

# Hamiltonian structure of reductions of the Benney system

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## Abstract

We show how to construct the Hamiltonian structures of any reduction of the Benney chain (dKP) starting from the family of conformal maps associated to it.

## Introduction

The Benney moment chain [4], given by the equations

$$A_t^k = A_x^{k+1} + kA^{k-1}A_x^0, \quad k = 0, 1, \dots,$$

with  $A^k = A^k(x, t)$ , is the most famous example of a chain of hydrodynamic type, which generalizes the classical systems of hydrodynamic type in the case when the dependent variables (and the equations they have to satisfy) are infinitely many.

A  $n$ -component reduction of the Benney chain is a restriction of the infinite dimensional system to a suitable  $n$ -dimensional submanifold, that is

$$A^k = A^k(u^1, \dots, u^n), \quad k = 0, 1, \dots$$

The reduced systems are systems of hydrodynamic type in the variables  $(u_1, \dots, u_n)$  that parametrize the submanifold:

$$u_t^i = v_j^i(u)u_x^j, \quad i = 1, \dots, n.$$

Benney reductions were introduced in [11], and there it was proved that such systems are integrable via the generalized hodograph transformation [20]. In particular, this method requires the system to be diagonalizable, that is, there exists a set of coordinates  $\lambda^1, \dots, \lambda^n$ , called *Riemann invariants*, such that the reduction takes diagonal form:

$$\lambda_t^i = v^i(\lambda) \lambda_x^i.$$

The functions  $v^i$  are called *characteristic velocities*.

A more compact description of the Benney chain can be given by introducing the formal series

$$\lambda = p + \sum_{k=0}^{+\infty} \frac{A^k}{p^{k+1}}.$$

In this picture, as follows from [16, 17], the Benney chain can be written as the single equation

$$\lambda_t = p \lambda_x - A_x^0 \lambda_p,$$

which is the equation of the second flow of the dispersionless KP hierarchy. This equation related with the Benney chain also appears in [18].

Clearly, in the case of a reduction, the coefficients of this series depend on a finite number of variables  $(u^1, \dots, u^n)$ . In this case, the series can be thought as the asymptotic expansion for  $p \mapsto \infty$  of a suitable function  $\lambda(p, u^1, \dots, u^n)$  depending piecewise analytically on the parameter  $p$ . It turns out [11, 12] that such a function satisfies a system of chordal Loewner equations, describing families of conformal maps (with respect to  $p$ ) in the complex upper half plane. The analytic properties of  $\lambda$  characterize the reduction. More precisely, in the case of an  $n$ -reduction the associated function  $\lambda$  possess  $n$  distinct critical points on the real axis, these are the characteristic velocities  $v^i$  of the reduced system, and the corresponding critical values can be chosen as Riemann invariants.

Some examples of such reductions, discussed below, have known Hamiltonian structures, but the most general result is far weaker, all such reductions are semi-Hamiltonian [20, 11].

The aim of this paper is to investigate the relations between the analytic properties of the function  $\lambda(p, u^1, \dots, u^n)$  and the Hamiltonian structures of the associated reduction. As is well known, such structures are associated to pseudo-riemannian metrics, and in particular, local Hamiltonian structures are associated to flat metrics.

Our approach is general, in the sense that it applies to all Benney reductions. Consequently, it reveals a unified structure for the Hamiltonian structure of such reduced systems. The main result of the paper provides the Hamiltonian structures of a Benney reduction directly in terms of the function  $\lambda(p, u^1, \dots, u^n)$  and its inverse with respect to  $p$ , denoted by  $p(\lambda, u^1, \dots, u^n)$ . The Hamiltonian operator then takes the form

$$\Pi^{ij} = \varphi_i \lambda''(v^i) \delta^{ij} \frac{d}{dx} + \Gamma_k^{ij} \lambda_x^k + \frac{1}{2\pi i} \sum_{k=1}^n \int_{C_k} \frac{\frac{\partial p}{\partial \lambda} \lambda_x^i}{(p(\lambda) - v^i)^2} \left( \frac{d}{dx} \right)^{-1} \frac{\frac{\partial p}{\partial \lambda} \lambda_x^j}{(p(\lambda) - v^j)^2} \varphi_k(\lambda) d\lambda,$$

where

$$\Gamma_k^{ij} \lambda_x^k = \frac{\varphi_j \lambda_x^i - \varphi_i \lambda_x^j}{(v^i - v^j)^2} \quad i \neq j,$$

$$\Gamma_k^{ii} \lambda_x^k = \varphi_i \left( \frac{1}{6} \frac{\lambda''''(v^i)}{\lambda''(v^i)} - \frac{1}{4} \frac{\lambda'''(v^i)^2}{\lambda''(v^i)^2} \right) \lambda_x^i + \frac{1}{2} \varphi_i' \lambda_x^i - \sum_{k \neq i} \frac{\lambda''(v^i)}{\lambda''(v^k)} \frac{\varphi_i \lambda_x^k}{(v^i - v^k)^2}.$$

Here  $\varphi_1, \dots, \varphi_n$  are arbitrary functions of a single variable,  $C_k$  are suitable closed contours on a complex domain, and

$$\lambda''(p) = \frac{\partial^2 \lambda}{\partial p^2}(p), \quad \lambda'''(p) = \frac{\partial^3 \lambda}{\partial p^3}(p), \quad \dots$$

The paper is organized as follows. In Section 1 we review the concepts of integrability for diagonalizable systems of hydrodynamic type and the Hamiltonian formalism for these systems, both in the local and nonlocal case. In Section 2 we introduce the Benney chain, its reductions, and we discuss the properties of these systems. Section 3 is dedicated to the representation of Benney reductions in the  $\lambda$  picture and to the relations with the Loewner evolution. The study of the Hamiltonian properties of reductions of Benney is addressed in Sections 4 and 5: in the former we use a direct approach, starting from the reduction itself, in the latter we describe these results from the point of view of the function  $\lambda$  associated with the reduction. In the last section we discuss two examples where calculations can be expressed in details.

## 1 Systems of hydrodynamic type

### 1.1 Semi-Hamiltonian systems

In  $(1+1)$  dimensions, *systems of hydrodynamic type* are quasilinear first order PDE of the form

$$u_t^i = v_j^i(u) u_x^j, \quad i = 1, \dots, n. \quad (1.1)$$

Here and below sums over repeated indices are assumed if not otherwise stated. We say that the system (1.1) is *diagonalizable* if there exist a set of coordinates  $\lambda^1, \dots, \lambda^n$ , called *Riemann invariants*, such that the matrix  $v_j^i(\lambda)$  takes diagonal form:

$$\lambda_t^i = v^i(\lambda) \lambda_x^i. \quad (1.2)$$

The functions  $v^i$  are called *characteristic velocities*. We recall that the Riemann invariants  $\lambda^i$  are not defined uniquely, but up to a change of coordinates

$$\tilde{\lambda}^i = \tilde{\lambda}^i(\lambda^i). \quad (1.3)$$

A diagonal system of PDEs of hydrodynamic type (1.2) is called *semi-Hamiltonian* [20] if the coefficients  $v^i(u)$  satisfy the system of equations

$$\partial_j \left( \frac{\partial_k v^i}{v^i - v^k} \right) = \partial_k \left( \frac{\partial_j v^i}{v^i - v^j} \right) \quad \forall i \neq j \neq k \neq i, \quad (1.4)$$

where  $\partial_i = \frac{\partial}{\partial \lambda^i}$ . The equations (1.4) are the integrability conditions both for the system

$$\frac{\partial_j w^i}{w^i - w^j} = \frac{\partial_j v^i}{v^i - v^j}, \quad (1.5)$$

which provides the characteristic velocities of the symmetries

$$u_\tau^i = w^i(u) u_x^i \quad i = 1, \dots, n$$

of (1.1), and for the system

$$(v^i - v^j) \partial_i \partial_j H = \partial_i v^j \partial_j H - \partial_j v^i \partial_i H,$$

which provides the densities  $H$  of conservation laws of (1.1). The properties of being diagonalizable and semi-Hamiltonian imply the integrability of the system:

**Theorem 1** [20] (*Generalized hodograph transformation*)

Let

$$\lambda_t^i = v^i \lambda_x^i \quad (1.6)$$

be a diagonal semi-Hamiltonian system of hydrodynamic type, and let  $(w^1, \dots, w^N)$  be the characteristic velocities of one of its symmetries. Then, the functions  $(\lambda^1(x, t), \dots, \lambda^N(x, t))$  determined by the system of equations

$$w^i = v^i x + t, \quad i = 1, \dots, N, \quad (1.7)$$

satisfy (1.6). Moreover, every smooth solution of this system is locally obtainable in this way.

## 1.2 Hamiltonian formalism

A class of Hamiltonian formalisms for systems of hydrodynamic type (1.1) was introduced by Dubrovin and Novikov in [6, 7]. They considered local Hamiltonian operators of the form

$$P^{ij} = g^{ij}(u) \frac{d}{dx} - g^{is} \Gamma_{sk}^j(u) u_x^k \quad (1.8)$$

and the associated Poisson brackets

$$\{F, G\} := \int \frac{\delta F}{\delta u^i} P^{ij} \frac{\delta G}{\delta u^j} dx \quad (1.9)$$

where  $F = \int g(u) dx$  and  $G = \int g(u) dx$  are functionals not depending on the derivatives  $u_x, u_{xx}, \dots$

**Theorem 2** [6] *If  $\det g^{ij} \neq 0$ , then the formula (1.9) with (1.8) defines a Poisson bracket if and only if the tensor  $g^{ij}$  defines a flat pseudo-riemannian metric and the coefficients  $\Gamma_{sk}^j$  are the Christoffel symbols of the associated Levi-Civita connection.*

Non-local extensions of the bracket (1.9), related to metrics of constant curvature, were considered by Ferapontov and Mokhov in [19]. Further generalizations were considered by Ferapontov in [9], where he introduced the nonlocal differential operator

$$P^{ij} = g^{ij} \frac{d}{dx} - g^{is} \Gamma_{sk}^j u_x^k + \sum_{\alpha} \epsilon_{\alpha} (w^{\alpha})_k^i u_x^k \left( \frac{d}{dx} \right)^{-1} (w^{\alpha})_h^j u_x^h, \quad \epsilon_{\alpha} = \pm 1. \quad (1.10)$$

The index  $\alpha$  can take values on a finite or infinite – even continuous – set.

**Theorem 3** *If  $\det g^{ij} \neq 0$ , then the formula (1.9) with (1.8) defines a Poisson bracket if and only if the tensor  $g^{ij}$  defines a pseudo-riemannian metric, the coefficients  $\Gamma_{sk}^j$  are the Christoffel symbols of the associated Levi-Civita connection  $\nabla$ , and the affinors  $w^{\alpha}$  satisfy the conditions*

$$[w^{\alpha}, w^{\beta}] = 0,$$

$$g_{ik} (w^{\alpha})_j^k = g_{jk} (w^{\alpha})_i^k,$$

$$\nabla_k (w^{\alpha})_j^i = \nabla_j (w^{\alpha})_k^i,$$

$$R_{kh}^{ij} = \sum_{\alpha} \left\{ (w^{\alpha})_k^i (w^{\alpha})_h^j - (w^{\alpha})_k^j (w^{\alpha})_h^i \right\},$$

where  $R_{kh}^{ij} = g^{is} R_{skh}^j$  are the components of the Riemann curvature tensor of the metric  $g$ .

In the case of zero curvature, operator (1.10) reduces to (1.9). Let us focus our attention on semi-Hamiltonian systems. In [9] Ferapontov conjectured that any diagonalizable semi-Hamiltonian system is always Hamiltonian with respect to suitable, possibly non local, Hamiltonian operators. Moreover he proposed the following construction to define such Hamiltonian operators:

1. Consider a diagonal system (1.2). Find the general solution of the system

$$\partial_j \ln \sqrt{g_{ii}} = \frac{\partial_j v^i}{v^j - v^i}, \quad (1.11)$$

which is compatible for a semi-Hamiltonian system, and compute the curvature tensor of the metric  $g$ .

2. If the non vanishing components of the curvature tensor can be written in terms of solutions  $w_\alpha^i$  of the linear system (1.5):

$$R_{ij}^{ij} = \sum_{\alpha} \epsilon_{\alpha} w_{\alpha}^i w_{\alpha}^j, \quad \epsilon_{\alpha} = \pm 1, \quad (1.12)$$

then it turns out that the system (1.1) is Hamiltonian with respect to the Hamiltonian operator

$$P^{ij} = g^{ii} \delta^{ij} \frac{d}{dx} - g^{ii} \Gamma_{ik}^j(u) u_x^k + \sum_{\alpha} \epsilon_{\alpha} w_{\alpha}^i u_x^i \left( \frac{d}{dx} \right)^{-1} w_{\alpha}^j u_x^j, \quad (1.13)$$

which is the form of (1.10) in case of diagonal matrices.

## 2 Benney reductions

A natural generalization of  $n$ -component systems of hydrodynamic type (1.1) can be obtained by allowing the number of equations and variables to be infinite. These systems are known as *hydrodynamic chains*, and the best known example is the Benney chain [4]:

$$A_t^k = A_x^{k+1} + k A^{k-1} A_x^0, \quad k = 0, 1, \dots \quad (2.1)$$

In this setting, the variables  $A^n$  are usually called moments. In [4] Benney proved that this system admits an infinite series of conserved quantities, whose densities are polynomial in the moments. The first few of them are

$$H^0 = A^0, \quad H^1 = A^1, \quad H^2 = \frac{1}{2} A^2 + \frac{1}{2} (A^0)^2 \dots$$

A  $n$ -component reduction of the Benney chain (2.1) is a restriction of the infinite dimensional system to a suitable  $n$ -dimensional submanifold in the space of the moments, that is:

$$A^k = A^k(u^1, \dots, u^n), \quad k = 0, 1, \dots \quad (2.2)$$

where  $u^i = u^i(x, t)$  are the new dependent variables. These are regarded as coordinates on the submanifold specified by (2.2), and all the equations of the chain have to be satisfied on this submanifold. In addition, we require the  $x$ -derivatives  $u_x^i$  to be linearly independent<sup>1</sup>, in the sense that

$$\sum_{i=1}^n \alpha_i(u^1, \dots, u^n) u_x^i = 0 \quad \Rightarrow \quad \alpha_i(u^1, \dots, u^n) = 0, \quad \forall i. \quad (2.3)$$

Thus, the infinite dimensional system reduces to a system with finitely many dependent variables (1.1). It was shown in [11] that all Benney reductions are diagonalizable and possess the semi-Hamiltonian property, hence they are integrable via the generalized hodograph

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<sup>1</sup>If this constraint is relaxed, solutions such as described in [15] may be obtained.

method. On the other hand, we may consider whether a diagonal system of hydrodynamic type

$$\lambda_t^i = v^i(\lambda)\lambda_x^i, \quad i = 1, \dots, n. \quad (2.4)$$

is a reduction of Benney (note that we do not impose the semi-Hamiltonian condition). A direct substitution in the chain (2.1) leads, after collecting the  $\lambda_x^i$  and making use of (2.3), to the system

$$v^i \partial_i A^k = \partial_i A^{k+1} + k A^{k-1} \partial_i A^0, \quad i = 1, \dots, n, \quad (2.5)$$

where  $\partial_i A^0 = \frac{\partial A^0}{\partial \lambda^i}$ . The consistency conditions

$$\partial_j \partial_i A^{k+1} = \partial_i \partial_j A^{k+1}, \quad i \neq j, \quad k = 0, 1, \dots$$

reduce to the  $\frac{3}{2}n(n-1)$  equations

$$\partial_i v^j = \frac{\partial_i A^0}{v^i - v^j} \quad (2.6a)$$

$$i \neq j,$$

$$\partial_{ij}^2 A^0 = \frac{2\partial_i A^0 \partial_j A^0}{(v^i - v^j)^2} \quad (2.6b)$$

which are called the *Gibbons-Tsarev system*. It has been shown that this system is in involution, hence it characterizes a  $n$ -component reduction of Benney. Moreover, if a solution of (2.6) is known, all the higher moments can be found, making recursive use of conditions (2.5).

**Theorem 4** [11] *A diagonal system of hydrodynamic type (2.4) is a reduction of the Benney moment chain (2.1) if and only if there exist a function  $A^0(\lambda^1, \dots, \lambda^n)$  such that  $A^0$  and the  $v^1, \dots, v^n$  of the system satisfy the Gibbons-Tsarev system (2.6). In this case, system (2.4) is automatically semi-Hamiltonian.*

It was noticed in [11] that a generic solution of the Gibbons-Tsarev system depends on  $n$  arbitrary functions of one variable. Essentially, this is due to the fact that in the system (2.6) the derivatives

$$\partial_i v^i, \quad \partial_{ii}^2 A^0 \quad (2.7)$$

are not specified. This leads to a freedom of  $2n$  functions of a single variable, which reduces to  $n$  allowing for the freedom of reparametrization (1.3) in the definition of Riemann invariants. Thus, for any fixed integer  $n$ , the Benney moment chain possesses infinitely many integrable  $n$ -component reductions, parametrized by  $n$  arbitrary functions of one variable.

In the next sections we will see how the knowledge of the ‘diagonal’ terms (2.7) plays an important role in determining the Hamiltonian structure of a Benney reduction. If these terms are specified, the Gibbons-Tsarev system becomes a system of pfaffian type, and a generic solution depends on  $n$  arbitrary constants.

**Example 2.1** The 2–component Zakharov reduction [22], is obtained by imposing on the moments the constraints

$$A^k = u^1 (u^2)^k, \quad k = 0, 1, \dots,$$

where  $(u^1, u^2)$  are the new dependent variables. The resulting classical shallow water wave system, first solved by Riemann, is known to be the dispersionless limit of the 2–component vector NLS equation. Under the change of dependent coordinates

$$\lambda^1 = u^2 + 2\sqrt{u^1} \quad \lambda^2 = u^2 - 2\sqrt{u^1},$$

the system takes the diagonal form (1.2), with velocities

$$v^1 = \frac{3}{4}\lambda^1 + \frac{1}{4}\lambda^2 \quad v^2 = \frac{1}{4}\lambda^1 + \frac{3}{4}\lambda^2.$$

It is easy to check that these velocities satisfy the Gibbons-Tsarev system with

$$A^0 = \frac{(\lambda^1 - \lambda^2)^2}{16}.$$

## 3 The $\lambda$ picture and chordal Loewner equations

### 3.1 Reductions in the $\lambda$ picture

A more compact description of the Benney chain can be given by introducing [16] a formal series

$$\lambda(p, x, t) = p + \sum_{k=0}^{\infty} \frac{A^k(x, t)}{p^{k+1}}. \quad (3.1)$$

It is well known that the moments satisfy the Benney chain (2.1) if and only if  $\lambda$  satisfies

$$\lambda_t = p\lambda_x - A_x^0 \lambda_p = \left\{ \lambda, \frac{1}{2} (\lambda^2)_{\geq 0} \right\}, \quad (3.2)$$

where  $(\cdot)_{\geq 0}$  denotes the polynomial part of the argument, and  $\{\cdot, \cdot\}$  is the canonical Poisson bracket on the  $(x, p)$  space. Equation 3.2 corresponds to the Lax equation of the second flow of the dispersionless KP hierarchy.

**Remark 1** If we introduce the inverse of the series  $\lambda$  with respect to  $p$ , and denote it as

$$p(\lambda) = \lambda + \sum_{k=0}^{\infty} \frac{H_k}{\lambda^{k+1}},$$

then it is easy to check that the following equation holds

$$p_t = \partial_x \left( \frac{1}{2} p^2 + A^0 \right).$$

Equivalently, its coefficients satisfy

$$H_t^k = \partial_x \left( H_x^{k+1} - \frac{1}{2} \sum_{i=0}^{k-1} H^i H^{k-1-i} \right),$$

which is the Benney chain written in conservation law form using the coordinate set  $H^n$ . It is easy to show that every  $H^k$  is polynomial in the moments  $A^0, \dots, A^k$ .

The use of the formal series (3.1) is to be understood as an algebraic model for describing the underlying integrable system in a more compact way. However, to describe the system in more detail we must impose more structure on  $\lambda$ . Following [12, 21], rather than considering a formal series in the parameter  $p$ , we instead consider a piecewise analytic function for the variable  $p$ . In particular, we let  $\lambda_+$  be an analytic function defined on  $Im(p) > 0$ , and  $\lambda_-$  an analytic function on  $Im(p) < 0$ . We also require the normalization

$$\lambda_{\pm} = p + O\left(\frac{1}{p}\right), \quad p \mapsto \infty. \quad (3.3)$$

Let us define, on the real axis, the jump function

$$f(p, x, t) = \frac{1}{2\pi i} (\lambda_-(p, x, t) - \lambda_+(p, x, t)),$$

and suppose  $f$  is a function of real  $p$  which is Holder continuous and satisfying the conditions

$$\int_{-\infty}^{+\infty} p^n f dp < \infty, \quad n = 0, 1, \dots$$

Then, using Plemelj's formula for boundary values of analytic functions, we may take

$$\lambda_{\pm}(p) = p - \pi \int_{-\infty}^{+\infty} \frac{f(p')}{p - p'} dp' \mp i\pi f(p).$$

What we obtained is that, with hypotheses above, the functions  $\lambda_+$  and  $\lambda_-$  are Borel sums of the series (3.1) in the upper and lower half plane respectively. On the other hand,  $\lambda_{\pm}$  will have, at  $p \mapsto \infty$ , the formal asymptotic series (3.1), where

$$A^n(x, t) = \int_{-\infty}^{+\infty} p^n f(p, x, t) dp.$$

Thus, to any solution of Benney's equations we can associate a pair of functions  $\lambda_{\pm}(p; x, t)$ . In particular, a real valued  $f$  leads to real valued moments. In this case, using the Schwarz reflection principle, we can restrict our attention to the function  $\lambda_+$ ; this is the case studied in [10, 21, 1, 2, 3].

On the other hand, it will be useful below to consider the analytic continuation of  $\lambda_+$  into the lower half plane, in the neighborhood of specified points in the real axis. Such a continuation may or may not coincide with  $\lambda_-$ , the Schwarz reflection of  $\lambda_+$ . In particular important examples such a continuation may be developed consistently, giving the structure of a Riemann surface.

**Remark 2** Other normalizations, more general than (3.3) are allowed, based on the fact that for any differentiable function  $\varphi$  of a single variable, the composed function  $\varphi(\lambda_+)$  remains a solution of (3.2), the associated reduction being the same. In concrete examples, it is sometimes more convenient to make use a different normalisation.

Let us consider now the relations between solutions of (3.2) and Benney reduction. In this case, we have that  $\lambda_+$  is associated with a  $n$  component reduction if and only if it depends on the variables  $x, t$  via  $n$  independent functions. As any reduction is diagonalizable, it is not restrictive to take as these variables the Riemann invariants. Thus, we have

$$\lambda_+(p, x, t) = \lambda_+(p, \lambda^1(x, t), \dots, \lambda^n(x, t)), \quad (3.4)$$

with

$$\lambda_t^i = v^i \lambda_x^i. \quad (3.5)$$

Remarkably, the characteristic velocities of the reduction turn out to be the critical points of the function  $\lambda_+$  associated with it. More precisely, we have

**Theorem 5** Let  $\lambda_+$ , solution of (3.2), satisfy conditions (3.4) with (3.5). Let us denote

$$\varphi^i(\lambda^1, \dots, \lambda^n) = \lambda_+(v^i, \lambda^1, \dots, \lambda^n), \quad i = 1 \dots n,$$

and suppose that the  $\rho^i$  are not constant functions. Then, the velocities  $v^i$  satisfy

$$\frac{\partial \lambda_+}{\partial p}(v^i) = 0, \quad i = 1 \dots n,$$

and the corresponding critical values  $\varphi^i$  can be chosen as Riemann invariants for the system (3.5).

*Proof* Considering equation (3.2) at  $p = v^i$ , we obtain the system of  $n$  equations

$$\varphi_t^i = v^i \varphi_x^i - A_x^0 \frac{\partial \lambda_+}{\partial p}(v^i).$$

As  $\lambda^1, \dots, \lambda^n$  can be chosen as coordinates, by the chain rule we get

$$\sum_{j=0}^n \frac{\partial \varphi^i}{\partial \lambda^j} \lambda_t^j = v^i \sum_{j=0}^n \frac{\partial \varphi^i}{\partial \lambda^j} \lambda_x^j - \frac{\partial \lambda_+}{\partial p}(v^i) \sum_{j=0}^n \frac{\partial A^0}{\partial \lambda^j} \lambda_x^j,$$

and this, after substituting (3.5) into it, is equivalent to

$$\frac{\partial \varphi^i}{\partial \lambda^j} (v^j - v^i) + \frac{\partial \lambda_+}{\partial p}(v^i) \frac{\partial A^0}{\partial \lambda^j} = 0 \quad i, j = 1 \dots, n, \quad (3.6)$$

due to the independence of the  $\lambda_x^j$ . Particularly, for  $i = j$  the system above reduces to

$$\frac{\partial \lambda_+}{\partial p}(v^i) \frac{\partial A^0}{\partial \lambda^i} = 0. \quad (3.7)$$

Further, if  $A^0$  does not depend on  $\lambda^i$ , the function  $\lambda_+$  is also independent of the same  $\lambda^i$ . In this case, the associated system (3.5) reduces to a  $n-1$  reduction. On the other hand, if the system is a proper  $n$ -component reduction then  $\partial_i A^0 \neq 0$  and the characteristic velocities are critical points for  $\lambda_+$ . Substituting back (3.7) into (3.6), we obtain  $\varphi^i = \varphi^i(\lambda^i)$ . Thus, if the critical values  $\varphi^i$  are not constant functions, it is possible to choose them as Riemann invariants.

□

The converse of the Theorem above is also true: if  $\lambda_+$  is a solution of (3.2) satisfying

$$\lambda_+(p, x, t) = \lambda_+(p, \lambda^1(x, t), \dots, \lambda^n(x, t)), \quad (3.8)$$

and with  $n$  distinct critical points  $v^1, \dots, v^m$ , then by evaluating equation (3.2) at  $p = v^i$  we obtain the diagonal system

$$\varphi_t^i = v^i \varphi_x^i,$$

where  $\varphi^i = \lambda_+(v^i)$ . Thus, critical points are characteristic velocities. Moreover, the existence of a function  $\lambda$  associated with a reduction selects a natural set of Riemann invariants, the critical values of  $\lambda$ . Unless otherwise stated, these are the coordinates we will consider below.

**Remark 3** *It might happen that the function  $\lambda_+$  possesses  $m$  critical points, with  $m > n$ . This is the case, for instance, in Remark 2, where critical points of the function  $\varphi$  have to be added. Then, substituting the critical points into (3.2) we obtain an  $m$  component diagonal system. However, in this case we have that  $m - n$  of the critical values have trivial dynamics for they are independent of  $x, t$ . Consequently, the  $m$  component system reduces to an  $n$  component one.*

**Example 3.2** *Consider  $u^i = u^i(x, t)$ ,  $i = 1, 2$ . The function*

$$\lambda_+ = p + \frac{u^1}{p - u^2}, \quad (3.9)$$

*rational in  $p$ , satisfies equation (3.2) if and only if  $u^1, u^2$  satisfy the 2 component Zakharov reduction of Example 2.1.*

## 3.2 Reductions and Loewner equations

It was shown in [13, 21] that the solution of the initial value problem of an  $n$  reduction is given by a Inverse Scattering Transform procedure (which leads to a particular form of Tsarev's generalized hodograph formula (1.7)), provided that

$$\frac{\partial \lambda_+}{\partial p}(p) \neq 0, \quad \text{Im}(p) > 0.$$

It is thus necessary that  $\lambda_+(p)$  be an *univalent conformal map* from the upper half plane to some image region. In [11, 12] it was proved that these conformal maps have to be solutions of a system of so called chordal Loewner equations. In fact, if a solution of equation (3.2) is associated with a  $n$ -component reduction of Benney, then conditions (3.4) holds. Substituting into equation (3.2), if  $v^i$  are the characteristic velocities associated with the reduction, we obtain

$$\sum_{i=1}^N \left( (v^i - p) \frac{\partial \lambda_+}{\partial \lambda^i} + \frac{\partial A^0}{\partial \lambda^i} \frac{\partial \lambda_+}{\partial p} \right) \lambda_x^i = 0. \quad (3.10)$$

As the  $\lambda_x^i$  are independent, then it follows that

$$\frac{\partial \lambda_+}{\partial \lambda^i} = \frac{\partial_i A^0}{p - v^i} \frac{\partial \lambda_+}{\partial p}, \quad i = 1, \dots, n. \quad (3.11)$$

This is a system of  $n$  chordal Loewner equations (see for example [8]). When the function  $\lambda_+$  is chosen with the normalization (3.3), this system describes the evolution of families of univalent conformal maps from the upper complex half plane to the upper half plane with  $n$  slits, when the end points of the slits are allowed to move along prescribed mutually non intersecting Jordan arcs. Using the implicit function theorem it is possible to show that the inverse function  $p$  satisfies an analogous system

$$\frac{\partial p}{\partial \lambda^i} = -\frac{\partial_i A^0}{p - v^i}, \quad i = 1, \dots, n. \quad (3.12)$$

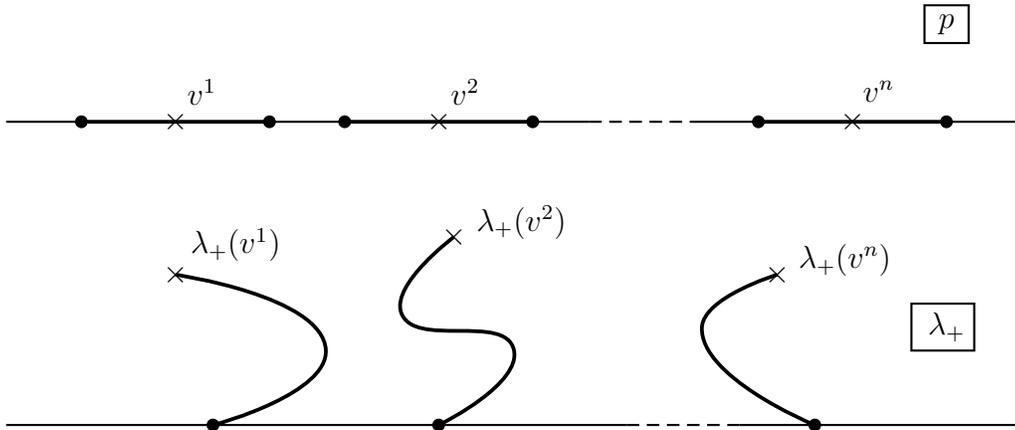


Figure 1:  $n$ -slit Loewner evolution on the upper half plane.

For  $n > 1$ , the consistency conditions of (3.11) (or (3.12) equivalently) turn out to be the Gibbons-Tsarev system. On the other hand, and more generally, we can consider a set of  $n$  Loewner equations,

$$\frac{\partial \lambda_+}{\partial \lambda^i} = \frac{a_i}{p - b^i} \frac{\partial \lambda_+}{\partial p}, \quad i = 1, \dots, n, \quad (3.13)$$

for arbitrary functions  $a_i, b^i$ . The consistency conditions

$$\frac{\partial^2 \lambda_+}{\partial \lambda^i \partial \lambda^j} = \frac{\partial^2 \lambda_+}{\partial \lambda^j \partial \lambda^i}$$

are then equivalent to the set of equations

$$\partial_i a_j = \partial_j a_i \quad (3.14)$$

$$\partial_i a_j = \frac{2a_i a_j}{(b^i - b^j)^2} \quad (3.15)$$

$$\partial_i b^j = \frac{a_i}{b^i - b^j}, \quad (3.16)$$

where  $i \neq j$ . The first of these equations implies locally the existence of a function  $A^0(\lambda^1, \dots, \lambda^n)$  such that

$$a_i = \partial_i A^0.$$

Consequently, equations (3.15), (3.16) become the Gibbons-Tsarev system (2.6), with  $b^i = v^i$ . So, to any solution of a system of  $n$  chordal Loewner equations there corresponds a  $n$ -component reduction of the Benney chain.

**Example 3.3** *The dispersionless Boussinesq reduction, which is a 2–component Gelfand-Dikii reduction, is given by*

$$\begin{aligned} A_t^0 &= A_x^1 \\ A_t^1 &= -A^0 A_x^0, \end{aligned}$$

can be described in the  $\lambda$  picture using the polynomial function

$$\lambda_+ = p^3 + 3A^0 p + 3A^1.$$

The characteristic velocities are

$$v^1 = -\sqrt{-A^0}, \quad v^2 = \sqrt{-A^0},$$

and the Riemann invariants are given by

$$\lambda^1 = \lambda_+(v^1) = 3A^1 + 2(-A^0)^{\frac{3}{2}}, \quad \lambda^2 = \lambda_+(v^2) = 3A^1 - 2(-A^0)^{\frac{3}{2}}.$$

After the renormalization

$$\tilde{\lambda}(\lambda_+) = \sqrt[3]{\lambda_+} = \sqrt[3]{p^3 + 3A^0 p + 3A^1}$$

we obtain a family of Schwarz-Christoffel maps as in Figure 3.3. It is easy to verify that the critical points of the function  $\tilde{\lambda}(\lambda_+)(p)$  are the same as  $\lambda_+(p)$ , while the corresponding new Riemann invariants are  $\tilde{\lambda}^i = \sqrt[3]{\lambda^i}$ .

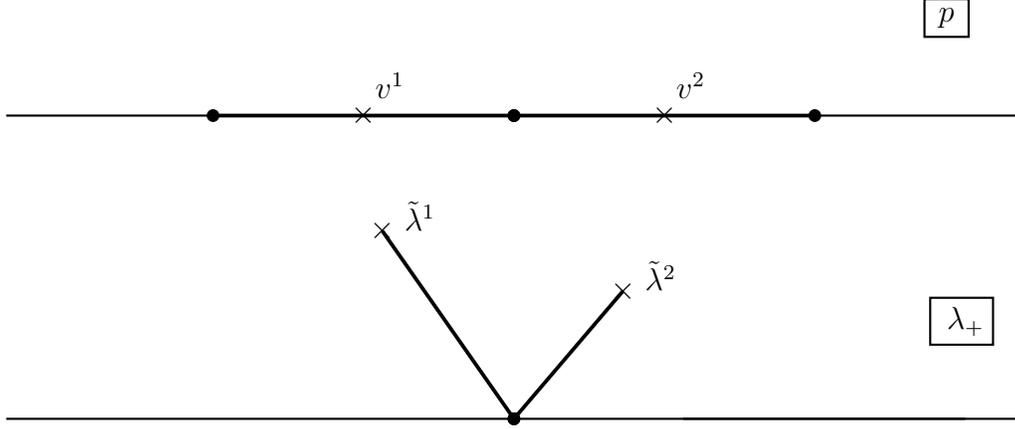


Figure 2: The dispersionless Boussinesq reduction.

As a consequence of the Loewner system (3.11) satisfied by a Benney reduction, it follows immediately that the critical points of  $\lambda_+(p)$  are simple. Indeed, taking the limit of the  $i$ -th equation of the system, for  $p \rightarrow v^i$  gives

$$1 = \frac{\partial^2 \lambda_+}{\partial p^2}(v^i) \partial_i A^0, \quad (3.17)$$

where we used the identity

$$\frac{\partial \lambda}{\partial \lambda^i} \Big|_{p=v^i} = \frac{d\lambda^i}{d\lambda^i} - \frac{\partial \lambda}{\partial p} \Big|_{p=v^i} \frac{\partial v^i}{\partial \lambda^i} = 1.$$

Thus,

$$\frac{\partial^2 \lambda_+}{\partial p^2}(v^i) \neq 0,$$

hence the  $v^i$  are simple.

Suppose now that  $\lambda_+$  admits an analytic continuation in some neighborhood of  $v^i$ . Henceforth, to simplify the notations, the subscript  $+$  will be dropped from  $\lambda_+$ , and we will denote both the analytic function  $\lambda_+$  and its analytic continuation simply by  $\lambda$ . Moreover, we will write

$$\lambda'(p) = \frac{\partial \lambda}{\partial p}(p), \quad \lambda''(p) = \frac{\partial^2 \lambda}{\partial p^2}(p), \quad \dots$$

Then, the function  $p(\lambda)$  has the series development near  $\lambda = \lambda^i$ ,

$$p(\lambda) = v^i + \frac{\sqrt{2}}{\sqrt{\lambda''(v^i)}} \sqrt{\lambda - \lambda^i} + O(\lambda - \lambda^i), \quad (3.18)$$

which becomes a Taylor expansion in the complex local parameter  $t = \sqrt{\lambda - \lambda^i}$ . Furthermore, we have

$$\begin{aligned} \frac{1}{\lambda'(p)} &= \frac{1}{\lambda''(v^k)} \frac{1}{p - v^k} - \frac{1}{2} \frac{\lambda'''(v^k)}{\lambda''(v^k)^2} \\ &+ \left( \frac{1}{4} \frac{\lambda'''(v^k)^2}{\lambda''(v^k)^3} - \frac{1}{6} \frac{\lambda''''(v^k)}{\lambda''(v^k)^2} \right) (p - v^k) + O((p - v^k)^2). \end{aligned} \quad (3.19)$$

This expansion will be useful in Section 5. Finally, we introduce two sets of contours, in the  $p$  and  $\lambda$  plane respectively, that we will need later for describing the Hamiltonian structure of the reductions. We define  $\Gamma_i$  as a closed and sufficiently small contour in the  $p$ -plane around  $v^i$ , and  $C_i$  as the image of  $\Gamma_i$  according to the analytical continuation of  $\lambda$ . Thus,  $\Gamma_i$  and  $C_i$  are well defined; in particular, it follows from expansion (3.18) that  $\lambda_i$  – the tip of the slit – is a square root branch point for  $p(\lambda)$ , hence  $C_i$  encircles it twice.

### 3.3 Symmetries of the Benney reductions

A well-known method [13] of obtaining a countable set of symmetries of the Benney reduction is based on the Lax representation of the dKP hierarchy,

$$\lambda_{t_n} = \{\lambda, h_n\} = (h_n)_p \lambda_x - (h_n)_x \lambda_p, \quad n = 1, 2, \dots$$

where  $h_n = \frac{1}{n} (\lambda^n)_{\geq 0}$ . We assume, unless otherwise stated, that  $\lambda$  is normalized as in (3.3). If the solution  $\lambda$  possesses  $n$  critical values  $(v^1, \dots, v^n)$ , the hierarchy can be reduced to

$$\lambda_{t_n}^i = w_n^i \lambda_x^i, \quad i = 1, \dots, n, \quad n = 1, 2, \dots,$$

with

$$w_n^i = \left( \frac{\partial h_n}{\partial p} \right) \Big|_{p=v^i}. \quad (3.20)$$

These are, by construction, components of the symmetries of the reductions, the first few of them being

$$w_1^i = 1, \quad w_2^i = v^i, \quad w_3^i = (v^i)^2 + A^0 \dots \quad i = 1, \dots, n.$$

Further, the above characteristic velocities can be obtained as the coefficients of the expansion at  $\lambda = \infty$  of the generating functions

$$W^i(\lambda) = \frac{1}{p(\lambda) - v^i}. \quad (3.21)$$

**Proposition 3.1** *The functions  $W^i(\lambda)$  are solutions of the linear system*

$$\frac{\partial_j W^i(\lambda)}{W^j(\lambda) - W^i(\lambda)} = \frac{\partial_j v^i}{v^j - v^i}. \quad (3.22)$$

Moreover, expanding  $W^i(\lambda)$  at  $\lambda = \infty$  we get

$$W^i(\lambda) = \sum_{n=1}^{\infty} \frac{w_n^i}{\lambda^n}, \quad (3.23)$$

where the coefficients of the series are the symmetries  $w_n^i$  given in (3.20).

**Remark 4** The first condition (3.22) holds in any normalization, but the expansion (3.23) assumes the normalization (3.3).

*Proof* In order to prove (3.22), knowing that

$$\frac{\partial p}{\partial \lambda^i} = \frac{\partial_i A^0}{v^i - p},$$

we can write

$$\begin{aligned} \partial_j W^i(\lambda) &= \partial_j \left( \frac{1}{p - v^i} \right) = \frac{-1}{(p - v^i)^2} \left( \frac{\partial p}{\partial \lambda^j} - \frac{\partial v^i}{\partial \lambda^j} \right) \\ &= -\frac{\partial_j A^0}{(p - v^i)^2} \left( \frac{1}{v^j - p} - \frac{1}{v^j - v^i} \right) \\ &= \frac{\partial_j A^0}{(p - v^i)(p - v^j)(v^j - v^i)}. \end{aligned}$$

On the other hand

$$W^j(\lambda) - W^i(\lambda) = \frac{(v^j - v^i)}{(p - v^i)(p - v^j)}.$$

and so

$$\frac{\partial_j W^i(\lambda)}{W^j(\lambda) - W^i(\lambda)} = \frac{\partial_j A^0}{(v^j - v^i)^2} = \frac{\partial_j v^i}{v^j - v^i}.$$

In order to prove (3.23), chosen the normalization (3.3), we have

$$w_n^i = \frac{1}{n} \lim_{p \rightarrow v_i} \frac{d}{dp} \left[ (p - v_i)^2 \frac{(\lambda^n)_+}{(p - v_i)^2} \right] = \frac{1}{n} \operatorname{res}_{p=v_i} \left( \frac{(\lambda^n)_+}{(p - v_i)^2} dp \right).$$

The function  $\frac{(\lambda^n)_+}{(p - v_i)^2}$  has poles only at  $p = v_i$  and  $p = \infty$  and therefore

$$\operatorname{res}_{p=v_i} \left( \frac{(\lambda^n)_+}{(p - v_i)^2} dp \right) = -\operatorname{res}_{p=\infty} \left( \frac{(\lambda^n)_+}{(p - v_i)^2} dp \right) = -\operatorname{res}_{p=\infty} \left( \frac{(\lambda^n)}{(p - v_i)^2} dp \right)$$

where the last identity is due to

$$\operatorname{res}_{p=\infty} \left( \frac{(\lambda^n)_-}{(p - v_i)^2} dp \right) = 0.$$

Changing variable we obtain

$$\begin{aligned} -\operatorname{res}_{p=\infty} \left( \frac{(\lambda^n)}{(p - v_i)^2} dp \right) &= -\frac{1}{2\pi i} \int_{\Gamma_\infty} \frac{\lambda^n \frac{dp}{d\lambda}}{(p(\lambda) - v_i)^2} d\lambda \\ &= \frac{n}{2\pi i} \int_{\Gamma_\infty} \frac{\lambda^{n-1}}{p(\lambda) - v_i} d\lambda, \end{aligned}$$

where  $\Gamma_\infty$  is a sufficiently small contour around  $p = \infty$ . Thus,

$$w_n^i = \int_{\Gamma_\infty} W^i(\lambda) \lambda^{n-1} d\lambda,$$

varying  $n$  we obtain the coefficients of the expansion (3.23).

□

**Remark 5** *It will be useful below to consider, as a generating function of the symmetries,*

$$w^i(\lambda) = \frac{\frac{\partial p}{\partial \lambda}}{(p(\lambda) - v^i)^2} = -\sum_{n=1}^{\infty} \frac{n w_n^i}{\lambda^{n+1}}, \quad (3.24)$$

which is nothing but the  $\lambda$ -derivative of (3.21).

We finally observe that, due to linearity of (3.22), the functions

$$z^i = \sum_{k=1}^n \int_{C_k} w^i(\lambda) \varphi_k(\lambda) d\lambda$$

still satisfy the system for the symmetries and therefore, applying the generalized hodograph method, we can write the general solution of the Benney reduction in the implicit form

$$z^i = v^i x + t, \quad i = 1, \dots, n.$$

The inverse scattering solutions found in [10, 21] are of this form.

## 4 Hamiltonian structure of the reductions

As it was shown in [11], any reduction of Benney is a diagonalizable and semi-Hamiltonian system of hydrodynamic type. However, very little is known about the Hamiltonian structure of these systems, whether local or nonlocal. A few examples are known explicitly. These include the Gelfand-Dikii and Zakharov reductions, which arise as dispersionless limits of known dispersive Hamiltonian systems.

In this section, we use Ferapontov's procedure sketched in Section 1.2 for semi-Hamiltonian diagonal systems in order to determine the metric associated with a generic reduction.

**Theorem 6** *The general solution of the system (1.11) in the case of Benney reductions is*

$$g_{ii} = \frac{\partial_i A_0}{\varphi_i(\lambda^i)} \quad (4.1)$$

where the functions  $\varphi_i(\lambda^i)$  are  $n$  arbitrary functions of one variable, the functions  $\lambda^1, \dots, \lambda^n$  being the Riemann invariants of the system.

*Proof.* From the system (1.11), and making use of both the Gibbons-Tsarev equations (2.6), which hold for any Benney reduction, we obtain

$$\partial_j \ln \sqrt{g_{ii}} = \frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j A_0}{(v_j - v_i)^2} = \partial_j \ln \sqrt{\partial_i A_0} \quad (4.2)$$

from which we obtain the general solution (4.1). □

In the case  $\varphi_i = 1$  the rotation coefficients

$$\beta_{ij} := \frac{\partial_i \sqrt{g_{jj}}}{\sqrt{g_{ii}}} \quad (4.3)$$

are symmetric:

$$\beta_{ij} = \frac{1}{2} \frac{\partial_i \partial_j A_0}{\sqrt{\partial_i A_0 \partial_j A_0}} = \frac{\sqrt{\partial_i A_0 \partial_j A_0}}{(v_i - v_j)^2}. \quad (4.4)$$

We now focus our attention on this case, that is we consider the Egorov (potential) metric

$$g_{ii} = \partial_i A_0. \quad (4.5)$$

**Remark 6** *We notice that the choice of potential metric is not restrictive, as any other metric (4.1) can be written in potential form under a change of coordinates*

$$\lambda^i \mapsto \varphi_i(\lambda^i), \quad (4.6)$$

which is exactly the freedom we have in the definition of the Riemann invariants. On the other hand, the choice of the Riemann invariants determines a unique metric which is potential in those coordinates.

In order to find the Christoffel symbols and the curvature tensor of the metric (4.5), one could compute these objects – as usual – starting from their definitions. However, for a Benney reduction, this procedure can be shortened. Indeed, using the Gibbons-Tsarev system (2.6), the connection and the curvature can be written as simple algebraic combinations of the quantities

$$v^i, \quad \partial_i A_0, \quad \delta(v^i), \quad \delta(\log \sqrt{\partial_i A_0}), \quad i = 1, \dots, n,$$

where we introduced the shift operator

$$\delta = \sum_{k=1}^n \frac{\partial}{\partial \lambda^k}.$$

We have

**Proposition 4.2** *The symbols*

$$\Gamma_k^{ij} = -g^{is}\Gamma_{sk}^j = -\frac{1}{2}g^{is}g^{jl}\left(\partial_s g_{lk} + \partial_k g_{ls} - \partial_l g_{sk}\right),$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols associated to the diagonal metric  $g_{ii} = \partial_i A_0$ , for a Benney reduction are given by

$$\Gamma_k^{ij} = 0, \quad i \neq j \neq k \quad (4.7a)$$

$$\Gamma_i^{ij} = -\Gamma_i^{ji} = \frac{1}{(v_i - v_j)^2}, \quad i \neq j \quad (4.7b)$$

$$\Gamma_k^{ii} = -\frac{\partial_k A_0}{\partial_i A_0} \frac{1}{(v_k - v_i)^2}, \quad i \neq k \quad (4.7c)$$

$$\Gamma_i^{ii} = \sum_{k \neq i} \frac{\partial_k A_0}{\partial_i A_0} \frac{1}{(v^i - v^k)^2} - \frac{\delta(\ln \sqrt{\partial_i A_0})}{\partial_i A_0}. \quad (4.7d)$$

*Proof* For the metric  $g_{ij} = \delta_{ij} \partial_i A_0$ , we get

$$\Gamma_k^{ij} = -\frac{1}{2} \frac{1}{\partial_i A_0 \partial_i A_0} \left( \delta_{jk} \partial_{ik} A_0 + \delta_{ij} \partial_{kj} A_0 - \delta_{ik} \partial_{jk} A_0 \right),$$

and equations (4.7) are obtained from these by substituting, whenever is allowed, the Gibbons-Tsarev equations (2.6b). □

In particular, the curvature can be expressed solely via the shift operator  $\delta$ , acting on  $v^i$  and  $\ln \sqrt{\partial_i A_0}$ .

**Proposition 4.3** *The non vanishing components of the curvature tensor of the metric (4.5) for a Benney reduction can be written in terms of the quantities  $\delta(v^i)$ ,  $\delta(\ln \sqrt{\partial_i A_0})$ . More precisely, we have the following identity*

$$R_{ij}^{ij} = \frac{\delta(\ln \sqrt{\partial_i A_0}) + \delta(\ln \sqrt{\partial_j A_0})}{(v^i - v^j)^2} - 2 \frac{\delta(v^i) - \delta(v^j)}{(v^i - v^j)^3}. \quad (4.8)$$

*Proof* Since the rotation coefficients of the metric (4.5) are symmetric, it is easy to check that

$$R_{ij}^{ij} = \delta(\beta_{ij}).$$

Using the Gibbons-Tsarev system (2.6) we get (4.8). Moreover, it is well known that for a semi-Hamiltonian system the other components of the Riemann tensor are identically zero. □

Formula (4.8) presents a compact way of describing the curvature tensor of the Poisson structure (1.13) associated with a Benney reduction. However, we should notice here that the knowledge of the curvature is not sufficient to write the Poisson bracket. Indeed, what we need is a decomposition (1.12) of the curvature in terms of the symmetries of the system. From formula (4.8) this decomposition looks non-trivial; we will address this problem in the next section using a different approach.

## 5 Hamiltonian structure in the $\lambda$ picture

The purpose of this Section is to derive an explicit formulation for the Hamiltonian structure of a reduction of Benney in terms of the function  $\lambda(p)$ , which defines the reduction itself. In particular, we will show how the differential geometric objects associated with Ferapontov's Poisson operator of type (1.13) can be expressed, in the case of a Benney reduction with associated  $\lambda$ , in terms of the set of data

$$v^i, \quad \lambda''(v^i), \quad \lambda'''(v^i), \quad \lambda''''(v^i), \quad i = 1, \dots, n,$$

where  $v^i$ , the characteristic velocities of the reduction, are the critical points of  $\lambda$ . Moreover, we will describe the quadratic expansion of the curvature associated with the metric.

Let us consider a Benney reduction with associated function  $\lambda(p)$ . In this case, as already mentioned, a set of Riemann invariants is naturally selected, the critical values of  $\lambda(p)$ . From (3.17) and (4.1), it is immediate to check that the components of the metric which is potential in those coordinates can be expressed in terms of  $\lambda$  as

$$g_{ii} = \partial_i A^0 = \frac{1}{\lambda''(v^i)} = \operatorname{res}_{p=v^i} \left( \frac{dp}{\lambda'(p)} \right). \quad (5.1)$$

This result was already known in the case of dispersionless Gelfand-Dikii [5] reductions. However, it holds for all Benney reductions.

### 5.1 Completing the Loewner system

We move now our attention from the metric to the Christoffel symbols and the curvature tensor, looking for a way to describing these objects in terms of  $\lambda$  and its critical points.

However, this step is not immediate. Indeed, from equations (4.7) and (4.8) we need to find an expression in the  $\lambda$  picture for the quantities

$$\delta(v^i), \quad \delta(\log \sqrt{\partial_i A^0}),$$

we will see that the right object to look at is

$$F(p) = \frac{\partial p}{\partial \lambda} + \sum_{i=1}^n \frac{\partial p}{\partial \lambda^i}, \quad (5.2)$$

obtained from the inverse function of  $\lambda$  with respect to  $p$ , that is

$$p = p(\lambda, \lambda^1, \dots, \lambda^n). \quad (5.3)$$

The function  $F$  is thus determined once the function  $\lambda(p, \lambda^1, \dots, \lambda^n)$  is known. Before discussing the Christoffel symbols and the curvature, we will consider the properties of this function in detail. First of all, using the Loewner equations (3.12) and the expression (3.17), we can write  $F(p)$  in the following form:

$$\begin{aligned} F(p) &= \frac{1}{\lambda'(p)} - \sum_{i=1}^n \frac{\partial_i A^0}{p - v^i} \\ &= \frac{1}{\lambda'(p)} - \sum_{i=1}^n \operatorname{res}_{p=v^i} \left( \frac{1}{\lambda'(p)} \right) \frac{1}{p - v^i}. \end{aligned} \quad (5.4)$$

From its expansion (3.19), the function  $\frac{1}{\lambda'(p)}$  in  $v^i$  has a simple pole, therefore,  $F(p)$  is analytic at  $p = v^i$ . Using this fact, we can prove the following

**Theorem 7** *Let  $\lambda(p, \lambda^1, \dots, \lambda^n)$  be a solution of equation (3.2), and let  $v^1, \dots, v^n$  be its critical points. Defining the function  $F(p)$  as above (5.2), we have*

$$F(v^i) = \delta(v^i) \quad (5.5)$$

$$\frac{\partial F}{\partial p}(v^i) = \delta(\ln \sqrt{\partial_i A^0}). \quad (5.6)$$

*Proof* We have already shown that  $F(p)$  is analytic at  $p = v^i$ . In order to prove (5.5) and (5.6), we consider the system of  $n + 1$  differential equations

$$\frac{\partial p}{\partial \lambda^i} = \frac{\partial_i A^0}{v^i - p} \quad i = 1, \dots, n \quad (5.7)$$

$$\frac{\partial p}{\partial \lambda} = \sum_{k=0}^n \frac{\partial_k A^0}{p - v^k} + F(p).$$

The conditions

$$\frac{\partial^2 p}{\partial \lambda^i \partial \lambda^j} - \frac{\partial^2 p}{\partial \lambda^j \partial \lambda^i} = 0 \quad i \neq j \quad (5.8)$$

give nothing but the Gibbons-Tsarev system, hence are satisfied for any reduction. So, we concentrate our attention on the remaining  $n$  consistency conditions

$$\frac{\partial^2 p}{\partial \lambda \partial \lambda^i} - \frac{\partial^2 p}{\partial \lambda^i \partial \lambda} = 0, \quad (5.9)$$

which – by construction – are satisfied, to obtain some information about  $F(p)$ . Expanding both sides we obtain

$$\begin{aligned} \frac{\partial^2 p}{\partial \lambda \partial \lambda^i} &= \frac{\partial_i A_0}{(p - v_i)^2} \frac{\partial p}{\partial \lambda} \\ &= \frac{\partial_i A_0}{(p - v_i)^2} \left[ F(p) + \sum_{k=1}^n \frac{\partial_k A_0}{p - v_k} \right], \\ \frac{\partial^2 p}{\partial \lambda^i \partial \lambda} &= \frac{\partial F}{\partial p} \frac{\partial p}{\partial \lambda^i} + \frac{\partial F}{\partial \lambda^i} + \sum_{k=1}^n \left[ \frac{\partial_k \partial_i A_0}{p - v_k} + \frac{\partial_k A_0}{(p - v_k)^2} \left( \partial_i v_k - \frac{\partial p}{\partial \lambda^i} \right) \right] \\ &= \frac{\partial F}{\partial p} \frac{\partial_i A^0}{v^i - p} + \frac{\partial F}{\partial \lambda^i} + \sum_{k=1}^n \left[ \frac{\partial_k \partial_i A_0}{p - v_k} + \frac{\partial_k A_0}{(p - v_k)^2} \left( \partial_i v_k + \frac{\partial_i A^0}{p - v^i} \right) \right], \end{aligned}$$

substituting the Gibbons-Tsarev equations (2.6) in the above formulae and rearranging (5.9), we find that

$$\frac{F(p) - \delta(v^i)}{(p - v^i)^2} + \frac{\frac{\partial F}{\partial p}(p) - 2\delta(\log \sqrt{\partial_i A^0})}{p - v^i} - \frac{1}{\partial_i A^0} \frac{\partial F(p)}{\partial \lambda^i} = 0. \quad (5.10)$$

Multiplying by  $(p - v^i)^2$  and taking the limit for  $p \rightarrow v^i$ , we get

$$F(v^i) = \delta(v^i).$$

Then, taking the residue of the right hand side of (5.10) at  $p = v^i$  gives

$$\frac{\partial F}{\partial p}(v^i) = \delta(\log \sqrt{\partial_i A^0}).$$

□

Thus, specifying the function  $F$  turns out to be the analogue, in the  $\lambda$  picture, of completing the system (2.6), from which we were able to express the Christoffel symbols and the curvature tensor of the metric.

It follows from its definition, that  $F(p)$  is obtained from  $\frac{1}{\lambda^{(p)}}$  by subtracting off its singularities at  $p = v^i$ . The importance of this function is that it describes the invariant properties

of the reduction. For instance, if a reduction admits a function  $\lambda$  associated with it such that  $F(p) = 1$ , then the reduction is Galilean invariant. The case  $F(p) = p$  corresponds instead to the scaling invariance of the system. As seen in example below, the function  $F$ , hence these invariances, are strongly related with the curvature of the Poisson bracket.

**Example 5.4** *The 2–component Zakharov reduction is known to possess both Galilean and scaling invariance. Using the technique above, the former can be explained by saying that, for the function  $\lambda(p)$  given in Example 3.2, it follows  $F(p) = 1$ . Thus, we get*

$$\delta(v^i) = \sum_{k=1}^n \frac{\partial v^i}{\partial \lambda^k} = 1,$$

where the Riemann invariants are the critical values of  $\lambda(p)$ . For the scaling invariance, one can proceed as follows: define the function  $\varphi(p) = \ln \lambda(p)$ , where  $\lambda(p)$  is the same as before. This new function has the same critical points  $v^i$  as  $\lambda$ , plus the poles of  $\lambda(p)$ , which play no role in the derivation of the reduction (see Remark 3). Hence, it is associated with the Zakharov reduction, and in this case  $F(p) = p$ . Thus

$$\delta(v^i) = \sum_{k=1}^n \frac{\partial v^i}{\partial \varphi^k} = v^i, \quad (5.11)$$

where the  $\varphi^i = \ln \lambda^i$  are the natural Riemann invariants associated with  $\varphi(p)$ , i.e. its critical values. It is elementary to show that (5.11) corresponds to the scaling invariance with respect to the  $\lambda^i$ .

**Remark 7** *System (5.7) was considered for the first time in connection with a Benney reduction by Kokotov and Korotkin [14], in the particular case of the  $N$ –component Zakharov reduction, where  $F(p) = 1$ .*

In the next section, we will use the function  $F$  to describe the Christoffel symbols and the curvature in terms of  $\lambda$ . Nevertheless, the latter can be expressed directly in terms of  $F$  using the following residue formula

**Proposition 5.4** *In terms of the function  $F$ , the non vanishing components of the Riemann tensor satisfy the following identity:*

$$R_{ij}^{ij} = \sum_{k=i,j} \operatorname{res}_{p=v_k} \left( \frac{F(p)}{(p-v^i)^2(p-v^j)^2} dp \right) = \frac{1}{2\pi i} \int_{\Gamma_i \cup \Gamma_j} \frac{F(p)}{(p-v^i)^2(p-v^j)^2} dp \quad (5.12)$$

where  $\Gamma_i$  and  $\Gamma_j$  are two sufficiently small contours around  $p = v^i$  and  $p = v^j$ .

*Proof.* From (4.8) and Theorem 7 it follows immediately that

$$R_{ij}^{ij} = \frac{1}{(v^i - v^j)^2} \left[ \left( \frac{\partial F}{\partial p}(v^i) + \frac{\partial F}{\partial p}(v^j) \right) - 2 \frac{F(v^i) - F(v^j)}{v^i - v^j} \right] \quad (5.13)$$

It is easy to check the chain of identities:

$$\begin{aligned}
& \frac{1}{(v^i - v^j)^2} \left[ \left( \frac{\partial F}{\partial p}(v^i) + \frac{\partial F}{\partial p}(v^j) \right) - 2 \frac{F(v^i) - F(v^j)}{v^i - v^j} \right] = \\
& \lim_{p \rightarrow v^i} \frac{d}{dp} \left[ \frac{F(p)}{(p - v^j)^2} \right] + \lim_{p \rightarrow v^j} \frac{d}{dp} \left[ \frac{F(p)}{(p - v^i)^2} \right] = \\
& \lim_{p \rightarrow v^i} \frac{d}{dp} \left[ (p - v^i)^2 \frac{F(p)}{(p - v^j)^2 (p - v^i)^2} \right] + \lim_{p \rightarrow v^j} \frac{d}{dp} \left[ (p - v^j)^2 \frac{F(p)}{(p - v^i)^2 (p - v^j)^2} \right] = \\
& \operatorname{res}_{p=v^i} \left( \frac{F(p)}{(p - v^i)^2 (p - v^j)^2} dp \right) + \operatorname{res}_{p=v^j} \left( \frac{F(p)}{(p - v^i)^2 (p - v^j)^2} dp \right).
\end{aligned}$$

In the last identity we used the fact that  $F(p)$  is regular at  $p = v^i$ , for all  $i$ .

□

## 5.2 Christoffel symbols and Curvature tensor

### 5.2.1 Potential metric

We are now able to complete the description of the Poisson bracket associated with a Benney reduction, in the case where the metric is potential in the coordinate used.

**Proposition 5.5** *The Christoffel symbols (4.7) of the potential metric (5.1) can be written, in terms of the function  $\lambda$ , as*

$$\Gamma_k^{ij} = 0, \quad i \neq j \neq k \quad (5.14a)$$

$$\Gamma_i^{ij} = -\Gamma_i^{ji} = \frac{1}{(v^i - v^j)^2}, \quad i \neq j \quad (5.14b)$$

$$\Gamma_k^{ii} = -\frac{\lambda''(v^i)}{\lambda''(v^k)} \frac{1}{(v_k - v^i)^2}, \quad i \neq k \quad (5.14c)$$

$$\Gamma_i^{ii} = \frac{1}{6} \frac{\lambda''''(v^i)}{\lambda''(v^i)} - \frac{1}{4} \frac{\lambda''''(v^i)^2}{\lambda''(v^i)^2}, \quad (5.14d)$$

*The curvature tensor (4.8) of the potential metric (5.1) can be written, in terms of the function  $\lambda(p)$  as*

$$\begin{aligned}
R_{ij}^{ij} &= \frac{3}{(v^i - v^j)^4} \left( \frac{1}{\lambda''(v^i)} + \frac{1}{\lambda''(v^j)} \right) + \frac{1}{(v^i - v^j)^3} \left( \frac{\lambda'''(v^i)}{\lambda''(v^i)^2} - \frac{\lambda'''(v^j)}{\lambda''(v^j)^2} \right) \\
&+ \frac{1}{(v^i - v^j)^2} \left( \frac{1}{4} \frac{\lambda'''(v^i)^2}{\lambda''(v^i)^3} - \frac{1}{6} \frac{\lambda''''(v^i)}{\lambda''(v^i)^2} + \frac{1}{4} \frac{\lambda'''(v^j)^2}{\lambda''(v^j)^3} - \frac{1}{6} \frac{\lambda''''(v^j)}{\lambda''(v^j)^2} \right) \\
&+ \sum_{k \neq i, j} \frac{1}{\lambda''(v^k)} \frac{1}{(v^i - v^k)^2 (v^j - v^k)^2}.
\end{aligned} \quad (5.15)$$

*Proof* Starting from the definition (5.2) of  $F$ , and using (3.19), we can write

$$F(v^i) = -\frac{1}{2} \frac{\lambda'''(v^i)}{\lambda''(v^i)^2} - \sum_{k \neq i} \frac{1}{\lambda''(v^k)} \frac{1}{v^i - v^k}, \quad (5.16)$$

$$\frac{\partial F}{\partial p}(v^i) = \frac{1}{4} \frac{\lambda''''(v^i)^2}{\lambda''(v^i)^3} - \frac{1}{6} \frac{\lambda''''(v^i)}{\lambda''(v^i)^2} + \sum_{k \neq i} \frac{1}{\lambda''(v^k)} \frac{1}{(v^i - v^k)^2}. \quad (5.17)$$

Then, by Theorem 7, and substituting the above expressions for  $F$  into (4.7) and (4.8), we obtain (5.14) and (5.15) respectively. □

**Remark 8** We note that (5.14d) is a constant multiple of the Schwarzian derivative of  $\lambda'(p)$ , evaluated at  $p = v^i$ .

Recalling the expression (5.1) for the the metric, form the Proposition above we find that the whole Poisson operator associated with a Benney reduction of symbol  $\lambda$  depends only on the critical points of  $\lambda$  and on the value of its second, third and fourth derivatives evaluated at these points.

We have now all we need to write the nonlocal tail of the Hamiltonian structure associated to the metric  $g_{ii} = \partial_i A_0$ .

**Proposition 5.6** *The non-vanishing components of the Riemann tensor of the metric (5.1) admit the following quadratic expansion*

$$R_{ij}^{ij} = \frac{1}{2\pi i} \int_C w^i(\lambda) w^j(\lambda) d\lambda. \quad (5.18)$$

where  $C = C_1 \cup \dots \cup C_n$  with  $C_i$  described as above, and the functions

$$w^i(\lambda) = \frac{\frac{\partial p}{\partial \lambda}}{(p(\lambda) - v^i)^2},$$

are the generating functions of the symmetries (3.24). Consequently the non local tail of the Hamiltonian structure associated to the metric  $g_{ii} = \partial_i A_0$  is given by

$$\frac{1}{2\pi i} \int_C w^i(\lambda) \lambda_x^i \left( \frac{d}{dx} \right)^{-1} w^j(\lambda) \lambda_x^j d\lambda.$$

*Proof* We prove the Proposition showing that the integral in (5.18) is the same as the right hand side of (5.15). First, writing the integral

$$\frac{1}{2\pi i} \int_C w^i(\lambda) w^j(\lambda) d\lambda$$

in terms of the variable  $p$  we obtain

$$R_{ij}^{ij} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\frac{1}{\lambda'(p)}}{(p-v^i)^2(p-v^j)^2} dp = \sum_{k=1}^n \operatorname{res}_{p=v^k} \left( \frac{\frac{1}{\lambda'(p)}}{(p-v^i)^2(p-v^j)^2} dp \right). \quad (5.19)$$

Using (3.19), the integrand can be expanded, for  $k = 1, \dots, n$ , as

$$\frac{1}{(p-v^i)^2(p-v^j)^2} \left( \frac{1}{\lambda''(v^k)} \frac{1}{p-v^k} - \frac{1}{2} \frac{\lambda'''(v^k)}{\lambda''(v^k)^2} + \left( \frac{1}{4} \frac{\lambda'''(v^k)^2}{\lambda''(v^k)^3} - \frac{1}{6} \frac{\lambda''''(v^k)}{\lambda''(v^k)^2} \right) (p-v^k) + \dots \right).$$

Thus, for  $k \neq i, j$  we get

$$\operatorname{res}_{p=v^k} \left( \frac{\frac{1}{\lambda'(p)}}{(p-v^i)^2(p-v^j)^2} dp \right) = \frac{1}{\lambda''(v^k)} \frac{1}{(v^k-v^i)^2(v^k-v^j)^2},$$

while

$$\begin{aligned} \operatorname{res}_{p=v^i} \left( \frac{\frac{1}{\lambda'(p)}}{(p-v^i)^2(p-v^j)^2} dp \right) &= \frac{3}{(v^i-v^j)^4} \frac{1}{\lambda''(v^i)} + \frac{1}{(v^i-v^j)^3} \frac{\lambda'''(v^i)}{\lambda''(v^i)^2} \\ &+ \frac{1}{(v^i-v^j)^2} \left( \frac{1}{4} \frac{\lambda'''(v^i)^2}{\lambda''(v^i)^3} - \frac{1}{6} \frac{\lambda''''(v^i)}{\lambda''(v^i)^2} \right), \end{aligned}$$

$$\begin{aligned} \operatorname{res}_{p=v^j} \left( \frac{\frac{1}{\lambda'(p)}}{(p-v^i)^2(p-v^j)^2} dp \right) &= \frac{3}{(v^j-v^i)^4} \frac{1}{\lambda''(v^j)} + \frac{1}{(v^j-v^i)^3} \frac{\lambda'''(v^j)}{\lambda''(v^j)^2} \\ &+ \frac{1}{(v^j-v^i)^2} \left( \frac{1}{4} \frac{\lambda'''(v^j)^2}{\lambda''(v^j)^3} - \frac{1}{6} \frac{\lambda''''(v^j)}{\lambda''(v^j)^2} \right), \end{aligned}$$

From these, formula (5.15) for the curvature tensor follows. The last statement of the Proposition is a consequence of the general theory of Ferapontov. □

**Remark 9** *Alternatively, one can prove the above result using starting from the function  $F$ , namely deforming the integral*

$$\frac{1}{2\pi i} \int_{\Gamma_i \cup \Gamma_j} \frac{F(p)}{(p-v^i)^2(p-v^j)^2} dp$$

which is shown in Proposition 5.4 to be equal to the curvature, into (5.18). In order to do so, it is sufficient to verify the following identity

$$-\frac{1}{2\pi i} \int_{\Gamma_i \cup \Gamma_j} \frac{\sum_{k=1}^n \frac{\partial_k A_0}{p-v^k}}{(p-v^i)^2(p-v^j)^2} dp = \frac{1}{2\pi i} \int_{\Gamma - (\Gamma_i \cup \Gamma_j)} \frac{\frac{1}{\frac{\partial \lambda}{\partial p}}}{(p-v^i)^2(p-v^j)^2} dp,$$

that can be proved by straightforward computation. Indeed, since the left hand side is the sum of residues at  $p = v^i$  and  $p = v^j$  of a function having a pole of order 3 at  $p = v^i, v^j$  we obtain

$$\begin{aligned}
l.h.s &= - \left( \lim_{p \rightarrow v^i} + \lim_{p \rightarrow v^j} \right) \frac{1}{2} \frac{d^2}{dp^2} \sum_{k=1}^n \left[ \frac{(p - v^i) \partial_k A_0}{(p - v^j)^2 (p - v^k)} \right] \\
&= \sum_{k \neq i, j} \frac{\partial_k A_0}{(v^i - v^j)^2} \left[ \frac{2}{(v^i - v^j)(v^i - v^k)} + \frac{1}{(v^i - v^k)^2} - \frac{2}{(v^i - v^j)(v^j - v^k)} + \frac{1}{(v^j - v^k)^2} \right] \\
&= \sum_{k \neq i, j} \frac{\partial_k A_0}{(v^i - v^k)^2 (v^j - v^k)^2}.
\end{aligned}$$

On the other hand the right hand side is the sum of residues at  $p = v^k$ , for  $k \neq i, j$  of a function having simple poles at  $p = v^k$ :

$$r.h.s. = \sum_{k \neq i, j} \lim_{p \rightarrow v^k} \frac{(p - v^k) \frac{1}{\partial p}}{(p - v^i)^2 (p - v^j)^2} = \sum_{k \neq i, j} \frac{\partial_k A_0}{(v^i - v^k)^2 (v^j - v^k)^2} = l.h.s.$$

Recalling the results above, we have the following

**Theorem 8** *The reduction of Benney associated with the function  $\lambda(p, \lambda^1, \dots, \lambda^n)$  is Hamiltonian with the Hamiltonian structure*

$$\Pi^{ij} = \lambda''(v^i) \delta^{ij} \frac{d}{dx} + \Gamma_k^{ij} \lambda_x^k + \frac{1}{2\pi i} \int_C \frac{\frac{\partial p}{\partial \lambda} \lambda_x^i}{(p(\lambda) - v^i)^2} \left( \frac{d}{dx} \right)^{-1} \frac{\frac{\partial p}{\partial \lambda} \lambda_x^j}{(p(\lambda) - v^j)^2} d\lambda, \quad (5.20)$$

where

$$\begin{aligned}
\Gamma_k^{ij} \lambda_x^k &= \frac{\lambda_x^i - \lambda_x^j}{(v^i - v^j)^2} \quad i \neq j, \\
\Gamma_k^{ii} \lambda_x^k &= \left( \frac{1}{6} \frac{\lambda''''(v^i)}{\lambda''(v^i)} - \frac{1}{4} \frac{\lambda'''(v^i)^2}{\lambda''(v^i)^2} \right) \lambda_x^i - \sum_{k \neq i} \frac{\lambda''(v^i)}{\lambda''(v^k)} \frac{\lambda_x^k}{(v^i - v^k)^2}
\end{aligned}$$

and  $C = C_1 \cup \dots \cup C_n$ . Here, the  $v^i$  are the critical points of  $\lambda$ , and the  $\lambda^i$  the critical values. In this coordinates, the metric  $g_{ij} = \frac{\delta^{ij}}{\lambda''(v^i)}$  is potential.

### 5.2.2 The general case

As we pointed out previously, any metric (4.1) associated with a reduction can be put in potential form, after a suitable change of the Riemann invariants. However, it is often convenient to write the expression of the Poisson operators generated by these metrics in terms

of the Riemann invariants selected by  $\lambda$ . Thus, we consider the metrics

$$g_{ii} = \frac{\partial_i A^0}{\varphi_i(\lambda^i)} = \frac{1}{\varphi_i(\lambda^i)\lambda''(v^i)}, \quad (5.21)$$

where  $\varphi_i = \varphi_i(\lambda^i)$  are arbitrary functions, and we proceed as before.

**Proposition 5.7** *The Christoffel symbols appearing in the Hamiltonian structure are given by*

$$\begin{aligned} \Gamma_k^{ij} &= 0, & i \neq j \neq k \\ \Gamma_i^{ij} &= \frac{\varphi_j}{(v^i - v^j)^2}, & i \neq j \\ \Gamma_j^{ij} &= -\frac{\varphi_i}{(v^i - v^j)^2}, & i \neq j \\ \Gamma_k^{ii} &= -\frac{\lambda''(v^i)}{\lambda''(v^k)} \frac{\varphi_i}{(v^k - v^i)^2}, & i \neq k \\ \Gamma_i^{ii} &= \varphi_i \left( \frac{1}{6} \frac{\lambda''''(v^i)}{\lambda''(v^i)} - \frac{1}{4} \frac{\lambda''''(v^i)^2}{\lambda''(v^i)^2} \right) + \frac{1}{2} \varphi_i'. \end{aligned}$$

Here we denote

$$\varphi_i' = \frac{d\varphi}{d\lambda^i}(\lambda^i).$$

The nonlocal tail appearing in the Hamiltonian structure is then given by

$$\frac{1}{2\pi i} \sum_{k=1}^n \int_{C_k} \frac{\frac{\partial p}{\partial \lambda} \lambda_x^i}{(p(\lambda) - v^i)^2} \left( \frac{d}{dx} \right)^{-1} \frac{\frac{\partial p}{\partial \lambda} \lambda_x^j}{(p(\lambda) - v^j)^2} \varphi_k(\lambda) d\lambda. \quad (5.22)$$

*Proof* The proof of the formula for the  $\Gamma_k^{ij}$  is a straightforward computation. Let us prove the second statement. Since nothing new is involved in such computations we will skip the details. For the metric (5.21), the non vanishing components of the curvature tensor are

$$\begin{aligned} R_{ij}^{ij} &= \frac{3}{(v^i - v^j)^4} (\varphi_i \partial_i A^0 + \varphi_j \partial_j A^0) - \frac{2}{(v^i - v^j)^3} (\varphi_i \partial_i v^i - \varphi_j \partial_j v^j) \\ &+ \frac{1}{(v^i - v^j)^2} \left( \varphi_i \partial_i \ln \sqrt{\partial_i A^0} + \varphi_j \partial_j \ln \sqrt{\partial_j A^0} \right) \\ &+ \sum_{k \neq i, j} \frac{\varphi_k \partial_k A^0}{(v^i - v^k)^2 (v^j - v^k)^2} + \frac{1}{2} \frac{\varphi_i' + \varphi_j'}{(v^i - v^j)^2}. \end{aligned} \quad (5.23)$$

Expression (5.23) can be written, in terms of  $\lambda$ , as

$$\begin{aligned}
R_{ij}^{ij} &= \frac{3}{(v^i - v^j)^4} \left( \frac{\varphi_i}{\lambda''(v^i)} + \frac{\varphi_j}{\lambda''(v^j)} \right) + \frac{1}{(v^i - v^j)^3} \left( \varphi_i \frac{\lambda'''(v^i)}{\lambda''(v^i)^2} - \varphi_j \frac{\lambda'''(v^j)}{\lambda''(v^j)^2} \right) \\
&+ \frac{1}{(v^i - v^j)^2} \left( \varphi_i \left( \frac{1}{4} \frac{\lambda''''(v^i)^2}{\lambda''(v^i)^3} - \frac{1}{6} \frac{\lambda''''(v^i)}{\lambda''(v^i)^2} \right) + \varphi_j \left( \frac{1}{4} \frac{\lambda''''(v^j)^2}{\lambda''(v^j)^3} - \frac{1}{6} \frac{\lambda''''(v^j)}{\lambda''(v^j)^2} \right) \right) \\
&+ \sum_{k \neq i, j} \frac{1}{\lambda''(v^k)} \frac{\varphi_k}{(v^i - v^k)^2 (v^j - v^k)^2} + \frac{1}{2} \frac{\varphi'_i + \varphi'_j}{(v^i - v^j)^2}, \tag{5.24}
\end{aligned}$$

The equivalence between (5.22) and the right hand side of (5.24) can be obtained by rewriting the integrals above in the  $p$ -plane,

$$\frac{1}{2\pi i} \sum_{k=1}^n \int_{\Gamma_k} \frac{\varphi_k(\lambda(p)) \frac{1}{\lambda'(p)}}{(p - v^i)^2 (p - v^j)^2} dp. \tag{5.25}$$

and using the same arguments of the main theorem, except that for every  $k$ , in the integral around  $\Gamma_k$  we have to consider also the contribution of the function  $\varphi_k(\lambda(p))$  which expands, at  $p = v^k$ , as

$$\varphi_k(\lambda(p)) = \varphi_k + \frac{\varphi'_k}{2} \lambda''(v^k) (p - v^k)^2 + \dots$$

□

It follows from the above that we have

**Theorem 9** *The reduction of Benney associated with the function  $\lambda(p, \lambda^1, \dots, \lambda^n)$  is Hamiltonian with the family of Hamiltonian structures*

$$\begin{aligned}
\Pi^{ij} &= \varphi_i \lambda''(v^i) \delta^{ij} \frac{d}{dx} + \Gamma_k^{ij} \lambda_x^k \\
&+ \frac{1}{2\pi i} \sum_{k=1}^n \int_{C_k} \frac{\frac{\partial p}{\partial \lambda} \lambda_x^i}{(p(\lambda) - v^i)^2} \left( \frac{d}{dx} \right)^{-1} \frac{\frac{\partial p}{\partial \lambda} \lambda_x^j}{(p(\lambda) - v^j)^2} \varphi_k(\lambda) d\lambda, \tag{5.26}
\end{aligned}$$

with

$$\begin{aligned}
\Gamma_k^{ij} \lambda_x^k &= \frac{\varphi_j \lambda_x^i - \varphi_i \lambda_x^j}{(v^i - v^j)^2} \quad i \neq j, \\
\Gamma_k^{ii} \lambda_x^k &= \varphi_i \left( \frac{1}{6} \frac{\lambda''''(v^i)}{\lambda''(v^i)} - \frac{1}{4} \frac{\lambda''''(v^i)^2}{\lambda''(v^i)^2} \right) \lambda_x^i + \frac{1}{2} \varphi'_i \lambda_x^i - \sum_{k \neq i} \frac{\lambda''(v^i)}{\lambda''(v^k)} \frac{\varphi_i \lambda_x^k}{(v^i - v^k)^2},
\end{aligned}$$

where  $\varphi_1, \dots, \varphi_n$  are arbitrary functions of a single variable. Here, the  $v^i$  are the critical points of  $\lambda$ , the coordinates  $\lambda^i$  the corresponding critical values, and the  $C_k$  are the contours defined above.

## 6 Finite nonlocal tail: some examples

In the expression (5.26) for the Poisson operator, the components of the curvature are expressed as integrals of functions around suitable contours in a complex domain. A natural question to ask is whether this integral can be reduced to a finite sum, and we will show now some examples where this is possible. For simplicity, we will consider the case when  $\varphi_1(\lambda) = \dots = \varphi_n(\lambda) = \lambda^k$ , for  $k \in \mathbb{Z}$ . In this case the curvature can be expressed as

$$R_{ij}^{ij} = \frac{1}{2\pi i} \int_C \frac{\left(\frac{\partial p}{\partial \lambda}\right)^2 \lambda^k}{(p(\lambda) - v^i)^2 (p(\lambda) - v^j)^2} d\lambda, \quad k \in \mathbb{Z}.$$

Essentially, the finite expansion appears whenever is possible to substitute the the contour  $C$  with a contour a round  $\lambda = \infty$  and a finite number of other marked points. We illustrate this special situation in two simple examples.

### 6.1 2-component Zakharov reduction

In this case (see examples 2.1, 3.2), since  $\lambda$  is a single-valued rational function of  $p$ , it is convenient to work in the  $p$ -plane. In order to calculate the curvature, the non vanishing components of the Riemann tensor are given by

$$R_{12}^{12} = \sum_{i=1}^2 \operatorname{res}_{p=v^i} \left( \frac{\lambda(p)^k \frac{1}{\lambda'(p)}}{(p - v_1)^2 (p - v_2)^2} dp \right), \quad k \in \mathbb{Z}.$$

The abelian differential

$$\frac{\lambda(p)^k \frac{1}{\lambda'(p)}}{(p - v_1)^2 (p - v_2)^2} dp$$

has poles at the points  $p = v_1$ ,  $p = v_2$ , as well as:

if  $k > 2$

$$p = \infty, \quad p = \frac{A_1}{A_0} \quad (\text{poles of } \lambda)$$

if  $k < 0$

$$p = s_1 = \frac{1}{2} \frac{A_1 + (A_1^2 - 4A_0^3)^{1/2}}{A_0}, \quad p = s_2 = \frac{1}{2} \frac{A_1 - (A_1^2 - 4A_0^3)^{1/2}}{A_0} \quad (\text{zeros of } \lambda),$$

while for  $k = 0, 1, 2$  there are no other poles. Since the sum of the residues of an abelian differential on a compact Riemann surface is zero, we can substitute the sum of residues at  $p = v_1, v_2$  with, respectively

- zero if  $k = 0, 1, 2$ ,

- minus the sum of residues at  $p = \infty$ ,  $p = \frac{A_1}{A_0}$  if  $k > 2$
- minus the sum of residues at  $p = s_1$ ,  $p = s_2$ , if  $k < 0$ .

Summarizing, we have

$$\begin{aligned}
R_{ij}^{ij} &= 0, & k = 0, 1, 2, \\
R_{ij}^{ij} &= - \left( \operatorname{res}_{p=\infty} + \operatorname{res}_{p=\frac{A_1}{A_0}} \right) \frac{\lambda(p)^k \frac{1}{\lambda'(p)}}{(p-v^i)^2(p-v^j)^2} dp, & k > 2, \\
R_{ij}^{ij} &= - \left( \operatorname{res}_{p=s_1} + \operatorname{res}_{p=s_2} \right) \frac{\lambda(p)^k \frac{1}{\lambda'(p)}}{(p-v^i)^2(p-v^j)^2} dp, & k < 0.
\end{aligned}$$

Moreover, as a counterpart in the  $\lambda$ -plane of the above formulae we have

$$\begin{aligned}
R_{ij}^{ij} &= 0, & k = 0, 1, 2, \\
R_{ij}^{ij} &= -2 \operatorname{res}_{\lambda=\infty} \left( w^1(\lambda) w^2(\lambda) \lambda^k d\lambda \right), & k > 2, \\
R_{ij}^{ij} &= -2 \operatorname{res}_{\lambda=0} \left( w^1(\lambda) w^2(\lambda) \lambda^k d\lambda \right), & k < 0,
\end{aligned}$$

and this shows that the residues have to be computed around marked points, which not depend on the dynamics of the reduction. Expanding  $w^1(\lambda)$  and  $w^2(\lambda)$  near  $\lambda = \infty$ , we get

$$\begin{aligned}
w^1(\lambda) &= - \sum_{k=1}^{\infty} \frac{k w_k^1}{\lambda^{k+1}} = \\
&= - \frac{1}{\lambda^2} - \frac{2v^1}{\lambda^3} - \frac{3(2A_0^3 + A_1^2 + 2A_1A_0^{\frac{3}{2}})}{A_0^2} \frac{1}{\lambda^4} - \frac{4(A_1^3 + 6A_1A_0^3 + 3A_0^{\frac{9}{2}} + 3A_1^2A_0^{\frac{3}{2}})}{A_0^3} \frac{1}{\lambda^5} + \dots \\
w^2(\lambda) &= - \sum_{k=1}^{\infty} \frac{k w_k^2}{\lambda^{k+1}} = \\
&= - \frac{1}{\lambda^2} - \frac{2v^2}{\lambda^3} - \frac{3(2A_0^3 + A_1^2 - 2A_1A_0^{\frac{3}{2}})}{A_0^2} \frac{1}{\lambda^4} - \frac{4(A_1^3 + 6A_1A_0^3 - 3A_0^{\frac{9}{2}} - 3A_1^2A_0^{\frac{3}{2}})}{A_0^3} \frac{1}{\lambda^5} + \dots
\end{aligned}$$

and near  $\lambda = 0$

$$\begin{aligned}
w^1(\lambda) &= \sum_{h=0}^{\infty} z_{-h}^1 \lambda^h = -\frac{2A_0^2(-\sqrt{A_1^2 - 4A_0^3} + A_1)}{\left(A_1 - \sqrt{A_1^2 - 4A_0^3} + 2A_0^{\frac{3}{2}}\right)^2 \sqrt{A_1^2 - 4A_0^3}} + \\
&\quad - \frac{8A_0^3 \left(\sqrt{A_1^2 - 4A_0^3}(A_0^3 - A_1^2) + A_1^3 - 3A_1A_0^3 + 2A_0^{\frac{9}{2}}\right)}{\left(A_1^2 - 4A_0^3\right)^{\frac{3}{2}} \left(A_1 - \sqrt{A_1^2 - 4A_0^3} + 2A_0^{\frac{3}{2}}\right)^3} \lambda + \dots \\
w^2(\lambda) &= \sum_{h=0}^{\infty} z_{-h}^2 \lambda^h = -\frac{2A_0^2(-\sqrt{A_1^2 - 4A_0^3} + A_1)}{\left(A_1 - \sqrt{A_1^2 - 4A_0^3} - 2A_0^{\frac{3}{2}}\right)^2 \sqrt{A_1^2 - 4A_0^3}} + \\
&\quad - \frac{8A_0^3 \left(\sqrt{A_1^2 - 4A_0^3}(A_0^3 - A_1^2) + A_1^3 - 3A_1A_0^3 - 2A_0^{\frac{9}{2}}\right)}{\left(A_1^2 - 4A_0^3\right)^{\frac{3}{2}} \left(A_1 - \sqrt{A_1^2 - 4A_0^3} - 2A_0^{\frac{3}{2}}\right)^3} \lambda + \dots
\end{aligned}$$

and taking into account that the coefficients of the expansion are characteristic velocities of symmetries, we easily obtain the quadratic expansion of the Riemann tensor. For  $k > 2$  we have

$$R_{12}^{12} = \sum_{i+j=k-1} (w_i^1 w_j^2 + w_j^1 w_i^2),$$

while for  $k < 0$ , we obtain

$$R_{12}^{12} = \sum_{i+j=k+1} (z_i^1 z_j^2 + z_j^1 z_i^2),$$

which can be put in the canonical form (1.12) after a linear change of basis of the symmetries. The expressions of these expansions in the Riemann invariants can be found by using formulae given in Example 2.1.

## 6.2 Dispersionless Boussinesq reduction

The case of the dispersionless Boussinesq reduction can be treated in a similar way. From Example 3.3, we will consider a function  $\lambda$  which is polynomial in  $p$ ,

$$\lambda = p^3 + 3A^0 p + 3A^1,$$

thus meromorphic on the Riemann sphere. The choice of a different normalisation reflects in the expansions below, where we have to consider an expansion in the local parameter  $t = \lambda^{-\frac{1}{3}}$ . For simplicity let us consider only the case  $k \geq 0$ . We observe that, apart from the poles at  $p = v_1$  and  $p = v_2$ , we have only an additional pole at infinity (starting from  $k = 2$ ). Following the same procedure used in the Zakharov case we obtain

$$\begin{aligned}
R_{ij}^{ij} &= 0, & k &= 0, 1, \\
R_{ij}^{ij} &= 3 \operatorname{res}_{t=0} (w^1(t) w^2(t) t^{-(3k+4)} dt), & k &> 2.
\end{aligned}$$

The expansions of  $w^1(t)$  and  $w^2(t)$  near  $t = 0$  are given by

$$w^1(t) = \sum_{k=0}^{\infty} k w_k^1 t^{k+4} = t^4 + 2(-A_0)^{\frac{1}{2}} t^5 + 4 \left( A_1 - (-A_0)^{\frac{3}{2}} \right) t^7 \\ + 5 \left( 2A_1(-A_0)^{\frac{1}{2}} - A_0^2 \right) t^8 + \dots$$

$$w^2(t) = \sum_{k=0}^{\infty} k w_k^2 t^{k+4} = t^4 - 2(-A_0)^{\frac{1}{2}} t^5 + 4 \left( A_1 + (-A_0)^{\frac{3}{2}} \right) t^7 \\ + 5 \left( -2A_1(-A_0)^{\frac{1}{2}} - A_0^2 \right) t^8 + \dots$$

From these formulas we immediately get the quadratic expansion of the Riemann tensor:

$$\begin{aligned} k = 0 : \quad R_{12}^{12} &= 0, \\ k = 1 : \quad R_{12}^{12} &= 0, \\ k = 2 : \quad R_{12}^{12} &= 3(v^1 + v^2) = 0, \end{aligned}$$

More generally, we have

$$R_{12}^{12} = \frac{3}{2} \sum_{i+j=k-1} (w_{3i}^1 w_{3j}^2 + w_{3j}^1 w_{3i}^2) \quad k > 2.$$

The expression in the Riemann invariants can be obtained from Example 3.3.

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