THE MODULAR CURVE AS THE SPACE OF STABILITY CONDITIONS OF A CY_3 ALGEBRA

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ABSTRACT. We prove that a connected component of the space of stability conditions of a CY_3 triangulated category generated by an A_2-collection of 3-spherical objects is isomorphic to the universal cover of the $\mathbb{C}^*$-bundle of non-zero holomorphic differentials on the moduli space of elliptic curves.

1. INTRODUCTION

The space of stability conditions $\mathrm{Stab}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ was introduced in [1]. As a set it has a description as the pairs $(\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a t-structure on $\mathcal{D}$, and $Z : K(\mathcal{A}) \cong K(\mathcal{D}) \to \mathbb{C}$ is a stability function on $\mathcal{A}$ known as the central charge. As the forgetful map $\mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ remembering just the central charge is a local homeomorphism [1, Prop 6.3], $\mathrm{Stab}(\mathcal{D})$ has the structure of a complex manifold. It carries an action of the group of autoequivalences $\mathrm{Aut}(\mathcal{D})$ and a free action of $\mathbb{C}$ for which $\mathbb{Z} \subset \mathbb{C}$ acts as the autoequivalence [1], the shift functor of $\mathcal{D}$.

In this paper we compute a connected component $\mathrm{Stab}^0(\mathcal{D})$ of the space of stability conditions of $\mathcal{D} = \mathcal{D}_{fd}(GA_2)$, the derived category of finite dimensional modules over the Ginzburg dg algebra of the A_2 quiver. This is a CY_3 triangulated category generated (cf [6, Sect 2]) by two objects S and T with

$$\mathrm{Hom}(S,S) \cong \mathbb{C} \cong \mathrm{Hom}(T,T) \qquad \mathrm{Ext}^1(S,T) \cong \mathbb{C}$$

We will call the heart $\mathcal{A}^0$ consisting of all modules supported in degree zero the standard heart. It is equivalent to the abelian category of finitely generated modules over the path algebra of the A_2 quiver, and its two simple objects are S and T. We study the connected component $\mathrm{Stab}^0(\mathcal{D})$ which contains stability conditions supported on the standard heart $\mathcal{A}^0$.

In section two we study the subquotient $\mathrm{Aut}^0(\mathcal{D})$ of $\mathrm{Aut}(\mathcal{D})$ of those autoequivalences preserving the connected component $\mathrm{Stab}^0(\mathcal{D})$ modulo those which act trivially on it. We show that the set of hearts supporting a stability condition in $\mathrm{Stab}^0(\mathcal{D})$ is an $\mathrm{Aut}^0(\mathcal{D})$-torsor and deduce that

Theorem 1.1. *$\mathrm{Aut}^0(\mathcal{D})$ is isomorphic to the braid group Br_3 on three strings.*

In section three we show how to define central charges using periods of a meromorphic differential λ on the universal family of framed elliptic curves $\mathcal{E} \to \widetilde{\mathcal{M}_{1,1}}$. Restricted to a fibre E, λ has a single pole of order 6 at the marked point p and double zeroes at each of the half-periods. Using the framing $\{\alpha, \beta\}$ and the basis $\{[S], [T]\}$ of $K(\mathcal{D})$ to identify the lattices $H_1(E\backslash p, \mathbb{Z}) \cong K(\mathcal{D})$, we prove

Theorem 1.2. *There is a biholomorphic map*

$$\begin{array}{ccc} \widetilde{\mathcal{M}_{1,1}} & \xrightarrow{f} & \mathrm{Stab}^0(\mathcal{D})/\mathbb{C} \\ & {\scriptstyle [\int_\alpha \lambda : \int_\beta \lambda]} \searrow & \downarrow {\scriptstyle [Z(S):Z(T)]} \\ & & \mathbb{P}\,\mathrm{Hom}(K(\mathcal{D}), \mathbb{C}) \end{array}$$

lifting the period map of λ. *It is equivariant with respect to the actions of* $\mathrm{PSL}(2,\mathbb{Z})$ *on the left by deck transformations and on the right by* $\mathrm{Aut}^0(\mathcal{D})/\mathbb{Z}$ *which are both determined by their induced actions on* $K(\mathcal{D})$.

As a corollary we obtain a Br_3-equivariant biholomorphism from the universal cover of the $\mathbb{C}^*$-bundle L^* of non-zero holomorphic differentials on $\mathcal{M}_{1,1}$ to $\mathrm{Stab}^0(\mathcal{D})$.

Remark 1.3. *In* [11] *the authors list 9 families of rank two connections on* $\mathbb{P}^1$ *having at least one irregular singularity which have precisely a one-parameter family of isomonodromic deformations described by one of the Painlevé equations. To each such family we associate a quiver* Q *as in* [3]*, where* $Q = A_2$ *corresponds to the family whose isomonodromic deformations are given by solutions to the first Painlevé equation. It is anticipated that similar considerations to those of this paper will give a description of the space of numerical stability conditions of* $\mathcal{D}_{fd}(GQ)$ *as the universal cover of a* $\mathbb{C}^*$*-bundle of meromorphic differentials over a moduli space of elliptic curves. We intend to return to this in future work.*

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2. Autoequivalences

In this section we prove Theorem 1.1. We show that every heart supporting a stability condition in $\mathrm{Stab}^0(\mathcal{D})$ is a translate of the standard heart $\mathcal{A}^0 = \mathrm{mod}(\mathbb{C}A_2)$ by a composite of a spherical twist and the shift functor [1]. We deduce that every element of $\mathrm{Aut}^0(\mathcal{D})$ is expressible in this way. The group of spherical twists $\mathrm{Sph}(\mathcal{D})$ is a subgroup of $\mathrm{Aut}^0(\mathcal{D})$ of index five, and we use a result of Seidel-Thomas that $\mathrm{Sph}(\mathcal{D}) \cong \mathrm{Br}_3$ to deduce that $\mathrm{Aut}^0(\mathcal{D}) \cong \mathrm{Br}_3$, the braid group on three strings.

Definition 2.1. *An object* $X \in \mathcal{D}$ *is spherical if* $\mathrm{Hom}_{\mathcal{D}}(X, X) \cong \mathbb{C} \oplus \mathbb{C}[-3]$. *For* X *spherical there is a twist functor* Φ_X *such that*

$$\Phi_X(Y) = \mathrm{Cone}(X \otimes \mathrm{Hom}(X, Y) \to Y)$$

There are two spherical objects S and T in $\mathcal{D}$ which are the simple objects in the standard heart $\mathcal{A}^0$. They form an A_2-collection [10, Def 1.1] as $\mathrm{Ext}^1(S, T) \cong \mathbb{C}$.

Theorem 2.2. [10, Thms 1.2, 1.3] *The spherical twists* Φ_S, Φ_T *satisfy the braid relations*

$$\Phi_S \Phi_T \Phi_S = \Phi_T \Phi_S \Phi_T$$

and generate a subgroup $\mathrm{Sph}(\mathcal{D})$ *of the group of autoequivalences* $\mathrm{Aut}(\mathcal{D})$ *isomorphic to the braid group on three strings* Br_3.

The braid group Br_3 has the following presentation by generators and relations [4, Sect 1.14]

$$\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

Its centre is the infinite cyclic subgroup generated by the element $u = (\sigma_1\sigma_2)^3$ [4, Thm 1.24] giving us the short exact sequence

$$1 \to \mathbb{Z} \to \mathrm{Br}_3 \to \mathrm{PSL}(2,\mathbb{Z}) \to 1$$

where the quotient map sends the generators σ_1, σ_2 to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

We note that the action of a spherical twist Φ_X on $K(\mathcal{D})$ is given by the formula

$$\Phi_X([Y]) = [Y] - \chi(X,Y)[X]$$

and so the map $\mathrm{Sph}(\mathcal{D}) \to \mathrm{PSL}(2,\mathbb{Z})$ sends a spherical twist to the matrix given by its action on the lattice $K(\mathcal{D})$ with respect to the basis $\{[S],[T]\}$.

We now study the combinatorial backbone of the space of stability conditions, namely (a connected component of) the *exchange graph* of hearts of $\mathcal{D}$.

Definition 2.3. *We say $\mathcal{A}'$ is a simple tilt of $\mathcal{A}$ at S if either*

- *$\mathcal{A}'$ is the left tilt of $\mathcal{A}$ with respect to the torsion pair*

$$\mathcal{T} = \langle S \rangle = \{S^{\oplus n} \mid n \in \mathbb{N}_0\} \qquad \mathcal{F} = \{X \mid \mathrm{Hom}_{\mathcal{A}}(S,X) = 0\}$$

- *$\mathcal{A}'$ is the right tilt of $\mathcal{A}$ with respect to the torsion pair*

$$\mathcal{T} = \{X \mid \mathrm{Hom}_{\mathcal{A}}(X,S) = 0\} \qquad \mathcal{F} = \langle S \rangle = \{S^{\oplus n} \mid n \in \mathbb{N}_0\}$$

The relevance of simple tilts is that they occur precisely at the codimension 1 components of the boundary of the space of stability conditions $U(\mathcal{A})$ supported on a given heart $\mathcal{A}$ by [2, Lemma 5.5]. Thus $\mathrm{Stab}(\mathcal{D})$ is glued together from the $U(\mathcal{A})$ according to the exchange graph.

Definition 2.4. *The exchange graph $\mathrm{EG}(\mathcal{D})$ of $\mathcal{D}$ has vertices the set of hearts $\mathcal{A} \subset \mathcal{D}$ and an edge between any two hearts related by a simple tilt. Define $\mathrm{EG}^0(\mathcal{D})$ to be the connected component containing the standard heart $\mathcal{A}^0$.*

We compute the four simple tilts of the standard heart $\mathcal{A}^0$.

Proposition 2.5. *Denote by E and X the unique non-trivial extensions up to isomorphism of S by T and T by $S[1]$ respectively. Let $(A,B)_C$ denote the abelian category generated by two simple objects A and B having a unique up to isomorphism non-trivial extension C of B by A, so that the standard heart $\mathcal{A}^0 = (T,S)_E$. Then*

$$R_S(\mathcal{A}^0) = (S[1],T)_{X[1]} \qquad L_T(\mathcal{A}^0) = (T[-1],S)_X$$
$$R_T(\mathcal{A}^0) = (E,T[1])_S \qquad L_S(\mathcal{A}^0) = (S[-1],E)_T$$

Moreover the tilted hearts are obtained from $\mathcal{A}^0$ by applying the following autoequivalences.

$$R_S(\mathcal{A}^0) = (\Phi_T\Phi_S\Phi_T)[3]\ (\mathcal{A}^0) \qquad L_T(\mathcal{A}^0) = ((\Phi_T\Phi_S\Phi_T)[3])^{-1}\ (\mathcal{A}^0)$$
$$R_T(\mathcal{A}^0) = (\Phi_S\Phi_T)[2]\ (\mathcal{A}^0) \qquad L_S(\mathcal{A}^0) = ((\Phi_S\Phi_T)[2])^{-1}\ (\mathcal{A}^0)$$

We will prove the statement about the left tilt at T, the rest being similar. The torsion pair in this case is

$$\mathcal{T} = \langle T \rangle \qquad \mathcal{F} = \{X \in \mathcal{A}^0 \mid \mathrm{Hom}_{\mathcal{A}^0}(T,X) = 0\} = \langle S \rangle$$

We will use the long exact sequence in cohomology with respect to the original t-structure $\mathcal{A}^0$, the groups being non-zero only in degrees 0 and 1.

Lemma 2.6. $T[1]$ *is simple in* $L_T(\mathcal{A}^0)$

Proof. Consider a short exact sequence in $L_T(\mathcal{A}^0)$

$$0 \to X \to T[-1] \to Y \to 0$$

giving a long exact sequence in $\mathcal{A}^0$.

$$0 \to H^0(X) \to H^0(T[-1]) \to H^0(Y) \to H^1(X) \to H^1(T[-1]) \to H^1(Y) \to 0$$

We have $H^0(T[-1]) = 0$ so $H^0(X) = 0$. Splitting the remaining 4-term exact sequence into two short exact sequences

$$0 \to H^0(Y) \to H^1(X) \to Z \to 0$$
$$0 \to Z \to T \to H^1(Y) \to 0$$

Z is either 0 or T as T is simple in $\mathcal{A}^0$. But there are no non-zero maps from $H^0(Y) \in \mathcal{F}$ to $H^1(X) \in \mathcal{T}$ so $H^1(X) \cong Z$ so X is either 0 or $T[-1]$ and so $T[-1]$ is simple.

□

Lemma 2.7. *S is simple in $L_T(\mathcal{A}^0)$.*

Proof. As $H^1(S) = 0$ we have as before

$$0 \to H^0(X) \to S \to Z \to 0$$
$$0 \to Z \to H^0(Y) \to H^1(X) \to 0$$

Thus as S is simple in $\mathcal{A}^0$, $H^0(X)$ is either 0 or S, and so Z is either S or 0. Then as there are no non-zero maps from $H^0(Y) \in \mathcal{F}$ to $H^1(X) \in T$, $H^1(X) = 0$ and so S is simple in $L_T(\mathcal{A}^0)$. □

We remark that all four simple tilts of $\mathcal{A}^0$ are isomorphic to $\mathcal{A}^0$ so the above is the local structure of the exchange graph at any vertex of the connected component $\mathrm{EG}^0(\mathcal{D})$.

Definition 2.8. *Let $\mathrm{Aut}^0(\mathcal{D})$ be the subquotient of $\mathrm{Aut}(\mathcal{D})$ consisting of all autoequivalences preserving the connected component $\mathrm{EG}^0(\mathcal{D})$ of the exchange graph modulo those acting trivially on it.*

We will see later that in fact $\mathrm{Aut}^0(\mathcal{D})$ is the subquotient preserving the connected component $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ modulo those acting trivially on it.

Proposition 2.9. *The vertices of the connected component $\mathrm{EG}^0(\mathcal{D})$ of the exchange graph are a torsor for $\mathrm{Aut}^0(\mathcal{D})$.*

Proof. From the above computation every heart in $\mathrm{EG}^0(\mathcal{D})$ can be obtained by applying an autoequivalence in $\langle \Phi_S, \Phi_T, [1] \rangle$ to the standard heart $\mathcal{A}^0$. Thus $\mathrm{Aut}^0(\mathcal{D})$ acts transitively on $\mathrm{EG}^0(\mathcal{D})$ and acts freely by definition. □

Lemma 2.10. *The centre of $\mathrm{Sph}(\mathcal{D})$ is generated by $[-5]$.*

Proof. As $\mathrm{Sph}(\mathcal{D}) \cong \mathrm{Br}_3$ the centre is generated by $\Phi = (\Phi_S \Phi_T)^3$. We compute Φ on S and T

$$S \mapsto X \mapsto T[-1] \mapsto T[-3] \mapsto E[-3] \mapsto S[-3] \mapsto S[-5]$$
$$T \mapsto T[-2] \mapsto E[-2] \mapsto S[-2] \mapsto S[-4] \mapsto X[-4] \mapsto T[-5]$$

Thus $\Phi = [-5]$ in $\mathrm{Aut}^0(\mathcal{D})$. □

We note that $\mathrm{Sph}(\mathcal{D})$ defines a subgroup of $\mathrm{Aut}^0(\mathcal{D})$ isomorphic to Br_3. The generators Φ_S and Φ_T are composites of two autoequivalences corresponding to simple tilts, e.g. $\Phi_S^{-1} = (\Phi_T \Phi_S \Phi_T[3])(\Phi_S \Phi_T[2])$ and so preserve the connected component of the exchange graph. If an element of $\mathrm{Sph}(\mathcal{D})$ acts trivially on $K(\mathcal{D})$

then it belongs to the centre which we have just seen is generated by a non-trivial element of $\mathrm{Aut}^0(\mathcal{D})$ so the only element of $\mathrm{Sph}(\mathcal{D})$ acting trivially is the identity.

Theorem 2.11. *The map* $\mathrm{Br}_3 \to \mathrm{Aut}^0(\mathcal{D})$ *given by* $(\sigma_1, \sigma_2) \mapsto (\Phi_S[1], \Phi_T[1])$ *is an isomorphism.*

Proof. As the exchange graph is an $\mathrm{Aut}^0(\mathcal{D})$-torsor we know that $\mathrm{Aut}^0(\mathcal{D}) = \langle \Phi_S, \Phi_T, [1]\rangle$. As the shift functor commutes with the spherical twists, we find that $((\Phi_S[1])(\Phi_T[1]))^3 = [-5][6] = [1]$ so $\mathrm{Aut}^0(\mathcal{D}) = \langle \Phi_S[1], \Phi_T[1]\rangle$. These two generators satisfy the braid relation as Φ_S, Φ_T do.

Now consider a word w in the generators $\Phi_S[1], \Phi_T[1]$ and their inverses which is equal to the identity of $\mathrm{Aut}^0(\mathcal{D})$. As $[1]$ is in the centre of $\mathrm{Aut}^0(\mathcal{D})$, we have $\Phi_S^{n_1} \dots \Phi_T^{n_k} = [-1]^{\sum n_i}$ in $\mathrm{Sph}(\mathcal{D})$. By the above lemma the centre of $\mathrm{Sph}(\mathcal{D})$ is generated by $(\Phi_S\Phi_T)^3 = [-5]$, so the right hand side is equal to $[-5]^{(\sum_i n_i)/5}$. As the braid relation is homogeneous, every element of $\mathrm{Sph}(\mathcal{D})$ has a well-defined word length in the generators Φ_S and Φ_T. But applying the word length homomorphism gives $\sum_i n_i = \frac{6}{5}\sum_i n_i$ so $\sum_i n_i = 0$. Thus the relations satisfied by the generators $\Phi_S[1], \Phi_T[1]$ of $\mathrm{Aut}^0(\mathcal{D})$ are precisely those satisfied by the generators Φ_S, Φ_T of $\mathrm{Sph}(\mathcal{D})$. □

To complete the picture we show that $\mathrm{Sph}(\mathcal{D})$ is a normal subgroup of index 5.

Proposition 2.12. *There is a short exact sequence*

$$1 \to \mathrm{Sph}(\mathcal{D}) \to \mathrm{Aut}^0(\mathcal{D}) \to \mathbb{Z}/5\mathbb{Z} \to 1$$

where the quotient map l is the modulo 5 word length map in the generators $\Phi_S[1]$ and $\Phi_T[1]$

Proof. $\mathrm{Sph}(\mathcal{D})$ is in the kernel of l as

$$l(\Phi_S) = l(\Phi_S[1]) - l([1]) = 1 - 6 = 0$$

Conversely the smallest power of $[1]$ in the kernel is $[5] = (\Phi_S\Phi_T)^{-3} \in \mathrm{Sph}(\mathcal{D})$ and so as $\mathrm{Aut}^0(\mathcal{D}) = \langle \Phi_S, \Phi_T, [1]\rangle$ the kernel is contained in $\mathrm{Sph}(\mathcal{D})$. □

Remark 2.13. *By Sabidussi's Theorem* [8, Thm 4], $\mathrm{EG}^0(\mathcal{D})$ *is isomorphic to the Cayley graph of the braid group* Br_3 *with respect to the generators* $\Delta = (\Phi_T\Phi_S\Phi_T)[3]$ *and* $\Sigma = (\Phi_S\Phi_T)[2]$ *which give the simple tilted hearts. Indeed this gives an alternative presentation of* Br_3 [4, Sect 1.14]

$$\langle \Sigma, \Delta \mid \Sigma^3 = \Delta^2\rangle$$

The quotient of $\mathrm{EG}^0(\mathcal{D})$ *by* $\mathrm{Sph}(\mathcal{D})$ *is the A_2 cluster exchange graph which is isomorphic to the Cayley graph of* $\mathbb{Z}/5\mathbb{Z}$*. This recovers a special case of a result of Keller and Nicolas* [5, Thm 5.6].

3. Stability conditions

In this section we prove Theorem 1.2. We derive the Picard-Fuchs equations satisfied by the periods of the family of meromorphic differentials λ on the fibres E of the universal family of framed elliptic curves $\mathcal{E} \to \widetilde{\mathcal{M}_{1,1}}$. Identifying the lattices $H_1(E, \mathbb{Z}) \cong K(\mathcal{D})$, the image in $\mathbb{P}\,\mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ of a certain branch of the period map is a double of the Schwarz triangle with angles $(\pi, \pi/3, \pi/2)$. We show that this coincides with the image under the local homeomorphism $\bar{Z} : \mathrm{Stab}^0(\mathcal{D})/\mathbb{C} \to \mathbb{P}\,\mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ of a fundamental domain for the action of $\mathrm{Aut}^0(\mathcal{D})/\mathbb{Z} \cong \mathrm{PSL}(2, \mathbb{Z})$ on $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$. We use our understanding of the exchange graph of $\mathcal{D}$ to lift the period map to our desired biholomorphism $f : \widetilde{\mathcal{M}_{1,1}} \to \mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$.

Definition 3.1. *On an elliptic curve $y^2 = z^3 + az + b$ define the meromorphic differential $\lambda = y\,dz$*

This has a pole of order 6 at the point at infinity and double zeroes at each of the three other branch points of y. This is the divisor of the function y^2. It is the unique differential up to scale with this property as the above divisor has degree zero.

Define the coordinates j and u on $\widetilde{\mathcal{M}_{1,1}}$ by

$$J = 1728/j \qquad j = 4u(1-u) \tag{1}$$

where J denotes the usual J-invariant. We note that the family of differentials $\lambda = \sqrt{z^3 - 3z + (4u-2)}\,dz$ satisfy $2\partial_u \lambda = \omega$, where $\omega = dz/y$ is the family of holomorphic differentials on $\widetilde{\mathcal{M}_{1,1}}$. Using this we show that the periods of λ satisfy hypergeometric equations in u and j.

Definition 3.2. *A hypergeometric differential equation is a second order ordinary differential equation on $\mathbb{P}^1$ of the form*

$$w(1-w)f'' + (\gamma - (\alpha+\beta-1)w)f' - \alpha\beta w = 0$$

with $\alpha, \beta, \gamma \in \mathbb{R}$.

It has regular singularities at 0, ∞ and 1 with exponents

$$\lambda = 1-\gamma \qquad \mu = \alpha - \beta \qquad \nu = \gamma - \alpha - \beta$$

Lemma 3.3. *The periods of λ satisfy the hypergeometric equation in j with exponents $(1, \frac{1}{3}, \frac{1}{2})$*

Proof. Suppose the periods of λ satisfy the hypergeometric equation in u

$$u(1-u)\partial_u^2 f + (\gamma - (\alpha+\beta-1)j)\partial_u f - \alpha\beta f = 0$$

Taking the derivative with respect to the dependent variable u, we find that the periods of ω must satisfy

$$u(1-u)\partial_u^2 f + (1-2u)\partial_u f + (\gamma - (\alpha+\beta-1)u)\partial_u f - (\alpha+\beta-1)f - \alpha\beta f = 0$$

which is hypergeometric of the form

$$u(1-u)\partial_u^2 f + ((\gamma+1) - ((\alpha+1)+(\beta+1)-1)u)\partial_u f - (\alpha+1)(\beta+1)f = 0$$

It is well-known the periods of ω satisfy the hypergeometric equation in j with exponents $(0, \frac{1}{3}, \frac{1}{2})$. Then by the quadratic transformation law for the change of variable given above [12, Eq 2], they satisfy the hypergeometric equation in u with exponents $(0, \frac{1}{3}, 0)$. By the above computation, the periods of λ satisfy the hypergeometric equation with exponents $(1, \frac{1}{3}, 1)$ and so reversing the change of variable gives the result. □

Remark 3.4. *The coordinate transformation (1) defines a double cover $B \to M_{1,1}$ of the coarse moduli space of elliptic curves. There is a family of elliptic curves on B whose total space is the complement of the three singular fibres of types (I_1, I_1, II^*) over $u = 0, 1$ and ∞ respectively of a rational elliptic surface $\Sigma_u \to \mathbb{P}^1_u$. This is the smooth part of Hitchin's fibration of the moduli space of meromorphic* SU(2)*-Higgs bundles on $\mathbb{P}^1_z$ with a single pole of order 4 at $z = \infty$ whose leading term is nilpotent. The meromorphic differential λ is the Seiberg-Witten differential of this integrable system, that is the exterior derivative of λ defines a holomorphic symplectic form on Σ.*

In fact Σ is a hyperkähler manifold [13]*, which was studied in* [3, Sect 9.3.3]*. In another complex structure Σ is isomorphic to the moduli space of flat* $\mathrm{SL}(2,\mathbb{C})$*-connections on $\mathbb{P}^1_z$ with a single pole at $z = \infty$ of Katz invariant* 5/2*. This complex* *manifold was studied in* [9, 11] *as the moduli space of initial conditions of the first Painlevé equation (cf Remark 1.3). Its image under the Riemann-Hilbert map is an affine cubic surface which is isomorphic as a complex variety to the cluster algebra of* A_2.

Now consider the moduli space of elliptic curves $\mathcal{M}_{1,1} \cong \mathbb{P}(2,3)\backslash\{\circ\}$ where $\circ$ is the point corresponding to $j = 0$. We make branch cuts on $\mathcal{M}_{1,1}$ along the line $\Im(j) = 0$ between $\circ$ and each of the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifold points $\times$, $*$ at $j = 1, \infty$. We deduce the image of this branch of the period map p of λ from the Schwarz triangle theorem.

Theorem 3.5. [7, p 206] *Suppose* f_1, f_2 *are linearly independent solutions to the hypergeometric equation with exponents* (λ, μ, ν). *Suppose further that their ratio* $s = f_1/f_2$ *restricted to the upper-half plane* $\mathfrak{h} \subset \mathbb{P}^1\backslash\{0, \infty, 1\}$ *is an injection. Then* s *maps* $\mathfrak{h}$ *biholomorphically onto the interior of a curvilinear triangle* $\Delta_{\lambda,\mu,\nu}$ *of angles* $(\lambda\pi, \mu\pi, \nu\pi)$.

The image is determined up to a Möbius map and so specified uniquely by the positions of the three vertices of the triangle Δ. By the Schwarz reflection principle we have

Corollary 3.6. *The image* $\Diamond = p(\mathcal{M}_{1,1})$ *is the double of the curvilinear triangle* $\Delta_{1,\frac{1}{3},\frac{1}{2}}$ *along the edge connecting the image of the two orbifold points* $\times$ *and* $*$.

We now define a fundamental domain $V = V(\bar{\mathcal{A}}^0)$ for the action of $\mathrm{Aut}^0(\mathcal{D})/\mathbb{Z}$ on $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ which maps bijectively under the local homeomorphism $\bar{Z}$ to $\Diamond$. Although the vertices of the quotient of the exchange graph $\overline{\mathrm{EG}^0}(\mathcal{D}) = \mathrm{EG}^0(\mathcal{D})/\mathbb{Z}[1]$ are indeed an $\mathrm{Aut}^0(\mathcal{D})/\mathbb{Z}$-torsor, the notion of a projective stability condition $\overline{\sigma} \in \mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ being supported at a given vertex $\bar{\mathcal{A}}$ of $\overline{\mathrm{EG}^0}(\mathcal{D})$ is not a priori well-defined. This is because points of $\mathrm{Stab}^0(\mathcal{D})$ in the same $\mathbb{C}$-orbit can be supported on different hearts, even modulo the shift functor. We *define* $\overline{\sigma}$ to be supported on $\bar{\mathcal{A}}$ using the following width function.

Definition 3.7. *Define the width* φ *of a stability condition* $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(\mathcal{D})$

$$\varphi(\sigma) = \phi^+(\sigma) - \phi^-(\sigma)$$

where $\phi^+(\sigma)$ *and* $\phi^-(\sigma)$ *denote the maximal and minimal phases respectively of an object in* $\mathcal{A}$.

The width is the angle of the image under Z of the cone $C(\mathcal{A}) \subset K(\mathcal{A})$ generated by classes of objects in $\mathcal{A}$.

Definition 3.8. *We say that* $\overline{\sigma} \in \mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ *is supported on* $\bar{\mathcal{A}}$ *if the width function is minimised on a lift* $\mathcal{A}$ *of* $\bar{\mathcal{A}}$.

Note that $\overline{\sigma}$ is supported on more than one $\bar{\mathcal{A}}$ where the width function is minimised on more than one such $\bar{\mathcal{A}}$. We will write $V(\bar{\mathcal{A}}) \subset \mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ for the subset supported uniquely on $\bar{\mathcal{A}}$, whose closure $\bar{V}(\bar{\mathcal{A}})$ is the subset supported on $\bar{\mathcal{A}}$.

Proposition 3.9. $V = V(\bar{\mathcal{A}}^0)$ *is the interior of a fundamental domain for the action on* $\mathrm{Aut}^0(\mathcal{D})/\mathbb{Z}$ *on* $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$

Proof. As the vertices of $\overline{\mathrm{EG}}^0(\mathcal{D})$ are an $\mathrm{Aut}^0(\mathcal{D})/\mathbb{Z}$-torsor, every point in the set $T = \coprod_{\bar{\mathcal{A}}} V(\bar{\mathcal{A}})$ belongs to a unique $V(\bar{\mathcal{A}})$. The points $\overline{\sigma}$ in $\bar{V}\backslash V$ lie on the three codimension 1 walls pictured below where $\overline{\sigma}$ is also supported on some other $\bar{\mathcal{A}}$ for some simple tilt $\mathcal{A}$ of $\mathcal{A}^0$. These walls of the $V(\bar{\mathcal{A}})$ are locally finite as there is only one other wall intersecting $\bar{V}$, namely $\bar{V}(L_S(\bar{\mathcal{A}}^0)) \cap \bar{V}(R_T(\bar{\mathcal{A}}^0))$. Thus the closure $\bar{T} = \coprod_{\bar{\mathcal{A}}} \bar{V}(\bar{\mathcal{A}})$. But $\bar{T}$ is clearly open and so is the entire connected component $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$. ∎

Remark 3.10. *The above proof shows that an autoequivalence* Φ *which preserves* $\overline{\mathrm{EG}}^0(\mathcal{D})$ *preserves the connected component* $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$*. Also if* Φ *acts trivially on* $\overline{\mathrm{EG}}^0(\mathcal{D})$ *then* Φ *fixes the central charge* $\bar{Z}$ *and so* Φ *acts trivially on* $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$*.*

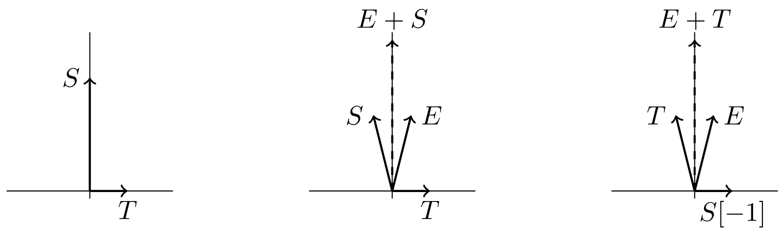

FIGURE 1. Typical stability conditions on the three boundary components of $V(\bar{\mathcal{A}}^0)$. The first $\bar{V}(\bar{\mathcal{A}}^0) \cap \bar{V}(R_S(\bar{\mathcal{A}}^0)) = \bar{V}(\bar{\mathcal{A}}^0) \cap \bar{V}(L_T(\bar{\mathcal{A}}^0))$ occurs where the only stable objects are S and T. The other two $\bar{V}(\bar{\mathcal{A}}^0) \cap \bar{V}(R_T(\bar{\mathcal{A}}^0))$ and $\bar{V}(\bar{\mathcal{A}}^0) \cap \bar{V}(R_T(\bar{\mathcal{A}}^0))$ lie in the region where S, T and E are stable.

This means that $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ is glued together from the $V(\bar{\mathcal{A}})$ according to the quotient of the exchange graph $\overline{\mathrm{EG}}^0(\mathcal{D})$ just as $\mathrm{Stab}^0(\mathcal{D})$ is glued from the $U(\mathcal{A})$ according to $\mathrm{EG}^0(\mathcal{D})$.

Proposition 3.11. *The image of* V *under the map* $\bar{Z}$ *is* $\lozenge$*.*

Proof. The boundary of V consists of stability conditions supported on one of the three walls which we picture below, whose image under $\bar{Z}$ is the boundary of $\lozenge$. □

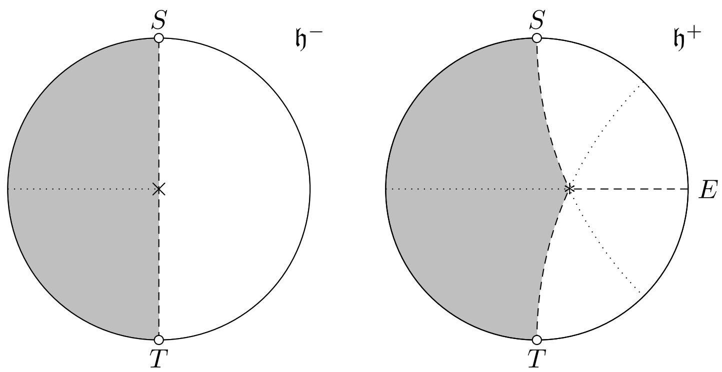

FIGURE 2. The fundamental domain $V(\bar{\mathcal{A}}^0) \cong \lozenge$ under the map $\bar{Z} : \mathrm{Stab}^0(\mathcal{D})/\mathbb{C} \to \mathbb{P}^1$. We picture $\mathbb{P}^1 = \mathfrak{h}^+ \cup \mathfrak{h}^-$ as the union of two discs where the imaginary part of the coordinate $\bar{Z}$ is positive and negative respectively. They are glued along the line $\bar{Z} \in \mathbb{R}$, which is the image of all walls of marginal stability in $\mathrm{Stab}(\mathcal{D})/\mathbb{C}$. The region $\mathfrak{h}^-$ where only two objects are stable contains the first wall passing through the image of $\times$. The region $\mathfrak{h}^+$ contains the other two walls of $V(\bar{\mathcal{A}}^0)$ which meet at the image of $*$. We label points on the boundary by the object whose central charge vanishes there.

Proof of Theorem 1.2. Using the identification $V(\mathcal{A}) \cong \Diamond = p(\mathcal{M}_{1,1})$, we can extend the branch of the period map to a map $f : \widetilde{\mathcal{M}_{1,1}} \to \mathrm{Stab}(\mathcal{D})/\mathbb{C}$ by equivariance. We only have to check continuity on the boundary of $\mathcal{M}_{1,1}$, i.e. the action of the monodromy on $H_1(E, \mathbb{Z})$ on crossing one of the two branch cuts in either direction is identical to the action of the four simple tilts on $K(\mathcal{D})$. But these both act by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

and their inverses. □

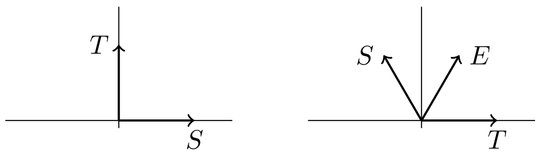

FIGURE 3. The image of the $\mathbb{Z}/2$- and $\mathbb{Z}/3$-orbifold points $\times$ and $*$

We denote by $\widetilde{L^\times}$ the total space of the $\mathbb{C}^*$-bundle of non-zero holomorphic differentials on $\widetilde{M}$. It is isomorphic to the complement of the discriminant locus in the space $\mathbb{C}^2_{a,b}$ of cubic polynomials $z^3 + az + b$. The fundamental group of $L^\times$ is isomorphic to the braid group Br_3 as the discriminant locus describes the trefoil knot.

Corollary 3.12. *There is a biholomorphic map*

$$\begin{array}{ccc} \widetilde{L^\times} & \xrightarrow{F} & \mathrm{Stab}^0(\mathcal{D}) \\ & {\scriptstyle (\int_\alpha \lambda, \int_\beta \lambda)} \searrow & \downarrow {\scriptstyle (Z(S), Z(T))} \\ & & \mathrm{Hom}(K(\mathcal{D}), \mathbb{C}) \end{array}$$

lifting the periods of the differential λ. It is equivariant with respect to the actions of Br_3 *on the left by deck transformations and on the right by* $\mathrm{Aut}(\mathcal{D})$.

Proof. We can lift the map $f : \widetilde{\mathcal{M}_{1,1}} \to \mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ to the desired F by equivariance with respect to the $\mathbb{C}$-actions on both sides. It is a bijection as both $\mathbb{C}$-actions are free, and holomorphic as it is locally given by the periods of λ. We know that the two braid groups act identically on $K(\mathcal{D})$ via their maps to $\mathrm{PSL}(2, \mathbb{Z})$ and so define identical actions on the $\mathbb{C}^*$-bundle $L^\times$. Also the actions of the central subgroup $\mathbb{Z} \subset \mathrm{Br}_3$ are identical by construction as it acts as $\mathbb{Z} \subset \mathbb{C}$. But given these data the actions are determined by a group homomorphism $\mathrm{PSL}(2, \mathbb{Z}) \to \mathbb{Z}$ giving a lifting of the Br_3-action on the $\mathbb{C}^*$ bundle $L^\times$ factoring through $\mathrm{PSL}(2, \mathbb{Z})$ to the universal cover. As the only such homomorphism is the trivial one the two Br_3 actions are identical. □

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