

STABILITY CONDITIONS AND THE A_2 QUIVER

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ABSTRACT. We compute the space of stability conditions on the CY_n version of the A_2 quiver for all n and relate it to the Frobenius-Saito structure on the unfolding space of the A_2 singularity.

1. INTRODUCTION

In this paper we study spaces of stability conditions $\text{Stab}(D_n)$ on the sequence of CY_n triangulated categories D_n associated to the A_2 quiver. Our main result is Theorem 1.1 below. There are several striking features. Firstly we obtain uniform results for all n : the space of stability conditions quotiented by the action of the spherical twists is independent of n , although the identification maps are highly non-trivial. Secondly, there is a close link between our spaces of stability conditions and the Frobenius-Saito structure on the unfolding space of the A_2 singularity: in fact this structure is precisely what encodes the identifications between our stability spaces for various n . A third interesting feature is that the space of stability conditions on the usual derived category of the A_2 quiver arises as a kind of limit of the spaces for the categories D_n as $n \rightarrow \infty$.

For any integer $n \geq 2$ we let $D_n = D_{CY_n}(A_2)$ denote the bounded derived category of the CY_n complex Ginzburg algebra associated to the A_2 quiver. It is a triangulated category of finite type over \mathbb{C} and is characterised by the following two properties:

- (a) It is CY_n , i.e. for any pair of objects $A, B \in D_n$ there are natural isomorphisms

$$\text{Hom}_{D_n}^*(A, B) \cong \text{Hom}_{D_n}^*(B, A[n])^\vee.$$

- (b) It is (strongly) generated by two spherical objects S_1, S_2 satisfying

$$(1) \quad \text{Hom}_{D_n}^*(S_1, S_2) = \mathbb{C}[-1].$$

We denote by D_∞ the usual bounded derived category of the A_2 quiver. It is again a \mathbb{C} -linear triangulated category, and is characterised by the property that it is generated by two exceptional objects S_1, S_2 satisfying (1) and

$$\text{Hom}_{D_\infty}^*(S_2, S_1) = 0.$$

The notation D_∞ is convenient: the point is that as n increases the Serre dual to the extension $S_1 \rightarrow S_2[1]$ occurs in higher and higher degrees until when $n = \infty$ it doesn't occur at all.

For $2 \leq n \leq \infty$ we let $\text{Stab}(D_n)$ denote the space of stability conditions on the category D_n . We let $\text{Stab}_*(D_n) \subset \text{Stab}(D_n)$ be the connected component containing stability conditions in which the objects S_1 and S_2 are stable. Let $\text{Aut}(D_n)$ denote the group of exact \mathbb{C} -linear autoequivalences of the category D_n , considered up to isomorphism of functors. We let $\text{Aut}_*(D_n)$ denote the subquotient consisting of autoequivalences which preserve the connected component $\text{Stab}_*(D_n)$ modulo those which act trivially on it. When $n < \infty$ we let $\text{Sph}_*(D_n)$ denote the subgroup of $\text{Aut}_*(D_n)$ generated by the Seidel-Thomas twist functors Tw_{S_1} and Tw_{S_2} corresponding to the spherical objects S_1 and S_2 .

The Cartan algebra of the Lie algebra \mathfrak{sl}_3 corresponding to the A_2 root system can be described explicitly as

$$\mathfrak{h} = \{(u_1, u_2, u_3) \in \mathbb{C} : \sum_i u_i = 0\}.$$

The complement of the root hyperplanes is

$$\mathfrak{h}^{\text{reg}} = \{(u_1, u_2, u_3) \in \mathfrak{h} : i \neq j \implies u_i \neq u_j\}.$$

There is an obvious action of the Weyl group $W = S_3$ permuting the u_i which is free on $\mathfrak{h}^{\text{reg}}$. The quotient \mathfrak{h}/W is isomorphic to \mathbb{C}^2 with co-ordinates (a, b) by setting

$$p(x) = (x - u_1)(x - u_2)(x - u_3) = x^3 + ax + b.$$

The image of the root hyperplanes $u_i = u_j$ is the discriminant

$$\Delta = \{(a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 = 0\}.$$

We can now state the main result of this paper.

Theorem 1.1. (a) For $2 \leq n < \infty$ there is an isomorphism of complex manifolds

$$\text{Stab}_*(D_n)/\text{Sph}_*(D_n) \cong \mathfrak{h}^{\text{reg}}/W.$$

Under this isomorphism the central charge map $\text{Stab}(D_n) \rightarrow \mathbb{C}^2$ corresponds to the multi-valued map $\mathfrak{h}^{\text{reg}}/W \rightarrow \mathbb{C}^2$ given by

$$\int_{\gamma_i} p(x)^{(n-2)/2} dx$$

for an appropriate basis of paths γ_i connecting the zeroes of the polynomial $p(x)$.

(b) For $n = \infty$ there is an isomorphism of complex manifolds

$$\mathrm{Stab}(\mathbb{D}_\infty) \cong \mathfrak{h}/W.$$

Under this isomorphism the central charge map $\mathrm{Stab}(\mathbb{D}_\infty) \rightarrow \mathbb{C}^2$ corresponds to the map $\mathfrak{h}/W \rightarrow \mathbb{C}^2$ given by

$$\int_{\delta_i} e^{p(x)} dx$$

for an appropriate basis of paths δ_i which approach ∞ in both directions along rays for which $x^3 \rightarrow -\infty$.

Theorem 1.1 gives a precise link with the Frobenius-Saito structure on the unfolding space of the A_2 singularity $x^3 = 0$. The corresponding Frobenius manifold is precisely $M = \mathfrak{h}/W$. The maps appearing in part (a) of our result are then the *twisted period maps* of M with parameter $\nu = (n - 2)/2$ (see Equation (5.11) of [4]). The map in part (b) is given by the *deformed flat co-ordinates* of M with parameter $\hbar = 1$ (see [3, Theorem 2.3]).

The $n < \infty$ case of Theorem 1.1 was first considered by R.P. Thomas in [15]: he obtained the $n = 2$ case and discussed the relationship with Fukaya categories and homological mirror symmetry. The $n = 2$ case was also proved in [1] and generalised to arbitrary ADE Dynkin diagrams. The $n = 3$ case of Theorem 1.1 was proved in [13], and was extended to all Dynkin quivers of A and D type in [2]. The first statement of part (a), that $\mathrm{Stab}(\mathbb{D}) \cong \mathfrak{h}^{\mathrm{reg}}/W$, was proved for all $n < \infty$ in [10].

The case $n = \infty$ of Theorem 1.1 was first considered by A.D. King [7] who proved that $\mathrm{Stab}(\mathbb{D}_\infty) \cong \mathbb{C}^2$. This result was obtained by several other researchers since then, and a proof was written down in [10]. The more precise statement of Theorem 1.1 (b) was conjectured by A. Takahashi [14].

Just as we were failing to get round to finishing this paper, A. Ikeda posted [6] on the arxiv which also proves Theorem 1.1 (a), and indeed generalizes it to the case of the A_k quiver for all $k \geq 1$. The methods we use here are quite different however so we feel this paper is also worth publishing.

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2. AUTO-EQUIVALENCES AND T-STRUCTURES

We start by recalling some results from [8, 13]. We use the word *heart* to mean the heart of a bounded t-structure. Recall that any such t-structure is determined by its

heart. For any $2 \leq n \leq \infty$ the category D_n has a canonical heart which is the extension-closed subcategory generated by S_1 and S_2 . The exchange graph $\text{EG}(D_n)$ has vertices the hearts in D_n and edges corresponding to simple tilts. We denote by $\text{EG}^o(D_n)$ the *principal component*, i.e. the connected containing the canonical heart $\mathcal{A} \subset D_n$. The vertices of $\text{EG}^o(D_n)$ are called *reachable* hearts. We say that a heart in D_n is *full* if it is equivalent to the canonical heart.

Remark 2.1. When $n > 2$, this canonical heart is equivalent to the category $\text{Rep}(A_2)$ of representations of the A_2 quiver; besides the simple objects S_1 and S_2 , it contains one more indecomposable object which we denote E ; there is a short exact sequence

$$(2) \quad 0 \longrightarrow S_2 \longrightarrow E \longrightarrow S_1 \longrightarrow 0.$$

When $n = 2$, the canonical heart is equivalent to the category of representations of the preprojective algebra: besides E there is another non-simple indecomposable fitting into a short exact sequence

$$(3) \quad 0 \longrightarrow S_1 \longrightarrow F \longrightarrow S_2 \longrightarrow 0.$$

The group of auto-equivalences of D_n acts on $\text{EG}(D_n)$ in the obvious way. An auto-equivalence is called *reachable* if it preserves the connected component $\text{EG}^o(D_n)$. We write $\text{Aut}_*(D_n)$ for the sub-quotient of the group of autoequivalences in D consisting of auto-equivalences which preserve the principal component, modulo those which act trivially on it. We will show that this agrees with the definition given in the introduction later (see Remark 4.3(b)).

Lemma 2.2. *Let $2 \leq n < \infty$ and define the following auto-equivalences of D_n :*

$$\Sigma = (\text{Tw}_{S_1} \text{Tw}_{S_2})[n-1], \quad \Upsilon = (\text{Tw}_{S_2} \text{Tw}_{S_1} \text{Tw}_{S_2})[2n-3].$$

Then we have

$$\Sigma(S_1, E, S_2) = (S_2[1], S_1, E), \quad \Upsilon(S_1, S_2) = (S_2, S_1[n-2]).$$

Proof. For any spherical object S we always have $\text{Tw}_S(S) = S[1-n]$, and for any pair of spherical objects we have the relation

$$\text{Tw}_{S_1} \circ \text{Tw}_{S_2} = \text{Tw}_{\text{Tw}_{S_1}(S_2)} \circ \text{Tw}_{S_1}.$$

The short exact sequence (2) shows that

$$\text{Tw}_{S_1}(S_2) = E, \quad \text{Tw}_E(S_1) = S_2[1], \quad \text{Tw}_{S_2}(E) = S_1.$$

Thus $\Sigma = \text{Tw}_E \circ \text{Tw}_{S_1}[n-1]$. Hence

$$\Sigma(S_1) = \text{Tw}_E(S_1) = S_2[1], \quad \Sigma(S_2) = \text{Tw}_{S_1}(S_2) = E.$$

It follows that $\Sigma(E)$ is the unique non-trivial extension of these two objects, namely S_1 .

Moving on to the second identity we know that Tw_{S_1} and Tw_{S_2} satisfy the braid relation (see Prop. 2.7 below). Hence

$$\Upsilon(S_1) = \Sigma(S_1[1]) = S_2, \quad \Upsilon(S_2) = \text{Tw}_{S_2}(E[n-2]) = S_1[n-2].$$

This completes the proof. \square

The following description of the tilting operation in D_n is the combinatorial underpinning of our main result.

Proposition 2.3. *Let $2 \leq n \leq \infty$, and consider hearts obtained by performing simple tilts of the standard heart $\mathcal{A} \subset D_n$.*

(a) *The right tilt of \mathcal{A} at the simple S_2 is another full heart:*

$$\mathcal{A} = \langle S_1, S_2 \rangle \rightarrow \langle S_2[1], E \rangle = \Sigma(\mathcal{A}).$$

(b) *If $n > 2$ then repeated right tilts at appropriate shifts of S_1 gives a sequence of hearts*

$$\mathcal{A} = \langle S_1, S_2 \rangle \rightarrow \langle S_1[1], S_2 \rangle \rightarrow \langle S_1[2], S_2 \rangle \rightarrow \cdots \rightarrow \langle S_1[n-2], S_2 \rangle = \Upsilon(\mathcal{A}).$$

Proof. This is easily checked by hand, or one can consult [8, Proposition 5.4]. \square

Remarks 2.4. (a) When $n > 3$ the intermediate hearts in the sequence in (b) are non-full. In fact, since

$$\text{Hom}_{D_n}^1(S_1[k], S_2) = 0 = \text{Hom}_{D_n}^1(S_2, S_1[k])$$

for $0 < k < n - 2$, each of these hearts is equivalent to the category of representations of the quiver with two vertices and no arrows.

(b) The cases $n = 2, 3, \infty$ of (b) all deserve special comment.

(i) When $n = \infty$ the sequence of non-full hearts is of course infinite.

(ii) When $n = 3$ the first tilt is already a full heart so no non-full hearts arise.

(iii) When $n = 2$, the statement of Prop. 2.3 (b) needs slight modification: there is now a non-trivial extension (3) and the right tilt of \mathcal{A} at S_1 is

$$\mathcal{A} = \langle S_1, S_2 \rangle \rightarrow \langle F, S_1[-1] \rangle = \Sigma^*(\mathcal{A}),$$

where $\Sigma^* = (\text{Tw}_{S_2} \text{Tw}_{S_1})[1]$. so again no non-full hearts arise.

Corollary 2.5. *The auto-equivalences Σ, Υ and $[1]$ are all reachable. In the group $\text{Aut}_*(D_n)$ we have relations*

$$\Sigma^3 = [1], \quad \Upsilon^2 = [n-2],$$

Proof. The reachability of Σ and Υ is immediate from the last result. Since the twist functors Tw_{S_i} are reachable it follows from the definition of Σ and Υ that the shift $[1]$ is also reachable. From Lemma 2.2 we know that the auto-equivalence $\Sigma^3[-1]$ fixes the objects S_1, S_2 . This is enough to ensure that it acts trivially on $\mathrm{EG}^0(\mathcal{D}_n)$ and hence defines the identity element in $\mathrm{Aut}_*(\mathcal{D}_n)$. Similarly for $\Upsilon^2[2-n]$. \square

Proposition 2.6. *For $2 \leq n \leq \infty$ the action of $\mathrm{Aut}_*(\mathcal{D}_n)$ on the set of full reachable hearts is free and transitive.*

Proof. It is free by definition. That it is transitive follows from the characterisation of the categories \mathcal{D}_n in terms of generators, or by the explicit description of tilts given in Prop. 2.3. \square

We denote by Br_3 the Artin braid group of the A_2 root system; it is the fundamental group of $\mathfrak{h}^{\mathrm{reg}}/W$. More concretely, Br_3 is the standard braid group on 3 strings and has a presentation

$$\mathrm{Br}_3 = \langle \sigma_1, \sigma_2 : \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle.$$

The centre of Br_3 is generated by the element $\tau = (\sigma_1 \sigma_2)^3$ and there is a short exact sequence

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\tau} \mathrm{Br}_3 \longrightarrow \mathrm{PSL}(2, \mathbb{Z}) \longrightarrow 1.$$

We can give the following description of the group $\mathrm{Aut}_*(\mathcal{D}_n)$.

Proposition 2.7. *Let $2 \leq n < \infty$.*

- (a) *The group $\mathrm{Aut}_*(\mathcal{D}_n)$ is generated by the subgroup $\mathrm{Sph}_*(\mathcal{D}_n)$ together with the shift functor $[1]$.*
- (b) *There is an isomorphism $\mathrm{Br}_3 \cong \mathrm{Sph}_*(\mathcal{D}_n)$ sending the generator σ_i to Tw_{S_i} .*
- (c) *The isomorphism in (b) sends the central element τ to $[3n-4]$.*
- (d) *The smallest power of $[1]$ contained in $\mathrm{Sph}_*(\mathcal{D}_n)$ is $[3n-4]$. Thus there is a short exact sequence*

$$1 \longrightarrow \mathrm{Sph}_*(\mathcal{D}_n) \longrightarrow \mathrm{Aut}_*(\mathcal{D}_n) \longrightarrow \mu_{3n-4} \longrightarrow 1.$$

Proof. Part (a) follows from the explicit description of tilts given in Prop. 2.3 since any element of $\mathrm{Aut}_*(\mathcal{D}_n)$ takes the canonical heart \mathcal{A} to a reachable full heart. Part (b) was proved by Seidel and Thomas [12]. Part (c) is immediate from Cor. 2.5. Part (d) then follows from the fact that τ generates the centre of Br_3 , since any shift $[d]$ lying in $\mathrm{Sph}_*(\mathcal{D})$ is necessarily central and hence corresponds to a multiple of τ . \square

It will be useful to introduce the quotient group

$$\mathbb{P} \mathrm{Aut}_*(\mathcal{D}_n) = \mathrm{Aut}_*(\mathcal{D}_n)/[1].$$

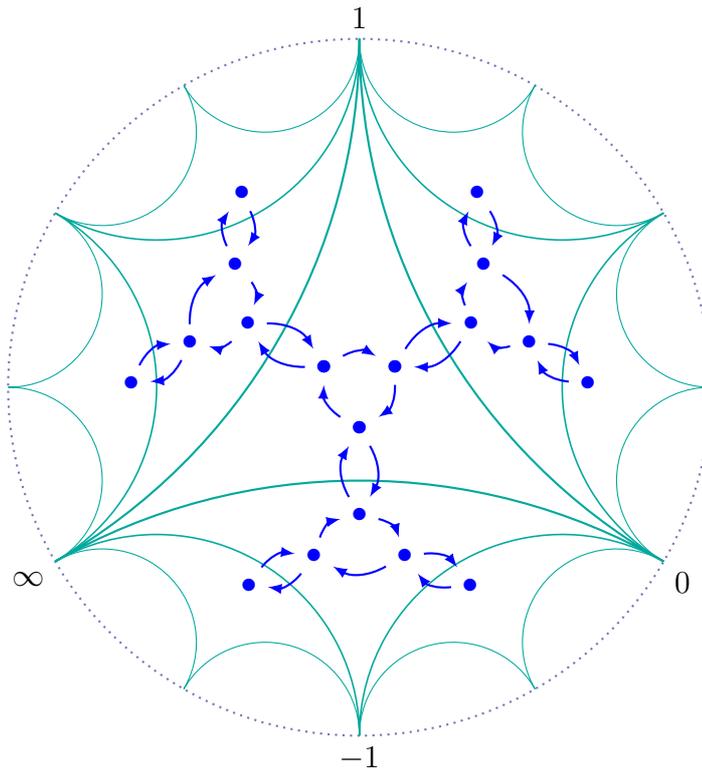


FIGURE 1. The projective exchange graph of D_3 drawn on the hyperbolic disc. The action of $\mathbb{P} \text{Aut}_*(D_3)$ corresponds to the standard action of $\text{PSL}(2, \mathbb{Z})$ on the disc.

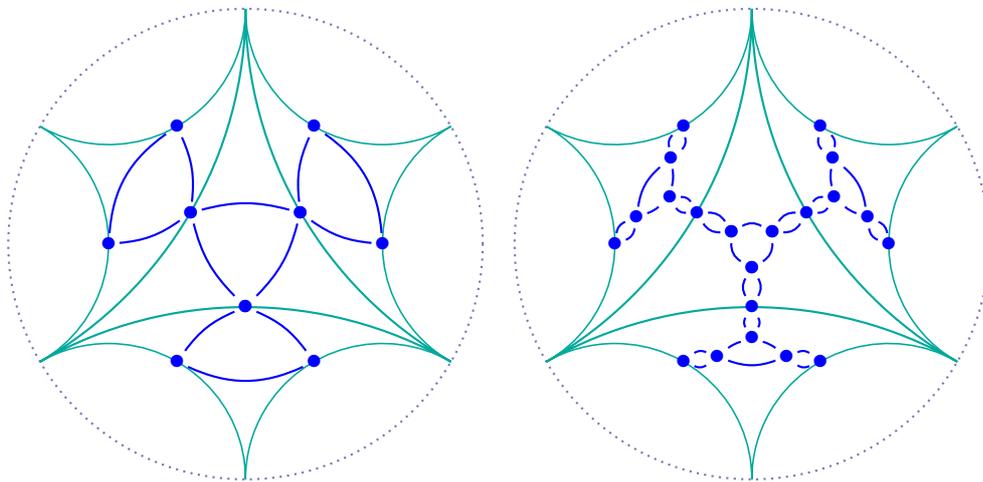


FIGURE 2. The projective exchange graphs of D_2 and D_4 drawn on the hyperbolic disc (orientations omitted).

When $2 \leq n < \infty$ it follows from Prop. 2.7 that

$$\mathbb{P} \operatorname{Aut}_*(D_n) = \operatorname{Sph}_*(D_n) / \langle [3n - 4] \rangle \cong \operatorname{Br}_3 / \langle \tau \rangle \cong \operatorname{PSL}(2, \mathbb{Z}).$$

Note that by Cor. 2.5, the autoequivalences Σ and Υ define elements of $\mathbb{P} \operatorname{Aut}_*(D_n)$ of orders 2 and 3 respectively.

For the category D_∞ the situation is much simpler.

Proposition 2.8. *There is an equality $\operatorname{Aut}_*(D_\infty) = \operatorname{Aut}(D_\infty)$. Moreover*

- (a) *The group $\operatorname{Aut}(D_\infty) \cong \mathbb{Z}$ with the Serre functor Σ being a generator.*
- (b) *There is a relation $\Sigma^3 = [1]$.*

Proof. This is easy and well-known. The Auslander-Reiten quiver for D_∞ is an infinite strip

$$\cdots \rightarrow E[-1] \rightarrow S_1[-1] \rightarrow S_2 \rightarrow E \rightarrow S_1 \rightarrow S_2[1] \rightarrow E[1] \rightarrow \cdots$$

and Σ moves along this to the right by one place. \square

It follows that $\mathbb{P} \operatorname{Aut}(D_\infty) \cong \mu_3$. Note that our use of the symbol Σ in Prop. 2.8 is reasonably consistent with our earlier use for the category D_n : for example the first part of Cor. 2.5 continues to hold in the $n = \infty$ case.

3. CONFORMAL MAPS

In this section we describe some explicit conformal maps which will be the analytic ingredients in the proof of our main result.

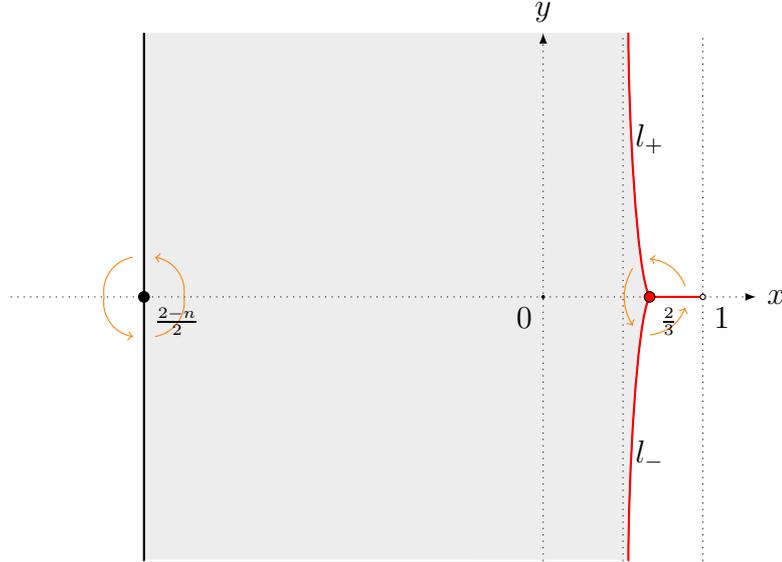


FIGURE 3. The region R_n

For $2 \leq n < \infty$ we consider the region $R_n \subset \mathbb{C}$ depicted in Figure 3. It is bounded by the line $\operatorname{Re}(z) = (2 - n)/2$ and by the curves ℓ_{\pm} . Here ℓ_{\pm} are the images under the map $z \mapsto (1/\pi i) \log(z)$ of the arcs of circles of Apollonius

$$(4) \quad r_{\pm} = \{z \in \mathbb{C} : |z^{\pm 1} + 1| = 1\}.$$

connecting 0 and $\omega^{\pm 1} = e^{\pm 2\pi i/3}$. We also consider splitting the region R_n into two halves R_n^{\pm} by dividing it along the line $\operatorname{Im}(z) = 0$. By convention we take R_n^+ to be the part lying above the real axis. Note that R_n^+ has three vertices: $(2 - n)/2, 2/3$ and ∞ . The Riemann mapping theorem ensures that there is a unique biholomorphism

$$f_n: \mathcal{H} \rightarrow R_n^+$$

which extends continuously over the boundary and sends $(0, 1, \infty)$ to $(\frac{2-n}{2}, \infty, \frac{2}{3})$. The Schwarz reflection principle then shows that R_n itself is biholomorphic to the open subset of \mathbb{P}^1 consisting of the complement of $[0, \infty]$.

Proposition 3.1. *For $n < \infty$ the functions f_n can be explicitly written as*

$$f_n(t) = \frac{1}{\pi i} \log \left(\frac{\phi_n^{(2)}(a, b)}{\phi_n^{(1)}(a, b)} \right)$$

where $t = -(27b^2)/(4a^3)$ and

$$\phi_n^{(i)}(a, b) = \int_{\gamma_i} (x^3 + ax + b)^{\frac{n-2}{2}} dx,$$

for appropriately chosen cycles γ_i .

Note that the function f_n only depends on $t = -(27b^2)/(4a^3)$ because rescaling (a, b) with weights $(4, 6)$ rescales both functions $\phi_n^{(i)}$ with weight

$$3(n - 2) + 2 = 3n - 4$$

and leaves their ratio unchanged.

Proof. We consider the Schwarzian derivative of the function $g_n(t) = \exp f_n(t)$. As the images in \mathbb{P}^1 of the three sides of R_n^+ under the exponential map are segments of circles, the image of the boundary of the upper-half plane \mathcal{H} under g_n is a curvilinear triangle in \mathbb{P}^1 . Thus by the proof of the Schwarz triangle theorem as e.g. in [9, p.207], the Schwarzian derivative of the function g_n is determined by its exponents α_i at the singular points $\{0, 1, \infty\}$.

In more geometric terms this means that the Schwarzian derivative is determined by the angles $\pi\alpha_i$ at which the images of the components of the boundary of \mathcal{H} meet at images

of the singular points $\{0, 1, \infty\}$. We know from the explicit description of the boundary of R_n^+ that the exponents at 0 and ∞ are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. The exponent at 1 is the difference between the real parts of the asymptotes of the boundary components of R_n^+ at infinity, namely

$$\frac{1}{2} - \frac{2-n}{2} = \frac{n-1}{2}.$$

The transformation law of the Schwarzian derivative is that of a projective connection; a function defines a section of the projective local system associated to its Schwarzian derivative. In particular g_n is given by the ratio of a pair of linearly independent solutions to any second-order linear differential equation with precisely three regular singularities at $(0, 1, \infty)$ at which the characteristic exponents of its solutions differ by $(\frac{1}{2}, \frac{n-1}{2}, \frac{1}{3})$ respectively.

Now consider the periods $\phi_n(z) = \phi_n(-3, 2(2z-1))$ where we have fixed the coefficient a of the polynomial $p(x)$. We prove in the Appendix that these periods satisfy the hypergeometric differential equation

$$(5) \quad z(1-z)\phi''(z) + (\gamma - (\alpha + \beta + 1)z)\phi'(z) - \alpha\beta\phi(z) = 0$$

with coefficients $(\alpha, \beta, \gamma) = (\frac{4-3n}{6}, \frac{8-3n}{6}, \frac{3-n}{2})$. This differential equation has three regular singularities at $(0, 1, \infty)$ with differences in characteristic exponents

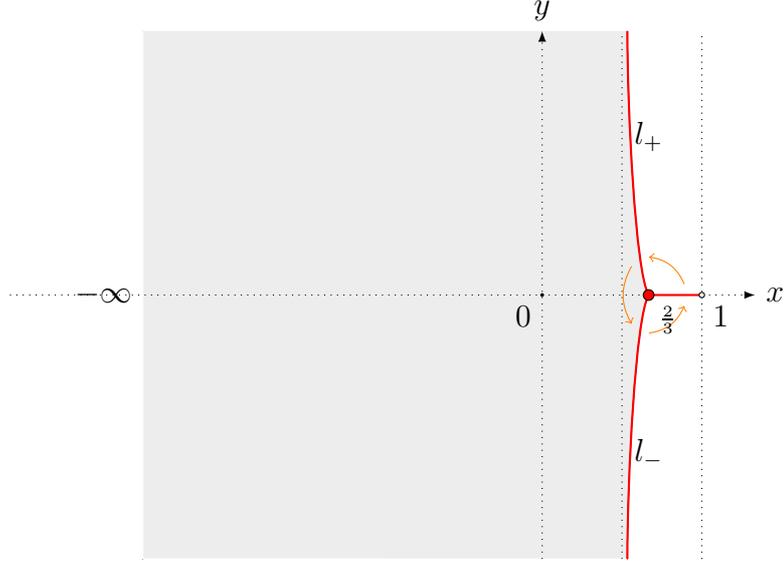
$$(1 - \gamma, \gamma - \alpha - \beta, \beta - \alpha) = \left(\frac{n-1}{2}, \frac{n-1}{2}, \frac{2}{3} \right).$$

We deduce that the periods $\phi_n(t)$ satisfy a differential equation with precisely three regular singularities at $(0, 1, \infty)$ with differences in exponents $(\frac{1}{2}, \frac{n-1}{2}, \frac{1}{3})$. Indeed the covering map $t = (2z-1)^2$ is ramified over $t = \{0, \infty\}$ and the two regular singularities at $z = \{0, 1\}$ are the preimages of the point $t = 1$. \square

In the case $n = \infty$ we consider the region R_∞ depicted in Figure 4. It is bounded by the same two curves ℓ_\pm . We again consider the half region R_∞^+ consisting of points of R_∞ with positive imaginary part. This region R_∞^+ has just two vertices: $2/3$ and ∞ . The Riemann mapping theorem ensures that there is a biholomorphism

$$f_\infty: \mathcal{H} \rightarrow R_\infty^+$$

which extends continuously over the boundary, and sends $(0, \infty)$ to $(\frac{2}{3}, \infty)$. This map is unique up to precomposing by a map of the form $t \mapsto \lambda \cdot t$ with λ real.


 FIGURE 4. The region R_∞

Proposition 3.2. *The function f_∞ can be written explicitly as*

$$f_\infty(t) = \frac{1}{\pi i} \log \left(\frac{\phi_\infty^{(2)}(a, b)}{\phi_\infty^{(1)}(a, b)} \right)$$

where $t = a^3$, b is arbitrary, and

$$\phi_\infty^{(i)}(a, b) = \int_{\gamma_i} e^{x^3+ax+b} dx,$$

for appropriately chosen cycles γ_i .

Note that the function f_∞ only depends on a because translating (a, b) in the b -direction rescales both functions $\phi_\infty^{(i)}$ and leaves their ratio unchanged.

Proof. The defining property of $f_\infty(t)$ shows that it is of the form $t^{1/3} \cdot m(t)$ at $t = 0$ and $t^{1/2} \cdot n(t)$ at $t = \infty$, where $m(t)$ and $n(t)$ are locally-defined analytic functions. Consider the function $g(a) = \exp(f_\infty(a^3))$ defined on the sector

$$\Sigma = \{a \in \mathbb{C} : 0 < \arg(a) < \pi/3\}.$$

Then $g(a)$ extends analytically over the boundary of $\Sigma \subset \mathbb{C}$, and in a neighbourhood of ∞ we can write $g(a) = \exp(a^{3/2}) \cdot q(a)$ for some locally-defined analytic function $q(a)$.

Consider now the Schwarzian derivative $\mathcal{S}(g)$ of the function g . It is analytic on a neighbourhood of the closure of Σ in \mathbb{C} . Moreover, since we can compose g with any Möbius

transformation without altering the Schwarzian, and since g maps the boundary ray $\mathbb{R}_{>0}$ to a circle, it follows that $\mathcal{S}(g)$ is real-valued on this ray. A similar argument applied to the function $g(e^{\pi i/3}a)$ shows that $\mathcal{S}(g)$ takes the other boundary ray $\mathbb{R}_{>0} \exp(\pi i/3)$ of Σ to itself (up to sign).

Consider the function $h(a) = \exp(a^{3/2})$. Then

$$\frac{h''(a)}{h'(a)} = \frac{3a^{3/2} + 1}{2a}.$$

So the Schwarzian is

$$\mathcal{S}(h) = \left(\frac{h''(a)}{h'(a)} \right)' - \frac{1}{2} \left(\frac{h''(a)}{h'(a)} \right)^2 = -\frac{9a^3 + 5}{8a^2}.$$

In particular, the Schwarzian $\mathcal{S}(h)$ has a simple pole at $a = \infty$.

It now follows that, up to sign, the Schwarzian $\mathcal{S}(g)$ takes Σ to itself and extends over the boundary. This then implies that $\mathcal{S}(g) = \lambda a$ for some real λ . By precomposing $f_\infty(a)$ by a rescaling of a we can reduce this to $\mathcal{S}(g) = a/3$. By general properties of the Schwarzian it follows that $g(a)$ is given by a ratio of solutions of the linear differential equation

$$y''(a) - \frac{a}{3} \cdot y(a) = 0,$$

a variant of the Airy equation. Since the solutions to this equation are precisely the functions $\phi_\infty^{(i)}(a)$ (as can easily be checked by differentiating under the integral sign), this completes the proof. \square

4. STABILITY CONDITIONS

We let $\text{Stab}_*(D_n)$ denote the connected component of the space of stability conditions on D_n containing stability conditions whose heart is the canonical one. We set

$$\mathbb{P}\text{Stab}_*(D_n) = \text{Stab}_*(D_n)/\mathbb{C}.$$

It is a complex manifold locally modelled on the projective space

$$\mathbb{P}^1 = \mathbb{P}\text{Hom}_{\mathbb{Z}}(K_0(D_n), \mathbb{C}).$$

If σ is a stability condition we set $\mathcal{S}(\sigma)$ to be the set of indecomposable semistable objects of σ . We also set $\mathbb{P}\mathcal{S}(\sigma)$ to be the set of such objects up to shift. We now define an open subset $\mathcal{U}_n \subset \mathbb{P}\text{Stab}_*(D_n)$ as follows.

Definition 4.1. A projective stability condition $\bar{\sigma} \in \mathbb{P}\text{Stab}_*(D_n)$ lies in \mathcal{U}_n if one of the following two conditions holds

- (a) $\mathbb{P}\mathcal{S}(\sigma) = \{S_1, S_2\}$ and $0 < \phi(S_2) - \phi(S_1) < (n-2)/2$,
 (b) $\mathbb{P}\mathcal{S}(\sigma) = \{S_1, S_2, E\}$ and
- (6) $0 \leq \phi(S_1) - \phi(S_2) < \phi(E[1]) - \phi(S_1), \quad 0 \leq \phi(S_1) - \phi(S_2) < \phi(S_2[1]) - \phi(E)$.

The point of this definition is the following result.

Proposition 4.2. *For any $2 \leq n \leq \infty$ the domain \mathcal{U}_n is a fundamental domain for the action of $\mathbb{P}\text{Aut}_*(D_n)$ on $\mathbb{P}\text{Stab}_*(D_n)$.*

Proof. Suppose a projective stability condition σ lies in the intersection $\mathcal{U}_n \cap \Phi^{-1}(\mathcal{U}_n) \subset \mathbb{P}\text{Stab}_*(D_n)$ for some $\Phi \in \text{Aut}_*(D_n)$. This means that $\sigma \in \mathcal{U}_n$ and also $\Phi(\bar{\sigma}) \in \mathcal{U}_n$. There are two cases to consider, corresponding to the two parts of Definition 4.1.

Suppose first that $\mathbb{P}\mathcal{S}(\sigma) = \{S_1, S_2\}$. Then Φ maps each S_i to an S_j up to shift. Given the Hom-spaces between S_1 and S_2 it is easy to see that if Φ defines a non-trivial element of $\mathbb{P}\text{Aut}_*(D_n)$ then we must have $n < \infty$ and

$$\Phi(S_1, S_2) = (S_2, S_1[n-2])$$

up to shift. But then for $\Phi(\sigma)$ to lie in \mathcal{U}_n we must have $n-2 - (\phi(S_2) - \phi(S_1)) < (n-2)/2$ which gives a contradiction.

The second case is when $\mathbb{P}\mathcal{S}(\sigma) = \{S_1, S_2, E\}$. Then Φ preserves this set of objects up to shift. Given the maps between them, and using Lemma 2.2, it follows that $\Phi \in \mathbb{P}\text{Aut}_*(D_n)$ lies in the order 3 subgroup generated by Σ . Noticing that the inequalities (6) are equivalent to

$$(7) \quad |Z(S_2)| < |Z(E)|, \quad |Z(S_1)| < |Z(E)|$$

where Z is the central charge for σ , they give a contradiction.

Now consider the union of the closures of the regions $\Phi(\mathcal{U}_n)$ for $\Phi \in \mathbb{P}\text{Aut}_*(D_n)$. This subset of $\mathbb{P}\text{Stab}_*(D_n)$ is closed because it is a locally-finite union of closed subsets. To prove that it is open consider a stability condition σ defining a point in the boundary of \mathcal{U}_n . Again, there are two possibilities, corresponding to the two parts of Definition 4.1. In the first case, $\mathbb{P}\mathcal{S}(\sigma) = \{S_1, S_2\}$ and $\phi(S_2) - \phi(S_1) = (n-2)/2$. Then a neighbourhood of σ is covered by the closures of the regions \mathcal{U}_n and $\Upsilon(\mathcal{U}_n)$. In the second case $\mathbb{P}\mathcal{S}(\sigma) = \{S_1, S_2, E\}$ and one or both of the two inequalities (6) is not strict. Then a neighbourhood of σ is covered by the closures of the regions \mathcal{U}_n , $\Sigma(\mathcal{U}_n)$ and $\Sigma^2(\mathcal{U}_n)$. This completes the proof. \square

Remarks 4.3. (a) When $n < \infty$ there are two special points in the boundary of \mathcal{U}_n : one is fixed by Σ and the other by Υ . In the case $n = \infty$ only the order 3 point fixed by Σ exists. These projective stability conditions are illustrated in Figure 5.

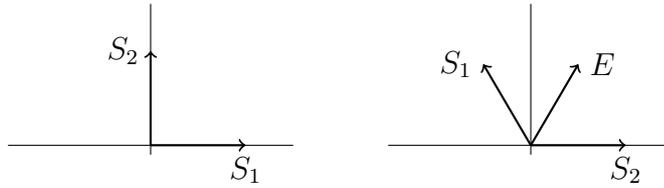


FIGURE 5. Stability conditions corresponding to orbifold points of $\mathbb{P}\text{Stab}_*(D_n)$

- (b) It follows from this result that an autoequivalence in $\text{Aut}(D_n)$ is reachable precisely if it preserves the connected component $\text{Stab}_*(D_n)$. Moreover by [11, Corollary 5.3]) such an autoequivalence acts trivially on $\text{Stab}_*(D_n)$ precisely if it acts trivially on the principal component $\text{EG}^o(D_n)$.

Proposition 4.4. *Let $2 \leq n \leq \infty$. Then the function*

$$g(\sigma) = \frac{1}{\pi i} \log \frac{Z(S_1)}{Z(S_2)}$$

defines a biholomorphic map between the regions \mathcal{U}_n and R_n .

Proof. The region \mathcal{U}_n consists of two parts, corresponding to conditions (a) and (b) of Definition 4.1. In the first part S_1 and S_2 are the only indecomposable semistable objects. This implies that $\phi(S_2) > \phi(S_1)$ since otherwise the extension E would also be semistable. Combined with the inequality in Definition 4.1 this gives

$$0 < \phi(S_2) - \phi(S_1) < (n - 2)/2.$$

Any stability condition for which S_1 and S_2 are the only indecomposable semistable objects is clearly determined up to the \mathbb{C} -action by $\log Z(S_2)/Z(S_1)$, and it is also easy to see that any possible value compatible with the above constraint is possible. So the image of this part of \mathcal{U}_n is precisely the strip $(2 - n)/2 < \text{Re}(z) < 0$.

In the second part of the region \mathcal{U}_n , all three objects S_1 , S_2 and E are semistable. The existence of nonzero maps $S_1 \rightarrow S_2[1]$ implies the image of this part of \mathcal{U}_n lies in the strip $0 \leq \text{Re}(z) < 1$. Now the inequalities (6) in Definition 4.1 (or equivalently (7)) imply that this image is one third of this region, divided by the order 3 subgroup generated by Σ in $\text{Aut}_*(D_n)$. To see it is precisely the left part of R_n , we only need to notice that the boundaries r_{\pm} , defined by (4), of R_n correspond to the points where $|Z(S_1) + Z(S_2)| = |Z(E)| = |Z(S_1)|$ and $|Z(S_1) + Z(S_2)| = |Z(E)| = |Z(S_2)|$. \square

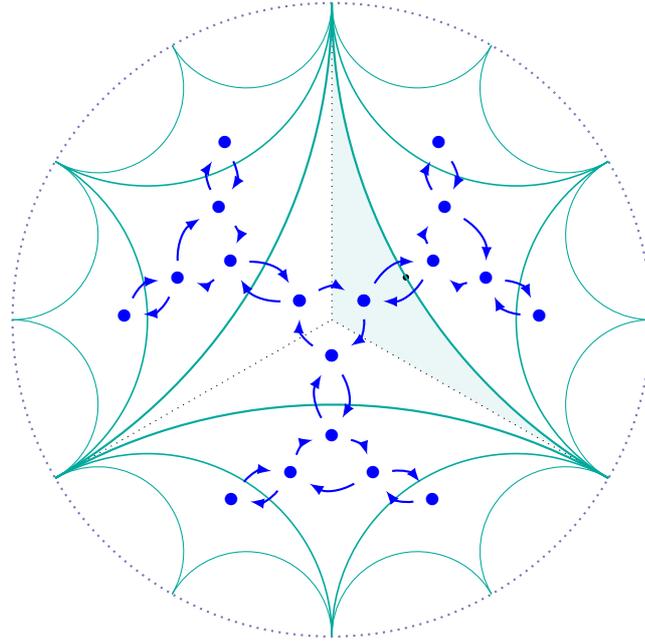


FIGURE 6. Exchange graph as the skeleton of space of stability conditions

We can now prove a projectivised version of Theorem 1.1. Recall that the quotient \mathfrak{h}/W is isomorphic to \mathbb{C}^2 with co-ordinates (a, b) by setting

$$p(x) = (x - u_1)(x - u_2)(x - u_3) = x^3 + ax + b.$$

The image of the root hyperplanes $u_i = u_j$ is the discriminant

$$\Delta = \{(a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 = 0\}.$$

Note that \mathfrak{h} has a natural \mathbb{C}^* action rescaling the u_i co-ordinates with weight 1. This acts on (a, b) with weights $(2, 3)$. We thus have

$$\mathbb{C}^* \backslash (\mathfrak{h} \setminus \{0\}) / W \cong \mathbb{P}(2, 3).$$

The weighted projective space $\mathbb{P}(2, 3)$ contains two orbifold points with stabilizer groups μ_2 and μ_3 respectively. The image of the discriminant is a single (non-orbifold) point which we also label Δ .

In the $n = \infty$ case we consider the quotient of \mathbb{P}^1 by μ_3 given by $[1 : z] \mapsto [1 : e^{2\pi i/3} z]$. We label the two orbifold points $\{\mu_3, \infty\}$.

Theorem 4.5. (a) For $2 \leq n < \infty$ the action of $\mathbb{P} \text{Aut}_*(D_n)$ on $\mathbb{P} \text{Stab}_*(D_n)$ is quasi-free and there is an isomorphism of complex orbifolds

$$\mathbb{P} \text{Stab}_*(D_n) / \mathbb{P} \text{Aut}_*(D_n) \cong \mathbb{P}(2, 3) \setminus \{\Delta\}.$$

(b) *The action of $\mathbb{P} \operatorname{Aut}(\mathbb{D}_\infty)$ on $\mathbb{P} \operatorname{Stab}(\mathbb{D}_\infty)$ is quasi-free and there is an isomorphism of complex manifolds*

$$\mathbb{P} \operatorname{Stab}(\mathbb{D}_\infty) / \mathbb{P} \operatorname{Aut}_*(\mathbb{D}_\infty) \cong \mathbb{C} / \mu_3.$$

Proof. We identify the upper half-plane arising in the last section with the upper half-plane in the coarse moduli space of the orbifold $\mathbb{P}(2, 3)$, in such a way that the points $(0, 1, \infty)$ correspond to (μ_2, Δ, μ_3) . Combining the map g of Prop. 4.4 and the inverse of the maps f_n of the last Section gives a biholomorphic map

$$\mathcal{U}_n \xrightarrow{g} R_n \xrightarrow{f_n^{-1}} P.$$

Here, we view \mathcal{U}_n as an open dense subset of $\mathbb{P} \operatorname{Stab}_*(\mathbb{D}_n) / \operatorname{Aut}_*(\mathbb{D}_n)$, and $P = \mathbb{P}(2, 3) \setminus ([\mu_2, \Delta] \cup [\Delta, \mu_3])$ is the union of two copies of the upper half-plane glued along the boundary component $[\mu_2, \mu_3]$.

By definition, the map g extends over the boundary of \mathcal{U}_n and sends the two types of boundary points (corresponding to parts (a) and (b) of Definition 4.1) to the boundaries on the left and right of Figures 3 – 4 respectively. Under the map f_n^{-1} these boundaries become identified with $[\mu_2, \infty]$ and $[\mu_3, \infty]$ respectively. The result then follows.

The case $n = \infty$ proceeds along similar lines. We identify the upper half-plane with the upper half-plane in the coarse moduli space of the resulting orbifold \mathbb{P}^1 / μ_3 . The composite $f_\infty^{-1} \circ g$ then identifies the dense open subset \mathcal{U}_∞ of $\mathbb{P} \operatorname{Stab}(\mathbb{D}_\infty) / \mathbb{P} \operatorname{Aut}(\mathbb{D}_\infty)$ with the union of two copies of the upper half-plane glued along one of the boundary components $[\mu_3, \infty]$. The rest of the argument is then as above. \square

Remark 4.6. Note that $\mathbb{P} \operatorname{Stab}(\mathbb{D}_\infty)$ is isomorphic to \mathbb{C} and the action of $\mathbb{P} \operatorname{Aut}(\mathbb{D}_\infty) \cong \mu_3$ corresponds to the usual action of μ_3 on \mathbb{C} by multiplication by a primitive third root of unity.

We can now lift Theorem 4.5 to obtain a proof of our main theorem.

Proof of Theorem 1.1. We have a diagram of complex manifolds and holomorphic maps

$$\begin{array}{ccc} \operatorname{Stab}_*(\mathbb{D}_n) / \operatorname{Sph}_*(\mathbb{D}_n) & & \mathbb{C}^2 \setminus \Delta \\ \downarrow & & \downarrow \\ \mathbb{P} \operatorname{Stab}_*(\mathbb{D}_n) / \mathbb{P} \operatorname{Sph}_*(\mathbb{D}_n) & \xrightarrow{\theta_n} & \mathbb{P}(2, 3) \setminus \Delta \end{array}$$

The vertical arrows are \mathbb{C}^* -bundles, and the horizontal arrow θ_n is the isomorphism of Theorem 4.5. We would like to complete the diagram by filling in an upper horizontal isomorphism satisfying the property claimed in Theorem 1.1. Note that by construction

the central charge map

$$\mathbb{P}\mathrm{Stab}(D_n) \rightarrow \mathbb{P}^1$$

corresponds under the isomorphism θ_n to the multi-valued map given by ratios of the functions $\phi_n^{(i)}(a, b)$. These functions lift to $\mathbb{C}^2 \setminus \Delta$ and are then scaled with weight $3n-4 > 0$ by the \mathbb{C}^* -action. There is therefore a unique way to fill in the upper arrow so that the multi-valued central charge map on $\mathrm{Stab}(D_n)$ corresponds to $\phi_n^{(i)}(a, b)$.

In the $n = \infty$ we have a similar diagram

$$\begin{array}{ccc} \mathrm{Stab}_*(D_\infty) & & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ \mathbb{P}\mathrm{Stab}_*(D_\infty) & \xrightarrow{\theta_\infty} & \mathbb{C} \end{array}$$

in which the vertical arrows are \mathbb{C} -bundles. The bundle on the right is just the projection $\mathbb{C}^2 \rightarrow \mathbb{C}$ given by $(a, b) \mapsto a$. By construction the central charge map

$$\mathbb{P}\mathrm{Stab}_*(D_\infty) \rightarrow \mathbb{P}^1$$

is given by ratios of the functions $\phi_\infty^{(i)}(a, b)$. These functions lift to \mathbb{C}^2 and are then scaled by weight e^b by translation by $(0, b)$. There is therefore a unique way to fill in the upper arrow so that the central charge map on $\mathrm{Stab}(D_\infty)$ corresponds to $\phi_\infty^{(i)}(a, b)$.

Remark 4.7. In the $n = \infty$ case the auto-equivalence group is \mathbb{Z} generated by Σ . The induced action on \mathfrak{h}/W is given by $(a, b) \mapsto (e^{2\pi i/3}a, b + \pi i/3)$. The element $\Sigma^3 = [1]$ then fixes a and acts by $b \mapsto b + \pi i$.

APPENDIX A. HYPERGEOMETRIC EQUATION FOR THE TWISTED PERIODS

In this section we prove that the twisted periods satisfy the hypergeometric differential equation (5) appearing in the proof of Theorem 3.1.

Let us fix $a \in \mathbb{C}$ and consider the function

$$f_a(h) = h^{-(\nu+1)} \int e^{h(x^3+ax)} dx.$$

Setting $t = h^{1/3} \cdot x$ we see that

$$f_a(h) = h^{-(\nu+\frac{4}{3})} \int e^{t^3+h^{2/3}at} dt.$$

Introduce the differential operator

$$D_h = h\partial_h + \nu + 1.$$

Then

$$(D_h + \frac{1}{3})f_a(h) = \frac{2}{3} \cdot h^{-(\nu+\frac{4}{3})} \int h^{\frac{2}{3}} \cdot at \cdot e^{t^3+h^{2/3}at} dt.$$

Repeating we obtain

$$(D_h - \frac{1}{3})(D_h + \frac{1}{3})f_a(h) = \frac{4}{9} \cdot h^{-(\nu+\frac{4}{3})} \int h^{\frac{4}{3}} \cdot (at)^2 \cdot e^{t^3+h^{2/3}at} dt,$$

and it follows that

$$\begin{aligned} & \left((D_h - \frac{1}{3})(D_h + \frac{1}{3}) + \frac{4a^3}{27} \cdot h^2 \right) f(a, h) = \\ & = h^{-(\nu+\frac{4}{3})} \cdot h^{\frac{4}{3}} \cdot \frac{4a^2}{27} \int (3t^2 + h^{\frac{2}{3}}a) \cdot e^{t^3+h^{2/3}at} dt = 0. \end{aligned}$$

Now consider the (inverse) Laplace transform

$$p_a(b) = \int e^{bh} f_a(h) dh = \int \int e^{h(x^3+ax+b)} \cdot h^{-(\nu+1)} dx dh.$$

Exchanging the order of integration and using

$$\int e^{h(y+b)} \cdot h^{-(\nu+1)} dh = (-\nu - 1)! \cdot (y + b)^\nu,$$

valid for $\text{Re}(\nu) > -1$ this becomes

$$p_a(b) = (-\nu - 1)! \cdot \int (x^3 + ax + b)^\nu dx.$$

Under the inverse transform $h\partial_h$ becomes $-b\partial_b - 1$ so the transform of the operator D_h is

$$M_b = (-b\partial_b + \nu).$$

The twisted periods therefore satisfy the differential equation

$$\left((-b\partial_b + \nu + \frac{1}{3})(-b\partial_b + \nu - \frac{1}{3}) + \frac{4a^3}{27} \cdot \partial_b^2 \right) p_a(b) = 0$$

which can be rewritten

$$(8) \quad \left(\frac{4a^3}{27} + b^2 \right) \partial_b^2 + (1 + \alpha + \beta)b\partial_b + \alpha\beta = 0$$

with $\alpha = -1/3 - \nu$ and $\beta = 1/3 - \nu$. Our derivation holds for $\text{Re}(\nu) > -1$ but by analytic continuation the result holds in general.

The differential equation (8) is second order in b with three regular singularities. To put it in hypergeometric form substitute

$$b = d(2z - 1) \text{ with } 4a^3 + 27d^2 = 0$$

so that the singularities lie at $z \in \{0, 1, \infty\}$. The equation now becomes

$$z(1 - z)\partial_z^2 + (\gamma - (1 + \alpha + \beta)z)\partial_z - \alpha\beta = 0,$$

with $\gamma = 1/2 - \nu$. Setting $\nu = (n - 2)/2$ gives the claim.

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