma 3 we have

$$c_2 = -\overline{c_1}, \quad c_3 = -c_1, \quad c_4 = \overline{c_1}, \tag{63}$$

and therefore clearly

$$\{c_k : 1 \leq k \leq 4\} = \{u_k : 1 \leq k \leq 4\}. \tag{64}$$

We make the final identification $c_k = u_k$, $1 \leq k \leq 4$, while deriving the parametrisation of ∂K_a.

We turn our attention to ∂K_a and for convenience of the reader, first re-introduce the function $\psi(\alpha)$, given in equation (8), which we proof to parametrise it. Consider the quartic equation

$$3X^4 + 4\alpha X^3 + (\alpha^2 - 1)X^2 - \frac{\nu^2}{4} = 0. \tag{65}$$

Its branching points are given by the zeros of $C(\alpha)$. Let $x = x(\alpha)$ be the unique algebraic function which solves the quartic analytically in the complex α-plane with $x(\alpha) \sim \frac{\nu}{2}\alpha^{-1}$ as $\alpha \to \infty$ and branch-cuts the diagonals $[u_1, u_3]$ and $[u_2, u_4]$ plus $[v_1, v_3]$ and $[v_2, v_4]$. Since equation (65) is of fourth order in X, we can compute all its branches explicitly. In particular, for $\alpha \in (v_1, \infty)$ all the branches of (65) are real and only one is positive, namely $x(\alpha)$. Since $X = 0$ is never a root of (65), it follows that $x(\alpha)$ does not coincide with any other branch of (65) at $\alpha = v_1$, so $x(\alpha)$ is single-valued at $\alpha = v_1$. Similarly it follows that $x(\alpha)$ is single-valued at v_k for $2 \le k \le 4$ and thus $x(\alpha)$ is analytic on the α-plane merely cut along $[u_1, u_3]$ and $[u_2, u_4]$.

An analogous argument shows that there exists a unique algebraic function $y = y(\alpha)$ which solves

$$y^2 = \alpha^2 + 6x\alpha + 6x^3 - 1 \tag{66}$$

on the same cut α-plane with $y(\alpha) \sim \alpha$ as $\alpha \to \infty$. We set

$$\psi(\alpha) = \tfrac{1}{2}\Re\left[\alpha y + \tfrac{1}{2}(1-\nu)\log(p_1) - \log(p_2) + \nu\log(p_3)\right], \tag{67}$$

where

$$p_1 = 1 - 2x\alpha - 2x^2, \quad p_2 = 2x + \alpha + y, \quad p_3 = \frac{x(\alpha^2 + 5x\alpha + 4x^2 - 1) + \frac{1}{2}\nu y}{x^2}.$$

In the following proposition we justify the parametrisation of ∂K_a given in Section 2.

Proposition 5. *Considering the function $\psi(\alpha)$ defined in (67), the following hold true.*

(1) The function $\psi(\alpha)$ is a harmonic function on the α-plane cut along $[u_1, u_3]$ and $[u_2, u_4]$, and $\psi(\alpha) \to 0$ as α approaches any of the branch points u_k, $1 \le k \le 4$ within the cut plane.

(2) The zero set $\{\phi(\alpha) = 0\}$ takes the form depicted in Figure 10, namely it consists of the four roots u_k, $1 \le k \le 4$ of $C(\alpha)$, and eight (mutually disjoint) level curves, ϵ_k, l_k, $1 \le k \le 4$. From each u_k emanate three level curves, $\epsilon_k, \epsilon_{k+1}$ and l_k, with l_k going to infinity asymptotic to $e^{\frac{\pi i}{4}(2k-1)}\infty$, for $1 \le k \le 4$.

(3) For $1 \le k \le 4$, the internal radial angle between ϵ_k and ϵ_{k+1} at u_k equals $\frac{2}{5}\pi$.

(4) Define

$$B_\Delta(\alpha) = x(-2 + 4x^2 + 6x\alpha + 2\alpha^2), \tag{68}$$

where $x = x(\alpha)$ is the algebraic function introduced in (65), then B_Δ is an analytic function on the cut α-plane with a well-defined limiting value $B_\Delta(\alpha) = f(\alpha)$ at the branching points $\alpha = u_k$, $1 \le k \le 4$, where f is defined in (60), and for $1 \le k \le 4$, the corner $\widetilde{c}_k$ and edge $\widetilde{e}_k$ of K are parametrised by

$$c_k = u_k, \qquad \widetilde{c}_k = (u_k, B_\Delta(u_k)), \tag{69a}$$

$$e_k = \epsilon_k, \qquad \widetilde{e}_k = \{(\alpha, B_\Delta(\alpha)) : \alpha \in \epsilon_k\}. \tag{69b}$$

Remark 5. *If $\nu = \frac{1}{3}$, then $\psi(e^{\frac{\pi i}{4}}\alpha) = -\psi(\alpha)$ and l_k simply equals the straight diagonal line $\{u_k + te^{\frac{\pi i}{4}(2k-1)} : t \in \mathbb{R}_+\}$ for $1 \le k \le 4$.*

Before proving Proposition 5, let us show how it implies Lemma 11 and thus Proposition 4.

Proof of Lemma 11. Due to Proposition 5, we know that

$$\Pi_a(\delta R) = e_1 \sqcup e_2 \sqcup e_3 \sqcup e_4 \sqcup \{c_1, c_2, c_3, c_4\}$$

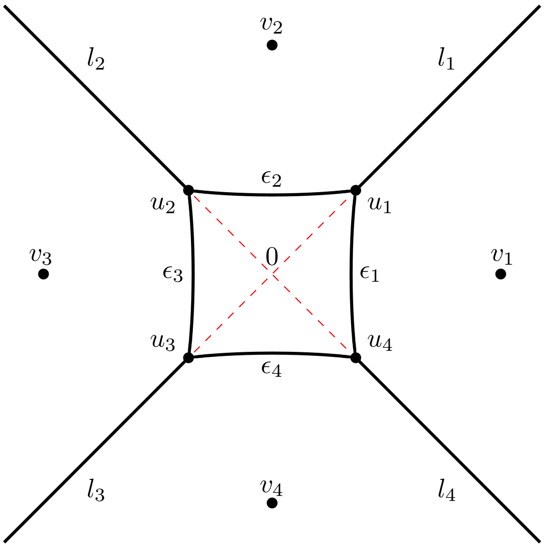


FIGURE 10. The level set $\{\psi(\alpha) = 0\}$ and zeros of the polynomial $C(\alpha)$, defined in (62), with the branch cuts $[u_1, u_3]$ and $[u_2, u_4]$ in dashed red.

is a Jordan curve in the α plane. Let us denote its interior by I. We know that R_a is connected, since R is connected, and that $\partial R_a \subseteq \Pi_a(\delta R)$, since K is compact. But there exists only one open non-empty connected set U with $\partial U \subseteq \Pi_a(\delta R)$, namely $U = I$. We conclude that $R_a = I$ equals the interior of the Jordan curve $\Pi_a(\delta R)$.

It follows in particular that $\Pi_a(\delta R) = \partial R_a$, thus $\Pi_a|_{\delta R} : \delta R \to \partial R_a$ is surjective. Furthermore it follows directly from part (4) of Proposition 5 that $\Pi_a|_{\delta R} : \delta R \to \partial R_a$ has a continuous inverse, namely B_Δ. The lemma follows. □

Proof of Proposition 5. Let us first recall Lemma 8, which states that

$$\delta R \subseteq \Omega \cap \{\Delta(\alpha, \beta) = 0\}.$$

Namely, on the border δR two or three turning points have coalesced, so that the resulting Boutroux curve $\widehat{\Gamma}$ is singular. To prove the proposition, we first describe a method to compute $\Omega \cap \{\Delta(\alpha, \beta) = 0\}$, and then we will specialise it to the subset of our interest, namely δR.

Suppose $(\alpha, \beta) \in \mathbb{C}^2$ is such that $\Delta(\alpha, \beta) = 0$, then there exist a unique turning point $\lambda = X$ of $V(\lambda; \alpha, \beta)$ which is not simple. Consider for the moment the generic case in which X is double and thus $V(\lambda)$ has two remaining simple turning points, say $x_{1,2}$. Then

$$V(\lambda; \alpha, \beta) = \lambda^{-2}(\lambda - x_1)(\lambda - x_2)(\lambda - X)^2,$$

yielding the following helpful relations,

$$3X^4 + 4\alpha X^3 + (\alpha^2 - 1)X^2 - \frac{\nu^2}{4} = 0, \tag{70a}$$

$$X(-2 + 4X^2 + 6X\alpha + 2\alpha^2) = \beta, \tag{70b}$$

$$x_1 + x_2 = -2(X + \alpha), \tag{70c}$$

$$(x_2 - x_1)^2 = 4(1 - 2X\alpha - 2X^2). \tag{70d}$$

Here we recognise equations (65) and (68) in (70a,70b).

Now clearly $\widehat{\Gamma}(\alpha,\beta)$ is a Boutroux curve if and only if

$$\Re\int_{x_*}^{X}\frac{\sqrt{(\lambda-x_1)(\lambda-x_2)(\lambda-X)}}{\lambda}d\lambda=0,$$

for any choice of $x_*\in\{x_1,x_2\}$ and choice of contour. This integral can be evaluated explicitly, yielding

$$\Psi=\tfrac{1}{2}\Re\left[\alpha y+\tfrac{1}{2}(1-\nu)\log(P_1)-\log(P_2)+\nu\log(P_3)\right], \tag{71}$$

with

$$P_1=\tfrac{1}{4}(x_2-x_1)^2,\quad P_2=X-\tfrac{1}{2}(x_1+x_2)+Y,\quad P_3=\frac{X(x_1x_2-\frac{1}{2}(x_1+x_2)X)+\frac{1}{2}\nu Y}{X^2},$$

where Y is a branch of

$$Y^2=(X-x_1)(X-x_2)=\alpha^2+6X\alpha+6X^3-1. \tag{72}$$

Firstly note that Ψ is invariant under interchanging of x_1 and x_2 and we may thus eliminate them from the formula using equations (70c,70d), yielding equation (67) with $x\to X$ and $y\to Y$.

In summary, we can compute $\Omega\cap\{\Delta(\alpha,\beta)=0\}$ as follows. Take a local branch $X=X(\alpha)$ of (70a) and choose the correct branch $Y=Y(\alpha)$ of (72) realising equation (71), define the algebraic function $\beta_\Delta(\alpha_0)$ by the left-hand side of (70b), then Ψ is (locally) a harmonic function in α and for any α_0 in its zero set we have

$$(\alpha_0,\beta_\Delta(\alpha_0))\in\Omega\cap\{\Delta(\alpha,\beta)=0\},$$

yielding a local parametrisation of $\Omega\cap\{\Delta(\alpha,\beta)=0\}$. By considering each of the four branches X of (70a) and following the above procedure we can in principle completely compute $\Omega\cap\{\Delta(\alpha,\beta)=0\}$. To prove the proposition this however will not be necessary.

In the case of our interest the relevant branch of (70a) is given by $X=x$, introduced in equation (65), and the corresponding correct branch of (72) is given by $Y=y$ introduced in (66). Let us first prove part (1) of the proposition. For α on the plane cut along $[u_1,u_3]$ and $[u_2,u_4]$ we have

$$V(\lambda;\alpha,B_\Delta(\alpha))=\lambda^{-2}(\lambda-x_1)(\lambda-x_2)(\lambda-X)^2,$$

for some up to permutation unique x_1 and x_2. Note that they cannot coincide on the cut plane and we may thus specify each uniquely by choosing $x_{1,2}=x_{1,2}(\alpha)$ analytically on the cut plane with

$$x_1=-\alpha-\sqrt{1-\nu}+\mathcal{O}(\alpha^{-1}),\quad x_2=-\alpha+\sqrt{1-\nu}+\mathcal{O}(\alpha^{-1}) \tag{73}$$

as $\alpha\to\infty$, due to equations (70c,70d).

Since $x_1\neq x_2$ on the cut plane it follows from equation (67), that the term p_1 in formula (71) does not vanish and thus $\Re\log(p_1)$ is finite and harmonic on the cut plane. Similarly, if $p_2=0$, then $Y^2=(X-\frac{1}{2}(x_1+x_2))^2$ which implies $x_1=x_2$. It follows that p_2 does not vanish and thus $\Re\log(p_2)$ is finite and harmonic on the cut plane. The same argument, using the identity $x_1x_2X^2=\frac{1}{4}\nu^2$, works for p_3 and we conclude that $\psi(\alpha)$ is a harmonic function on the entire cut plane.

The branching points of $X(\alpha)$ are characterised by the coalescing of one of the two simple turning points $\{x_1,x_2\}$ with the double turning point $X(\alpha)$: at $\alpha=u_1,u_4$ the turning point x_2 coalesces with X and at $\alpha=u_2,u_3$ the turning point x_1 coalesces with X.

The local behaviour of $\psi(\alpha)$ near the branching points is easily computed. Firstly, let us note that ψ has the following symmetries,

$$\psi(-\alpha)=\psi(\alpha),\quad \psi(\overline{\alpha})=\overline{\psi(\alpha)} \tag{74}$$

and we thus merely have to compute the behaviour near the branching point $\alpha = u_1$ where x_2 and X coalesce. By direct computation we have the corresponding Puiseux series

$$x_1 = \eta_1 + \mathcal{O}(\alpha - u_1), \tag{75}$$

$$x_2 = \eta_2 - 2r(\alpha - u_1)^{\frac{1}{2}} + \mathcal{O}(\alpha - u_1), \tag{76}$$

$$X = \eta_2 + r(\alpha - u_1)^{\frac{1}{2}} + \mathcal{O}(\alpha - u_1), \tag{77}$$

as $\alpha \to u_1$, where

$$\eta_1 = -3\eta_2 - 2u_1, \quad \eta_2 = \frac{1 + 3\nu^2 - u_1^4}{4u_1(2 + u_1^2)},$$

for a unique root r of $r^2 = -\frac{1}{3}\eta_2$. Therefore

$$\begin{aligned}\psi(\alpha) &= -\frac{4}{15}\Re\left[\frac{(x_2 - x_1)^{\frac{1}{2}}}{x_2}(X - x_2)^{\frac{5}{2}}\right](1 + \mathcal{O}(\alpha - u_1)^{\frac{1}{2}})\\ &= -\frac{4}{15}3^{\frac{5}{4}}\Re\left[\kappa(\alpha - u_1)^{\frac{5}{4}}\right](1 + \mathcal{O}(\alpha - u_1)^{\frac{1}{2}})\end{aligned} \tag{78}$$

where the branch of $z^{\frac{5}{4}}$ is taken real and positive on $\mathbb{R}_+$ with branch cut $\{\arg z = -\frac{3\pi i}{4}\}$, for a unique $\kappa = \kappa(\nu)$ which satisfies

$$\kappa^4 = -(\eta_2 - \eta_1)^2\eta_2. \tag{79}$$

In particular ψ vanishes near u_1 and due to symmetries (74) we know that ψ vanishes near the other branching points as well, finishing the proof of part (1) of the proposition.

We now turn our attention to the zero set $\{\psi(\alpha) = 0\}$. We first compute it locally near the branching point u_1. To this end, note that $\kappa = \kappa(\nu)$ in equation (78) is an algebraic function in ν. It is the unique branch of equation (79) satisfying $\arg\kappa(\frac{1}{3}) = \frac{3}{16}\pi$. By direct computation it can be shown that $\arg\kappa : (0, \frac{1}{3}] \to (\frac{1}{6}\pi, \frac{3}{16}\pi]$.

The right-hand side of (78) can be expanded into a complete Puiseux series and since the argument of κ is bounded by $\frac{1}{20}\pi < \arg\kappa < \frac{9}{20}\pi$, it follows that there are three level curves of $\{\psi(\alpha) = 0\}$ emanating radially from $\alpha = u_1$ with angles

$$\frac{4}{5}\left(\frac{1}{2}\pi - \arg\kappa + m\pi i\right) \quad (m \in \{-1, 0, +1\}). \tag{80}$$

Due to the symmetries (74), it follows that each branching point u_k has precisely three level curves emanating from it. We call them, going around u_k in anti-clockwise direction starting from the branch-cut, $\epsilon_k, l_k^*, \epsilon_{k+1}^*$. Note in particular that the radial angle at u_k: between ϵ_k and l_k^* equals $\frac{4}{5}\pi$, between l_k^* and ϵ_{k+1}^* equals $\frac{4}{5}\pi$ and between ϵ_{k+1}^* and ϵ_k equals $\frac{2}{5}\pi$, for $1 \leq k \leq 4$, due to equation (80).

Clearly

$$\psi(\alpha) = \tfrac{1}{2}\Re[\alpha^2] + \mathcal{O}(\log|\alpha|) \quad (\alpha \to \infty)$$

and it is relatively straightforward to show that there are precisely four level curves $\{\psi(\alpha) = 0\}$ emanating from $\alpha = \infty$ along the asymptotic directions $e^{\frac{\pi i}{4}(2k-1)}\infty$, $1 \leq k \leq 4$. We call these level curves l_k, $1 \leq k \leq 4$, in accordance with Figure 10.

The final piece of information to deduce the correctness of Figure 10 is that $\psi(\alpha)$ is strictly monotonic along the branch cuts: for $k = 1, 2$ the lower (upper) branch of ψ along $[0, u_k]$ is strictly increasing (decreasing) as α traverses along it in the direction of u_k whereas for $k = 3, 4$ the lower (upper) branch of ψ along $[0, u_k]$ is strictly decreasing (increasing) as α traverses along it in the direction of u_k. Indeed this implies that both the upper and lower branches of ψ along the branch cuts are nonzero, except for at the branching points $u_k, 1 \leq k \leq 4$. Thus the $\epsilon_k, \epsilon_k^*, l_k, l_k^*$ make up all the level curves of $\{\psi(\alpha) = 0\}$, and since ψ is harmonic, it is easy to deduce that we must have $l_k = l_k^*$ and $\epsilon_k = \epsilon_k^*$ for $1 \leq k \leq 4$, where $\epsilon_1^* := \epsilon_5^*$, yielding part (2) of the proposition. Furthermore, since $\epsilon_k = \epsilon_k^*$, it follows that the internal radial angle between ϵ_k and ϵ_{k+1} at u_k equals $\frac{2}{5}\pi$, for $1 \leq k \leq 4$, establishing part (3).

Due to Lemma 15, we know that the isomorphism class of the Stokes complex $\mathcal{C}(\alpha, B_\Delta(\alpha))$ is constant along each of the level curves ϵ_k, l_k, $1 \leq k \leq 4$. To prove part (4), we have to determine the isomorphism class on each level curve. We first compute the isomorphism classes on the l_k, $1 \leq k \leq 4$, after which the isomorphism classes at the branching points u_k and curves ϵ_k, $1 \leq k \leq 4$, are easily deduced.

We proceed in computing the isomorphism class on l_1. Firstly, due to Lemma 13, we know that the inner Stokes complex is connected along l_1. By equations (73), setting $\lambda = -\alpha + t$ with $t = \mathcal{O}(1)$ and $x_k = -\alpha + \widetilde{x}_k$ for $k = 1, 2$, we have

$$V(\lambda) = \widetilde{V}(t)(1 - \mathcal{O}(\alpha^{-1})), \quad \widetilde{V}(t) := (t - \widetilde{x}_1)(t - \widetilde{x}_2),$$

as $\alpha \to e^{\frac{\pi}{4}i}\infty$ along l_1. Furthermore note that $\widetilde{X} = X + \alpha$ and $\widetilde{0} = 0 + \alpha$ are asymptotic to $e^{\frac{\pi}{4}i}\infty$ in the same limit. The Stokes complex of the leading order potential $\widetilde{V}(t)$ is depicted in Figure 11 under $\widetilde{L}_1$.

Similarly, setting $\lambda = |\alpha|^{-1}s$ with $s = \mathcal{O}(1)$ and writing $\widehat{X} = |\alpha|^{-1}X$, we have

$$V(\lambda) = \widehat{V}(s)(1 - \mathcal{O}(\alpha^{-1})), \quad \widehat{V}(s) := i\frac{(s - \widehat{X})^2}{s^2},$$

as $\alpha \to e^{\frac{\pi}{4}i}\infty$ along l_1. Furthermore $\widehat{x}_{1,2} = |\alpha|x_{1,2}$ are asymptotic to $e^{-\frac{3\pi}{4}i}\infty$ in the same limit. The Stokes complex of the leading order potential $\widehat{V}(s)$ is depicted in Figure 11 under $\widehat{L}_1$.

There is only one isomorphism class consistent with above two limiting behaviours, namely the class L_1 defined in Figure 11. Thus on l_1 the Stokes complex falls in the class L_1.

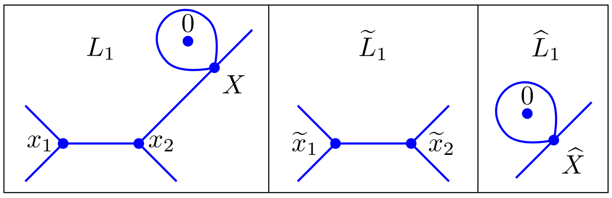


FIGURE 11. Isomorphism classes of Stokes complexes L_1, $\widetilde{L}_1$ and $\widehat{L}_1$ corresponding respectively to the potentials $V(\lambda; \alpha, B_\Delta(\alpha))$ with $\alpha \in l_1$, $\widetilde{V}(t)$ with $\alpha = e^{\frac{\pi i}{4}}\infty$ and $\widehat{V}(s)$ with $\alpha = e^{\frac{\pi i}{4}}\infty$.

As $\alpha \to u_1$ along l_1, we know that x_2 merges with X, and thus the resulting Stokes complex of $V(\lambda)$ at $(\alpha, \beta) = (u_1, B_\Delta(u_1))$ is given by C_1, defined in Figure 9. We conclude that $c_1 = u_1$ and hence $\widetilde{c}_1 = (u_1, B_\Delta(u_1))$ by equation (61). Using the symmetries (63) we obtain equation (69a) for all $1 \leq k \leq 4$.

Next we consider the Stokes complex of $V(\lambda; \alpha, B_\Delta(\alpha))$ along ϵ_2. Again we know it's inner Stokes complex must be connected due to Lemma 13. Furthermore, we have

- as $\alpha \to u_1 = c_1$ along ϵ_2, the turning point x_2 merges with X and the resulting Stokes complex falls in the class C_1;
- as $\alpha \to u_2 = c_2$ along ϵ_2, the turning point x_1 merges with X and the resulting Stokes complex falls in the class C_2;
- The isomorphism class of the Stokes complex is invariant under reflection in the imaginary axes (and interchanging of marked points $\infty_1 \leftrightarrow \infty_2$ and $\infty_3 \leftrightarrow \infty_4$).

Indeed the third easily follows from the symmetries given in equation (74). Clearly, there is only one isomorphism class of Stokes complexes which satisfies these three conditions, namely E_2 defined in Figure 9, with $\mu_{1,2} = x_{1,2}$ and $\mu_3 = X$. Thus the Stokes complex along ϵ_2 falls in the class E_2 and we have

$$\epsilon_2 \subseteq e_2, \quad \{(\alpha, B_\Delta(\alpha)) : \alpha \in \epsilon_2\} \subseteq \widetilde{e}_2. \tag{81}$$

Since $\widetilde{e}_2$ is a smooth non-self intersecting curve with end-points $\widetilde{c}_1$ and $\widetilde{c}_2$, and the same holds true for $\{(\alpha, B_\Delta(\alpha)) : \alpha \in \epsilon_2\}$, equation (81) implies $\{(\alpha, B_\Delta(\alpha)) : \alpha \in \epsilon_2\} = \widetilde{e}_2$ and thus $\epsilon_2 = e_2$. Analogously we prove equation (69b) for the remaining $k \in \{1, 3, 4\}$ and with part (4) thus established, the proposition is proven. □

We define

$$\mathcal{S}_a = \mathcal{S} \circ \Pi_a^{-1} : K_a \to Q,$$

which is a homeomorphism that maps R_a diffeomorphically onto Q°. We may now justifiably call e_k,$1 \leq k \leq 4$ and c_k,$1 \leq k \leq 4$ the edges and corners of K_a. Note that $\mathcal{S}_a$ maps c_k to $\widehat{c}_k$ and e_k onto $\widehat{e}_k$, see Figure 4.

5. Bulk Asymptotic

In the present section we use the WKB analysis of the equation (4) to provide an asymptotic formula for the roots of the generalised Hermite polynomials. The reader should recall, see e.g. the Section 2, the definition of the domains K, K_a (i.e. the projection of K on the α-plane, $K_a = \Pi_a(K)$), the function $\mathcal{S}$ and the rectangle $Q = \left[-\frac{1}{2}(1-\nu)\pi, \frac{1}{2}(1-\nu)\pi\right] \times [-\nu\pi, \nu\pi]$, which is the image of K under $\mathcal{S}$.

In order to state and prove our results, for sake of simplicity we choose $p \geq q$, p, q either equal or co-prime, and thus fix a ratio $\frac{m}{n} = \frac{p}{q}$. Hence, in the whole section, the numbers m, n take values in the sequences $m = xq, n = xp, x \in \mathbb{N}$. Correspondingly $\nu = \frac{p}{2q+p} \in (0, \frac{1}{3}]$ is fixed and the large parameter E belongs to the sequence $(2q+p)x$. However, all results are essentially unchanged if we let ν vary on $[\nu_0, \frac{1}{3}]$, for some fixed $\nu_0 > 0$.

Definition 12. *Given an integer number $m \in \mathbb{N}^*$ we let $I_m = \{-m+1, -m+3, \ldots, m-1\} \subseteq \mathbb{Z}$. For each $j \in I_m, k \in I_n$, we let $(\alpha_{j,k}, \beta_{j,k}) \in K$ be the unique solution of $S(\alpha, \beta) = (\frac{\pi j}{E}, \frac{\pi k}{E})$.*

Definition 13. *A filling fraction is a real number $\sigma \in (0,1)$. We let $I_m^\sigma = I_m \cap [\sigma(-m+1), \sigma(m-1)]$ and define $Q^\sigma \subset Q$ as the closed rectangle $[-\frac{\pi\lfloor\sigma(m-1)\rfloor}{E}, \frac{\pi\lfloor\sigma(m-1)\rfloor}{E}] \times [-\frac{\pi\lfloor\sigma(n-1)\rfloor}{E}, \frac{\pi\lfloor\sigma(n-1)\rfloor}{E}]$, which in the large E limit converges to $\sigma \cdot Q$.*

Finally we define $K^\sigma = \mathcal{S}^{-1}(Q^\sigma)$ and $K_a^\sigma = \Pi_a(K^\sigma)$ as the projection of K^σ on the α-plane.

Our main result is the following asymptotic characterisation of the solutions (α, β) of the inverse monodromy problem characterising the roots of generalised Hermite polynomials.

Theorem 7. *Fix $\sigma \in (0,1)$. Then there exists an $R_\sigma > 0$ such that for E large enough the following hold true:*

(1) In each ball of centre $(\alpha_{j,k}, \beta_{j,k})$, $(j,k) \in I_m^\sigma \times I_n^\sigma$ and radius $R_\sigma E^{-2}$ there exists a unique point (α, β) such that the anharmonic oscillator (4) satisfies the inverse monodromy problem characterising the roots of generalised Hermite polynomials.

(2) In the ϵ neighbourhood of K^σ with radius $R_\sigma E^{-2}$, there are exactly $\lfloor\sigma m\rfloor \times \lfloor\sigma n\rfloor$ points (α, β) such that the anharmonic oscillator (4) satisfies the inverse monodromy problem characterising the roots of generalised Hermite polynomials.

The proofs of the Theorem 2 and of the Corollary 1 stated in the Results Section follows directly from the above Theorem.

Proof of Theorem 2. A point $a \in \mathbb{C}$ is a root of $H_{m,n}$ if and only if there exists β such that $(E^{-\frac{1}{2}}a, \beta)$ provides a solution to the inverse monodromy problem for the scaled oscillator (4). Since the projection $\Pi_a : K^\sigma \to K^\sigma_a$ is a local diffeomorphism (Proposition 4), the thesis follows from Theorem 7. □

Proof of Corollary 1. By the above Theorem, we need to solve $S(\alpha_{j,k}, \beta_{j,k}) = (\frac{\pi j}{E}, \frac{\pi k}{E})$ for j, k bounded. Since j, k are bounded, up to a uniform (and immaterial) $O(E^{-2})$ error we can solve the above equation using the first order Taylor expansion of $\mathcal{S}$ at $(\alpha, \beta) = (0, 0)$ (recall that $\mathcal{S}(0, 0) = (0, 0)$). More precisely, if we let $(J)_{ij}$ be the Jacobian of $\mathcal{S}$ at $(\alpha, \beta) = 0$, we have that $\alpha_{j,k}$ is, up to the $O(E^{-2})$ error, the first component of the solution of the linear equation $J.(\alpha, \beta) = (\frac{\pi j}{E}, \frac{\pi k}{E})$. That is

$$\alpha_{j,k} = (\det J)^{-1} \frac{\pi i}{E} \big(J_{22} j - J_{12} k\big) + O\big(E^{-2}\big) \,.$$

The thesis follows from the above formula when substituting the actual values of $(J)_{ij}$ obtained by computing the relevant elliptic integrals. □

In order to prove Theorem 7 we will use the multidimensional Rouché theorem, which we state below

Theorem 8 ([1]). *Let D, E be bounded domains in $\mathbb{C}^n$, $\overline{D} \subset E$, and let $f(z), g(z)$ be holomorphic maps $E \to \mathbb{C}^n$ such that*

- $f(z) \neq 0$, $\forall z \in \partial D$,
- $|g(z)| < |f(z)|$, $\forall z \in \partial D$,

then $w(z) = f(z) + g(z)$ and $f(z)$ have the same number (counted with multiplicities) of zeroes inside D. Here $|f(z)|$ is any norm on $\mathbb{C}^n$.

Proof of Theorem 7. The proof is essentially an improved version of the proof of the analogous result for poles of the Tritronquee solution of Painleve I [20].

For sake of definiteness, we use on $\mathbb{C}^2$ the euclidean norm and we denote it by $\| \, \|$.

We denote by $\widetilde{K}^\sigma$ the epsilon neighbourhood of K^σ where the radius epsilon is $R_\sigma E^{-2}$, for some $R_\sigma > 0$ to be determined.

Since $\widetilde{K}^\sigma$ is shrinking to a compact subset of the interior of K, namely K^σ, it follows that $\widetilde{K}^\sigma$ eventually possesses the W property as per Definition 6. Therefore by Theorem 3 and Theorem 4, for E large enough the solutions of the inverse monodromy problem, restricted to $\widetilde{K}^\sigma$, coincide with the zeros of the function

$$w : \widetilde{K}^\sigma \to \mathbb{C}^2 \,, \; w = (Wr[\chi_+, \psi_0], Wr[\psi_1^R, \psi_1^L]) \,.$$

Here the solutions $\chi_+, \psi_0, \psi_1^R, \psi_1^L$ of equation (4) are defined in Theorem 3 and Theorem 4.

Moreover since $|e^{E \oint_{\gamma_i} \sqrt{V}}| = 1$ for all $(\alpha, \beta) \in K^\sigma$, for any fixed R_σ we can find an E_σ such that for all $E \geq E_\sigma$,

$$|e^{\oint_{\gamma_i} \sqrt{V}}| \leq 2 \quad \forall (\alpha, \beta) \in \widetilde{K}^\sigma. \tag{82}$$

We conclude, after Theorem 3 and Theorem 4, that there exist C_σ, E_σ such that for all $E \geq E_\sigma$, the following estimates hold,

$$\left| Wr[\chi_+, \psi_0] + 1 + e^{E \oint_{\gamma_1} \sqrt{V}} \right| \leq \frac{C_\sigma}{E} \quad \forall (\alpha, \beta) \in \widetilde{K}^\sigma, \tag{83}$$

$$\left| Wr[\psi_1^R, \psi_1^L] + 1 + e^{E \oint_{\gamma_2} \sqrt{V} - i\pi n} \right| \leq \frac{C_\sigma}{E} \quad \forall (\alpha, \beta) \in \widetilde{K}^\sigma. \tag{84}$$

We define

$$f : \widetilde{K}^\sigma \to \mathbb{C}^2, \; f = -(e^{E \oint_{\gamma_1} \sqrt{V}} + 1, e^{E \oint_{\gamma_2} \sqrt{V} - i\pi n} + 1) = -(e^{iEs_1(\alpha,\beta) - i\pi m} + 1, e^{iEs_2(\alpha,\beta) - i\pi n} + 1)$$

and notice that its zeros coincide with the points $(\alpha_{j,k}, \beta_{j,k}), (j,k) \in I_m^\sigma \times I_n^\sigma$.

In order to prove the thesis by means of the Rouché theorem, it is sufficient to find an R_σ such that

(1) $\forall (j,k) \in I_m^\sigma \times I_n^\sigma$, $\|f\| > \sqrt{2}E^{-1}C_\sigma$ on $\Sigma_{R_\sigma}^{(j,k)}$, where $\Sigma_{R_\sigma}^{(j,k)}$ is the sphere of centre $(\alpha_{j,k}, \beta_{j,k})$, $j \in I_m^\sigma, k \in I_n^\sigma$ and radius $R_\sigma E^{-2}$;
(2) $\|f\| > \sqrt{2}E^{-1}C_\sigma$ on the boundary of $\widetilde{K}^\sigma$.

We do this by proving the corresponding estimates on the images of these sets under $\mathcal{S}$.

We choose the co-ordinates $(\hat{x}, \hat{y})$ on Q^σ and we embed Q^σ in $\mathbb{C}^2$ by allowing $\hat{x}, \hat{y}$ to be complex numbers. We define the function $\hat{f} = -(e^{iE\hat{x}-i\pi m}, e^{iE\hat{y}-i\pi n})$ and notice that $\hat{f} = f \circ \mathcal{S}$.

We begin by proving (1). The following statement follows from a simple computation: if $Y > 0$ is big enough then for each $Y' > Y$ and each $(j,k) \in I_m \times I_n$, on the spherical shell $\widehat{B}(j,k)_{Y,Y'} \subset \mathbb{C}^2$ of centre $(\frac{\pi j}{E}, \frac{\pi k}{E})$, internal radius radius YE^{-2} and external radius $Y'E^{-2}$, the estimate $\|\hat{f}\| > \sqrt{2}E^{-1}C_\sigma$ is satisfied for E big enough. Let us then choose a $Y > 0$ for which the above property holds and consider $S^{-1}(\widehat{B}(j,k)_{Y,Y'})$ for $(j,k) \in I_m^\sigma \times I_n^\sigma$ and some $Y' > Y$. Denoting $J(j,k)$ the Jacobian of $\mathcal{S}$ at $(\alpha_{j,k}, \beta_{j,k})$ and $A(j,k)$ its inverse (recall that S restricted to K^σ is a diffeomorphism), we have that -up to a negligible $O(E^{-4})$ contribution- the counter-image of a sphere of radius $E^{-2}Y$ is just the ellipsoid with semi-axis $\sqrt{\lambda_i(j,k)}YE^{-2}, i = 1,2$ where $\lambda_i(j,k) > 0$'s are eigenvalues of the matrix $A^\dagger(j,k)A(j,k)$. Since the compact set K^σ is a subset of the open domain $R = K \setminus \partial K$ where $\mathcal{S}$ is a diffeomorphism, the eigenvalues λ_i's are uniformly bounded. It means that there is a R_σ and a $Y' > Y$ such that $\Sigma_{R_\sigma}^{(j,k)} \subset \mathcal{S}^{-1}(\widehat{B}(j,k)_{Y,Y'})$ for each $(j,k) \in I_m^\sigma \times I_n^\sigma$. Hence $\|f\| > \sqrt{2}E^{-1}C_\sigma$ for each of such $\Sigma_{R_\sigma}^{(j,k)}$, and thus the thesis is proven.

Part (2) can be proven similarly. For Y, E big enough, in the boundary of the ε neighbourhood of Q^σ with radius $E^{-2}Y$ the estimate $\|\hat{f}\| > \sqrt{2}E^{-1}C_\sigma$ is satisfied. By the same reasoning as above, we can find a R_σ such that the estimate $\|f\| > \sqrt{2}E^{-1}C_\sigma$ is satisfied on the boundary of $\widetilde{K}^\sigma$.

□

Remark 6. *The Theorems 7,2,1 hold true unchanged if we let ν vary on the interval $[\nu_0, \frac{1}{3}]$ for some fixed $\nu_0 > 0$. By this we mean that the constant R_σ in the theorems above can be chosen to be independent on ν.*

Appendix A. Elliptic Integrals

In this appendix we collect a number of explicit formulae for the elliptic integrals under consideration in the paper, which can be derived using standard elliptic function theory. The formulae are explicitly given in terms of the zeros of $V(\lambda; \alpha, \beta)$. For (α, β) close to $(0,0)$ the zeros $\lambda_k = \lambda_k(\alpha, \beta)$ of $V(\lambda; \alpha, \beta)$ do not coalesce and are analytic in (α, β). They are determined by equations (49), up to permutation, and we fix them unambiguously by the initial conditions (43) at $(\alpha, \beta) = (0,0)$.

Firstly, we have the following explicit formulas for s_1 and s_2,

$$s_1 = +\frac{2i}{\sqrt{(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_2)}} F(\lambda_1, \lambda_2, \lambda_3, \lambda_4) + \frac{1}{2} i\pi(1 - \nu),$$
$$s_2 = -\frac{2}{\sqrt{(\lambda_4 - \lambda_3)(\lambda_2 - \lambda_1)}} F(\lambda_4, \lambda_1, \lambda_2, \lambda_3) + i\pi\nu,$$

with

$$
\begin{aligned}
F(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = &-\tfrac{1}{4}(\lambda_4-\lambda_2)(\lambda_3-\lambda_2)(3\lambda_1-\lambda_2+\lambda_3+\lambda_4)\mathcal{K}(m) \\
&+\tfrac{1}{4}(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\mathcal{E}(m) \\
&+(\lambda_4-\lambda_2)\Pi(n_1,m)+2\lambda_1\lambda_3(\lambda_4-\lambda_2)\Pi(n_2,m),
\end{aligned}
$$

where $\mathcal{K}(m)$, $\mathcal{E}(m)$ and $\Pi(n,m)$ denote the standard complete elliptic integrals of the respective first, second and third kind with parameter $m=k^2$ and elliptic characteristic n, in the above formula equal to

$$
m=\frac{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_2)},\quad n_1=-\frac{\lambda_4-\lambda_3}{\lambda_3-\lambda_2},\quad n_2=-\frac{(\lambda_4-\lambda_3)\lambda_2}{(\lambda_3-\lambda_2)\lambda_4}.
$$

The formula for s_1 holds for (α,β) close to $(0,0)$ with all branches chosen principal, and is globally correct (on an open environment of R) via appropriate analytic continuation. The formula for s_2 holds for (α,β) close to $(0,0)$, with all branches chosen principal except the one for $\Pi(n_2,m)$, namely

$$
\Pi(n_2,m)=\begin{cases}\Pi^{(p)}(n_2,m) & \text{if } \Im n_2>0,\\ \Pi^{(p)}(n_2,m)+\frac{i\pi}{\sqrt{(n_2-1)(1-m/n_2)}} & \text{if } \Im n_2\le 0,\end{cases}
$$

where $\Pi^{(p)}(n_2,m)$ denotes the principal branch, so that $\Pi(n_2,m)$ is analytic in n_2 on an open environment of $(1,\infty)$ for $m\in\mathbb{C}\setminus[1,\infty)$. It is globally correct (on an open environment of R) via appropriate analytic continuation.

Considering the Jacobian

$$
J_{(s_1,s_2)}(\alpha,\beta)=\begin{pmatrix}\frac{\partial s_1}{\partial\alpha} & \frac{\partial s_1}{\partial\beta}\\ \frac{\partial s_2}{\partial\alpha} & \frac{\partial s_2}{\partial\beta}\end{pmatrix} \tag{85}
$$

we have the following explicit expressions

$$
\begin{aligned}
\frac{\partial s_1}{\partial\alpha} &= +\frac{2i}{\sqrt{(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)}}\left[(\lambda_3-\lambda_2)(\lambda_4-\lambda_1)\mathcal{E}(m_{11})-(\lambda_1\lambda_2+\lambda_3\lambda_4)\mathcal{K}(m_{11})\right],\\
\frac{\partial s_2}{\partial\alpha} &= -\frac{2}{\sqrt{(\lambda_4-\lambda_3)(\lambda_2-\lambda_1)}}\left[(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)\mathcal{E}(m_{21})-(\lambda_2\lambda_3+\lambda_1\lambda_4)\mathcal{K}(m_{21})\right],\\
\frac{\partial s_1}{\partial\beta} &= -\frac{2i}{\sqrt{(\lambda_3-\lambda_1)(\lambda_4-\lambda_2)}}\mathcal{K}(m_{12}),\\
\frac{\partial s_2}{\partial\beta} &= +\frac{2}{\sqrt{(\lambda_3-\lambda_1)(\lambda_4-\lambda_2)}}\mathcal{K}(m_{22}),
\end{aligned}
$$

with parameters

$$
m_{12}=\frac{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_2)},\quad m_{22}+m_{12}=1,\quad m_{11}+m_{22}^{-1}=1,\quad m_{11}m_{21}=1,
$$

which holds near $(\alpha,\beta)=(0,0)$ with all branches taken principally, and holds globally via appropriate analytic continuation. In particular

$$
|J_{(s_1,s_2)}(\alpha,\beta)|\equiv 2\pi i \tag{86}
$$

due to Legendre's identity.

Appendix B. Boutroux Curves

We have the following equivalent characterisations of Boutroux curves.

Lemma 13. *Let $(\alpha,\beta)\in\mathbb{C}^2$, then the following are equivalent:*

(i) $\widehat{\Gamma}(\alpha,\beta)$ is a Boutroux curve,
(ii) each external vertex of the Stokes complex $\mathcal{C}(\alpha,\beta)$ has valency one,
(iii) the inner Stokes complex corresponding to $V(\lambda;\alpha,\beta)$ is connected.

Proof. Let us note that (iii) trivially implies (i) and it is easy to see that (ii) implies (iii) using the fact that the Stokes complex is connected, see Proposition 2. It remains to be shown that (i) implies (ii).

Suppose $\widehat{\Gamma}(\alpha,\beta)$ is a Boutroux curve. By Proposition 2, each external vertex has valency at least one. Suppose the valency of any particular external vertex, say $e^{\frac{1}{4}(2k-1)\pi i}$, is greater than one. Then there exist distinct Stokes lines l_1 and l_2, both asymptotic to $e^{\frac{1}{4}(2k-1)\pi i}\mathbb{R}_+$, such that

$$\Re\int_{\Lambda_1}^{\Lambda_2}\sqrt{V(\lambda)}d\lambda=0, \tag{87}$$

for any choice of $\Lambda_1\in l_1$, $\Lambda_2\in l_2$ and choice of connecting contour. Let $C,R>0$ be such that

$$|\sqrt{V(\lambda)}-\lambda|\leq C,$$

for $|\lambda|>R$. Then, for any $\Lambda_1\in l_1$ and $\Lambda_2\in l_2$ with $|\Lambda_1|,|\Lambda_2|>R$, equation (87) with a choice of contour lying in $\{|\lambda|>R\}$, implies

$$|\Re(\Lambda_2^2-\Lambda_1^2)|\leq 2C|\Lambda_2-\Lambda_1|. \tag{88}$$

We choose $\Lambda_j=re^{i\theta_i(r)}\in l_j$ for $r\gg 0$, with $\theta_j(r)\sim\frac{1}{4}(2k-1)\pi$ as $r\to+\infty$, for $j\in 1,2$. Note that $\theta_1(r)-\theta_2(r)\notin 2\pi\mathbb{Z}$ for any r. Now inequality (88) translates to

$$r\leq 2C\frac{|e^{i(\theta_2(r)-\theta_1(r))}-1|}{|\cos{(2\theta_2(r))}-\cos{(2\theta_1(r))}|},$$

but the right-hand side converges to C as $r\to+\infty$ and we have arrived at a contradiction. We infer that each external vertex must has valency one, which completes the proof of the lemma. ☐

We proceed with discussing deformations of Stokes complexes, and in particular Boutroux curves. Firstly we consider generic deformations of Stokes lines in the following lemma.

Lemma 14 (deformations of Stokes lines)**.** *Let $(\alpha^*,\beta^*)\in\mathbb{C}^2$ and consider the Stokes complex of $V(\lambda;\alpha^*,\beta^*)$. Let μ^* be a turning point of $V(\lambda;\alpha^*,\beta^*)$ and let l^* be a Stokes line with μ^* as one of its endpoints. Let $\gamma_0:\mathbb{R}_{\geq 0}\to\mathbb{C}$ be a homeomorphism onto $l^*\cup\{\mu^*\}$ such that its restriction to $\mathbb{R}_+$ is a diffeomorphism onto l^*. Then, for any $0<T<\infty$, there exists*

- *a simply connected open environment $W\subseteq\mathbb{C}^2$ of (α^*,β^*),*
- *an analytic function $\mu(\alpha,\beta)$ on W with $\mu(\alpha^*,\beta^*)=\mu^*$, and*
- *a smooth mapping $\gamma:W\times[0,T)\to\mathbb{C},(\alpha,\beta,t)\mapsto\gamma_{(\alpha,\beta)}(t)$,*

such that $\gamma_{(\alpha^*,\beta^*)}([0,T))=\gamma_0([0,T))$, and, for all $(\alpha,\beta)\in W$,

- *$\mu(\alpha,\beta)$ is a turning point of $V(\lambda;\alpha,\beta)$,*
- *$\gamma_{(\alpha,\beta)}(0)=\mu(\alpha,\beta)$, and*
- *$\gamma_{(\alpha,\beta)}:[0,T)\to\mathbb{C}$ is, when restricted to $(0,T)$, a diffeomorphism onto part of a unique Stokes line $l=l(\alpha,\beta)$ of $V(\lambda;\alpha,\beta)$.*

Furthermore, if l^* is asymptotic to $e^{\frac{1}{4}(2k-1)\pi i}\mathbb{R}_+$ for some $k\in\mathbb{Z}_4$, then the above also holds for $T=+\infty$, so that $\gamma_{(\alpha,\beta)}:[0,\infty)\to\mathbb{C}$ is, when restricted to $(0,\infty)$, a diffeomorphism onto a unique Stokes line $l=l(\alpha,\beta)$ of $V(\lambda;\alpha,\beta)$ asymptotic to $e^{\frac{1}{4}(2k-1)\pi i}\mathbb{R}_+$, for all $(\alpha,\beta)\in W$.

Proof. The proof is a straightforward but technical exercise in analysis. We leave it to the interested reader. □

Lemma 15 (generic deformations of Boutroux curves)**.** *Let T be a both locally and globally simply connected metric space, together with a continuous mapping*

$$(\alpha, \beta) : T \to \mathbb{C}^2, t \mapsto (\alpha(t), \beta(t))$$

such that $(\alpha(t), \beta(t)) \in \Omega$ for all $t \in T$. Further suppose that turning points of $V(\lambda; \alpha(t), \beta(t))$ do not merge or split on T, i.e. there exists an $m \in \{2, 3, 4\}$ and continuous functions $\mu_j : T \to \mathbb{C}$, for $1 \leq j \leq m$, such that $\{\mu_j(t) : 1 \leq j \leq m\}$ are the turning points of $V(\lambda; \alpha(t), \beta(t))$ and $|\{\mu_j(t) : 1 \leq j \leq m\}| = m$ for all $t \in T$.
Then the isomorphism class of the Stokes complex $\mathcal{C}(\alpha(t), \beta(t))$ is constant on T.

Proof. Note that it is enough to show that, for every $t^* \in T$, there exists an open environment $B \subseteq T$ of t^* such that the Stokes complex $\mathcal{C}(\alpha(t), \beta(t))$ is isomorphic to $\mathcal{C}(\alpha(t^*), \beta(t^*))$ for $t \in B$.

Let $t^* \in T$ and write $(\alpha^*, \beta^*) = T(t^*)$. We denote $V^*(\lambda) = V(\lambda; \alpha^*, \beta^*)$ and $V_t(\lambda) = V(\lambda; \alpha(t), \beta(t))$ for $t \in T$. Let us pick one of the turning points, say $\mu_j(t)$, $1 \leq j \leq m$. It is convenient to number the Stokes lines of $V_t(\lambda)$ emanating from $\mu_j(t)$ uniquely. Let $r_j \geq 1$ be the degree of the turning point and $l_1^*, \ldots, l_{r_j+2}^*$ be the Stokes lines of $V^*(\lambda)$ emanating from μ_j^*[4]. We may choose a unique $\theta_1^* \in [0, \frac{1}{r_j+2}2\pi)$, write $\theta_s^* = \theta_1^* + \frac{s-1}{r_j+2}2\pi$, and order the Stokes lines such that l_s^* emanates from μ_j^* with angle θ_s^* for $1 \leq s \leq r_j + 2$. It is now easy to see that there exists a unique continuous mapping $\theta_1 : T \to \mathbb{R}$ with $\theta_1(t^*) = \theta_1^*$, so that, for every $t \in T$ and $1 \leq s \leq r_j + 2$, there exists a unique Stokes line $l_s(t)$ of $V_t(\lambda)$ which emanates from $\mu_j(t)$ with angle $\theta_s = \theta_1(t) + \frac{s-1}{r_j+2}2\pi$. We call $l_s(t)$ the continuous extension to T of l_s^* with respect to the turning point μ_j^* and angle θ_j^*. Note that every Stokes line of $V^*(\lambda)$ has a unique continuous extension to T with respect to every turning point from which it emanates for every of the angles by which it does so. In particular, a priori it is not excluded that an internal Stoke line might have two different continuous extensions to T.

Let $1 \leq s \leq r_j + 2$. Suppose l_s^* is asymptotic to $e^{\frac{1}{4}(2k-1)\pi i}\mathbb{R}_+$ for a $k \in \mathbb{Z}_4$, then we may apply Lemma 14 to find an open environment $B_{j,s} \subseteq T$ of t^* such that the Stokes line $l_s(t)$ is asymptotic to $e^{\frac{1}{4}(2k-1)\pi i}\mathbb{R}_+$ for all $t \in B_{j,s}$.

Suppose instead that l_s^* is an internal Stokes line and has endpoints $\{\mu_j^*, \mu_k^*\}$. It might be that $\mu_j^* = \mu_k^*$, in which case there is an $1 \leq s' \leq r_j + 2$ with $s' \neq s$ such that $l_s^* = l_{s'}^*$. Regardless of this, let $\theta_\diamond^* \in [0, 2\pi)$ by the angle by which l_s^* emanates from μ_k^* and denote by $l_\diamond(t)$ the unique continuous extension of l_s^* to T with respect to μ_k^* and angle $\theta_\diamond^*$. We wish to show that there exists an open environment $B_{j,s} \subseteq T$ of t^* such that $l_s(t) = l_\diamond(t)$, and thus $l_s(t)$ is an internal Stokes line with endpoints $\{\mu_j(t), \mu_k(t)\}$, for all $t \in B_{j,s}$.

Assume, for the sake of contradiction, that such a set does not exist. Then there exists a sequence $(t_n)_{n\geq 1}$ in T with $t_n \to t^*$ as $n \to \infty$ such that $l_s(t_n) \neq l_\diamond(t_n)$ for all $n \geq 1$. Fix a point $\Lambda \in l_s(t^*) = l_\diamond(t^*) = l_s^*$. Using Lemma 14, it is easy to see that there exists a sequence $(\Lambda_n)_{n\geq 1}$ in $\mathbb{C}$ with $\Lambda_n \to \Lambda$ as $n \to \infty$ such that $\Lambda_n \in l_s(t_n)$ for $n \geq 1$. Similarly there exists a sequence $(\Lambda_n^\diamond)_{n\geq 1}$ in $\mathbb{C}$ with $\Lambda_n^\diamond \to \Lambda$ as $n \to \infty$ such that $\Lambda_n^\diamond \in l_\diamond(t_n)$ for $n \geq 1$. In fact, it is not difficult to see that we may choose these sequences such that $\Lambda_n^\diamond - \Lambda_n$ is approximately orthogonal to the Stokes line l_s^* through Λ for large n, namely

$$\Lambda_n^\diamond - \Lambda_n = \epsilon_n |\Lambda_n^\diamond - \Lambda_n| \overline{\sqrt{V^*(\Lambda)}} + o(\Lambda_n^\diamond - \Lambda_n) \tag{89}$$

as $n \to \infty$, for some irrelevant choice of signs $\epsilon_n \in \{\pm 1\}$, $n \geq 1$.

[4]Here we allow for duplicity of a Stokes line if all its endpoints equal $\mu_j(t)$.

Since $(\alpha(t), \beta(t)) \in \Omega$ for $t \in T$, we know, by Lemma 13, that the internal Stokes complex of $V_{t_n}(\lambda)$ is connected and in particular

$$\Re \int_{\Lambda_n}^{\Lambda_n^\diamond} \sqrt{V_{t_n}(\lambda)} d\lambda = 0, \tag{90}$$

for every choice of contour avoiding critical points and $n \geq 1$. For large n, say $n \geq N \gg 0$, we may choose as contour the line segment between Λ_n and $\Lambda_n^\diamond$ and find a c, independent of n, such that

$$\left| \int_{\Lambda_n}^{\Lambda_n^\diamond} \sqrt{V_{t_n}(\lambda)} d\lambda - (\Lambda_n^\diamond - \Lambda_n)\sqrt{V_{t_n}(\Lambda)} \right| \leq c|\Lambda_n^\diamond - \Lambda_n|^2$$

for $n \geq N$, so that equation (90) yields

$$|\Re(\Lambda_n^\diamond - \Lambda_n)\sqrt{V_{t_n}(\Lambda)}| \leq c|\Lambda_n^\diamond - \Lambda_n|^2$$

for $n \geq N$. Therefore, using (89), we obtain

$$|\Re \overline{\sqrt{V_0(\Lambda)}} \sqrt{V_{t_n}(\Lambda)}| = o(1)$$

as $n \to \infty$. However, the left-hand side converges to $|V_0(\Lambda)|$ as $n \to \infty$ and therefore $V_0(\Lambda) = 0$. Clearly Λ is not a turning point of $V^*(\lambda)$ and we have arrived at a contradiction. We conclude that there exists an open environment $B_{j,s} \subseteq T$ of t^* such that $l_s(t) = l_\diamond(t)$, and thus $l_s(t)$ is an internal Stokes line with endpoints $\{\mu_j(t), \mu_k(t)\}$, for all $t \in B_{j,s}$.

Now

$$B = \bigcap_{1 \leq j \leq m, 1 \leq s \leq r_j + 2} B_{j,s}$$

is an open environment of t^* and the Stokes complex $\mathcal{C}(\alpha(t), \beta(t))$ is isomorphic to $\mathcal{C}(\alpha^*, \beta^*)$ for $t \in B$. □

Lemma 16. *Let $(\alpha^*, \beta^*) \in \Omega$ be such that $V(\lambda; \alpha^*, \beta^*)$ has one double turning point and two simple turning points, say $\mu_1^*, \mu_2^* \in \mathbb{C}^*$. Choose a simply connected open environment $W \subseteq \mathbb{C}^2$ of (α^*, β^*) such that, for $j \in \{1, 2\}$, there exists a unique analytic function $\mu_j : W \to \mathbb{C}^*$ with $\mu_j(\alpha^*, \beta^*) = \mu_j^*$, such that $\mu_j(\alpha, \beta)$ is a simple turning point of $V(\lambda; \alpha, \beta)$ for $(\alpha, \beta) \in W$.*
Then there exists an open environment $W_0 \subseteq W$ of (α^*, β^*), such that, for all $(\alpha, \beta) \in W_0 \cap \Omega$, if $\Delta(\alpha, \beta) \neq 0$, then the two turning points of $V(\lambda; \alpha, \beta)$, not equal to μ_1 or μ_2, are connected by a unique Stokes line which is homotopic to the straight line segment between the two in $\mathbb{C}^*$ minus critical points.

Proof. Note that, for $(\alpha, \beta) \in \Omega$, considering the Stokes $\mathcal{C}(\alpha, \beta)$, the sum of the valencies of the vertices equals twice the number of edges minus 4. Using this identity in conjunction with Lemma 14, the lemma follows quite directly. We leave the details to the reader. □

The following lemma is proven similarly to the above.

Lemma 17. *Let $(\alpha^*, \beta^*) \in \Omega$ be such that $V(\lambda; \alpha^*, \beta^*)$ has one triple turning point and one simple turning point, say $\mu_1^* \in \mathbb{C}^*$. Choose a simply connected open environment $W \subseteq \mathbb{C}^2$ of (α^*, β^*) such that there exists a unique analytic function $\mu_1 : W \to \mathbb{C}^*$ with $\mu_1(\alpha^*, \beta^*) = \mu_1^*$, such that $\mu_1(\alpha, \beta)$ is a simple turning point of $V(\lambda; \alpha, \beta)$ for $(\alpha, \beta) \in W$.*
Then there exists an open environment $W_0 \subseteq W$ of (α^*, β^*), such that, for all $(\alpha, \beta) \in W_0 \cap \Omega$ not equal to (α^*, β^*),

- *if $\Delta(\alpha, \beta) \neq 0$, then the three simple turning points $\{\mu_2, \mu_3, \mu_4\}$ of $V(\lambda; \alpha, \beta)$, not equal to $\mu_1(\alpha, \beta)$, can be labelled such that μ_2 and μ_3 are connected by a Stokes line which is homotopic to the straight line segment between the two in $\mathbb{C}^*$ minus critical points, and the same for the pair $\{\mu_3, \mu_4\}$;*

- *if $\Delta(\alpha,\beta) = 0$, then the two turning points $\{\mu_2, \mu_3\}$, one simple and one double, of $V(\lambda;\alpha,\beta)$, not equal to $\mu_1(\alpha,\beta)$, are connected by a Stokes line which is homotopic to the straight line segment between the two in $\mathbb{C}^*$ minus critical points.*

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Grupo de Física Matemática e Departamento da Matemática da Universidade de Lisboa, Campo Grande Edifício C6, 1749-016 Lisboa, Portugal.

E-mail address: dmasoero@gmail.com

SISSA, via Bonomea 265, 34136 Trieste, Italy.
E-mail address: proffels@sissa.it