Research

Subject Areas:

mathematical physics, quantum physics, differential equations

Keywords:

spectral problems, affine Toda field theory, Bethe ansatz

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The Bethe ansatz and the Tzitzéica–Bullough–Dodd equation

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The theory of classically integrable nonlinear wave equations and the Bethe ansatz systems describing massive quantum field theories defined on an infinite cylinder are related by an important mathematical correspondence that still lacks a satisfactory physical interpretation. In this paper, we shall extend this link to the case of the classical and quantum versions of the Tzitzéica–Bullough–Dodd model.

1. Introduction

An interesting correspondence was discovered in Dorey & Tateo [1] between the functional approach to the spectral theory of ordinary differential equations, mainly developed by Sibuya, Voros and co-workers [2,3], and a series of works by Bazhanov et al. [4–6], where earlier results of Baxter [7–9] were extended to the conformal field theories (CFTs) governing the continuum limits of certain integrable models (IMs) on a two-dimensional lattice.

It turns out that the initial observation of Dorey & Tateo [1] was the first hint of a very general mathematical scheme—an 'ODE/IM correspondence'—involving wide classes of multi-parameter ordinary differential equations. Their generalized eigenvalue problems turn out to be constrained by the same Bethe ansatz equations (BAEs) as arise in the CFT limits of certain integrable vertex models, related to Lie algebras [10–16].

A precise relationship between elements of this new scheme and the $\mathcal{PT}$-symmetric quantum mechanical models of Bender and co-workers [17,18] was established in Dorey & Tateo [19]. This has led to proofs of spectral reality in certain cases [20], which were further generalized in Shin [21], and insights into the loss of reality in others [22]. Applications of the correspondence are also being found in various problems arising in condensed matter physics [23–25].

A generalization of this correspondence to off-critical, or massive, quantum field theories (QFTs) was subsequently obtained by Gaiotto *et al.* [26] following a different chain of discoveries, relating to string and gauge theories. As a net outcome of a surprising series of mathematical connections, also bringing in the anti-de Sitter (AdS)/CFT correspondence, gluon scattering amplitudes in $\mathcal{N} = 4$ super Yang–Mills can now be studied using powerful tools from the theory of IMs [27].

A further important step for the understanding of the relationship between the AdS/CFT-related results and the original formulation of the ODE/IM correspondence was made by Lukyanov & Zamolodchikov [28]. They showed how to link the classical sinh-Gordon equation to the quantum massive sine(h)-Gordon model, generalizing the earlier works on the ODE/IM correspondence. In doing this, they brought into play all the formalism developed in more than 50 years of soliton theory and the study of exactly solvable nonlinear partial differential equations (of which sinh-Gordon is an exemplar), in particular the role played by the association between a nonlinear classical field theory and certain sets of linear differential equations.

In this paper, we shall discuss another example of an off-critical generalization of the ODE/IM correspondence, which will be built starting from the definition of a particular model on the classical side. This model is often referred to as the Bullough–Dodd (BD) model [29], though the special properties of the corresponding partial differential equation were noticed as long ago as 1907, by Tzitzeica [30]. Like the sinh-Gordon model, it is an example of an affine Toda field theory, or two-dimensional Toda chain, as will be explained in §2. The quantum version of the BD model, also known as the Izergin–Korepin model [31], plays an important role in the framework of massive two-dimensional QFTs corresponding to integrable perturbations of the minimal series of CFTs [32]. Together with its purely imaginary coupling version, it has been directly studied using the exact S-matrix approach [33]. It is also related to the scaling limit of important statistical–mechanical systems such as the q-state Potts models [34] and the dilute A_n models [35].

The analysis in this paper follows the steps taken previously [13,28]. Some further details of the BD model and the ODE/IM correspondence can be found in Faldella [36], whereas a more comprehensive presentation encompassing the extension to other affine Toda field theories will be the subject of a forthcoming publication.

2. The Bullough–Dodd model

The one-dimensional Toda chain is defined as

$$\partial_t^2 \eta^k = 2 \, e^{(2\eta^{k+1} - 2\eta^k)} - 2 \, e^{(2\eta^k - 2\eta^{k-1})}, \tag{2.1}$$

where k enumerates and labels the fields. This definition can be simply modified to yield the $(1 + 1)$-dimensional variant of (2.1) [37],

$$\partial_t^2 \eta^k - \partial_x^2 \eta^k = 2 \, e^{(2\eta^{k+1} - 2\eta^k)} - 2 \, e^{(2\eta^k - 2\eta^{k-1})}. \tag{2.2}$$

Varying the periodicity conditions and symmetries of the fields η^k, the system (2.2) leads to a whole family of nonlinear equations that are solvable by means of the inverse scattering transform [38]. This involves an associated linear system, which has to reproduce the nonlinear chain through a compatibility condition. It turns out that the correct choice for (2.2) is [37]

$$\left. \begin{aligned} X\Psi &\equiv (\partial_t + V - i\lambda C_1 - i\lambda^{-1} C_2)\Psi = 0 \\ T\Psi &\equiv (\partial_x + W + i\lambda C_1 - i\lambda^{-1} C_2)\Psi = 0, \end{aligned} \right\} \tag{2.3}$$

and

where λ is the *spectral parameter*,

$$
\left.
\begin{array}{l}
V_{ij} = \partial_t \eta_i \delta_{ij}; \quad W_{ij} = \partial_x \eta_i \delta_{ij}; \quad C_{1ij} = C_{2ji} = C_j \delta_{i-1,j} \\[6pt]
C_i = e^{(\eta^{i+1} - \eta^i)},
\end{array}
\right\}
\tag{2.4}
$$

and

and δ_{ij} is the Kronecker symbol which, in a case of a closed chain consisting of N elements, is defined as

$$
\delta_{ij} = \begin{cases} 1, & \text{when } i \equiv j \bmod N; \\ 0, & \text{in all other cases.} \end{cases}
\tag{2.5}
$$

It is also possible to give a Lagrangian formulation for the system (2.2),

$$
\mathcal{L} = \sum_{k=1}^{N} \left(\frac{1}{2} \partial_\mu \eta^k \partial^\mu \eta^k - e^{(2\eta^k - 2\eta^{k-1})} + 1 \right)
\tag{2.6}
$$

with $\mu \in (t, x)$. Then, it is straightforward to verify that

$$
\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \eta^k)} - \frac{\partial \mathcal{L}}{\partial \eta^k} = 0 \Longrightarrow \partial_t^2 \eta^k - \partial_x^2 \eta^k = 2 e^{(2\eta^{k+1} - 2\eta^k)} - 2 e^{(2\eta^k - 2\eta^{k-1})}.
$$

Typically, the further condition $\sum_k \eta^k = 0$ is imposed on the fields,[1] which results in the affine Toda field theory based on the $a_{N-1}^{(1)}$ affine Dynkin diagram. For $N = 2$, with $\eta^1 = -\eta^2 = \eta$, the equations then reduce to

$$
\eta_{tt} - \eta_{xx} + 4 \sinh(2\eta) = 0 \quad \text{(the sinh-Gordon equation).}
\tag{2.7}
$$

For $N = 3$, and with $\eta^3 = -\eta^1 - \eta^2$,

$$
\left.
\begin{array}{l}
\eta_{tt}^1 - \eta_{xx}^1 + 2 e^{(2\eta^1 - 2\eta^2)} - 2 e^{(-4\eta^1 - 2\eta^2)} = 0 \\[6pt]
\eta_{tt}^2 - \eta_{xx}^2 + 2 e^{(4\eta^1 + 2\eta^2)} - 2 e^{(2\eta^1 - 2\eta^2)} = 0,
\end{array}
\right\}
\tag{2.8}
$$

and

which are the equations of motion for the $a_2^{(1)}$ affine Toda field theory. The $\mathbb{Z}_2$ symmetry of the $a_2^{(1)}$ affine Dynkin diagram allows a further symmetry to be imposed, namely $\eta^1 = -\eta^2 = \eta$. Then, (2.8) becomes

$$
\eta_{tt} - \eta_{xx} + 2 e^{4\eta} - 2 e^{-2\eta} = 0 \quad \text{(the BD equation).}
\tag{2.9}
$$

Equation (2.9) can also be recovered directly from the compatibility of 3×3 matrix operators that are defined as [37]

$$
T = \partial_x + \begin{pmatrix} \eta_t & -i\lambda^{-1} e^{-\eta} & i\lambda e^{2\eta} \\ i\lambda e^{-\eta} & 0 & -i\lambda^{-1} e^{-\eta} \\ -i\lambda^{-1} e^{2\eta} & i\lambda e^{-\eta} & -\eta_t \end{pmatrix}
\tag{2.10}
$$

and

$$
X = \partial_t + \begin{pmatrix} \eta_x & i\lambda^{-1} e^{-\eta} & i\lambda e^{2\eta} \\ i\lambda e^{-\eta} & 0 & i\lambda^{-1} e^{-\eta} \\ i\lambda^{-1} e^{2\eta} & i\lambda e^{-\eta} & -\eta_x \end{pmatrix}.
\tag{2.11}
$$

If we were to introduce light-cone coordinates as

$$
z = x + t; \quad \bar{z} = x - t,
\tag{2.12}
$$

and redefine $\eta \to \eta/2$, (2.9) would become

$$
\partial_z \partial_{\bar{z}} \eta(z, \bar{z}) + e^{-\eta(z, \bar{z})} - e^{2\eta(z, \bar{z})} = 0.
\tag{2.13}
$$

---

[1]This condition has the effect of excluding a zero mode in the 'mass spectrum' of the theory:

$$
m_n^2 = 4 \sin^2 \left(\frac{\pi n}{N} \right), \quad n = 0, 1, \dots, N - 1.
$$

The nonlinear equation (2.13) is the starting point for the construction of objects which are in correspondence with the quantum world. However, it is important to stress that, for this purpose, the coordinates z and $\bar{z}$ should be considered as living in an 'auxillary space', which is *not* the same as the space on which the QFT under discussion will be defined. In fact, it will be better to adopt a Euclidean metric on this auxillary space so that z and $\bar{z}$ are complex conjugates of each other, though this will often be relaxed and z and $\bar{z}$ regarded as independent complex variables. To pave the way for the correspondence, it is also convenient to adopt a modified version of (2.13). This will be referred to as the modified Bullough–Dodd (mBD) equation and is

$$\partial_z \partial_{\bar{z}} \eta + e^{-\eta(z,\bar{z})} - p(z, s^{3M}) p(\bar{z}, s^{3M}) \, e^{2\eta(z,\bar{z})} = 0, \quad \text{(the mBD equation)} \tag{2.14}$$

where $M > 0$ and the function $p(z, s^{3M})$ is defined to be

$$p(z, s^{3M}) = z^{3M} - s^{3M}, \tag{2.15}$$

and s will turn out to take the role of a scale parameter. Equations (2.13) and (2.14) are related by a change of variables and a redefinition of the field, because, defining

$$w(z) = \int^z \sqrt{p(z')} \, dz' \quad \text{and} \quad \bar{w}(\bar{z}) = \int^{\bar{z}} \sqrt{p(\bar{z}')} \, d\bar{z}', \tag{2.16}$$

and shifting the field as $\eta \to \eta + \frac{1}{3} \ln(p(z) p(\bar{z}))$, the mBD equation reduces to the BD equation. The mBD model has, in general, no explicit rotational symmetry. Instead, it has a discrete symmetry dictated by the form of the 'potential' function $p(z)$,

$$z \to e^{2\pi i/3M} z, \quad \bar{z} \to e^{-2\pi i/3M} \bar{z}. \tag{2.17}$$

The solutions of the mBD equation (2.14) which will be relevant for the current analysis must respect this symmetry, be continuous at every finite non-zero $z, \bar{z}$, and grow more slowly than exponentially at $z, \bar{z} \to \infty$. As in Lukyanov & Zamolodchikov [28] for the (modified) sinh-Gordon case, we will consider a one-parameter family of such solutions, characterized by a logarithmic behaviour near the origin. Before listing in detail the exact conditions that are required, polar coordinates can be introduced to show these properties more neatly:

$$z = \rho \, e^{i\phi}, \quad \bar{z} = \rho \, e^{-i\phi}. \tag{2.18}$$

In most of the following, z and $\bar{z}$ will be treated as independent complex variables, so the solution η will be a function of the independent variables (ρ, ϕ), according to (2.18).

It is now possible to list the properties of the sought-after solutions, as follows:

(i) periodicity:

$$\eta\left(\rho, \phi + \frac{2\pi}{3M}\right) = \eta(\rho, \phi); \tag{2.19}$$

or, better, the solutions $\eta(\rho, \phi)$ are single-valued functions on a cone with the apex angle $2\pi/3M$,

$$\mathcal{C}_{2\pi/3M} : \phi \sim \phi + \frac{2\pi}{3M}, \quad 0 \leq \rho < \infty; \tag{2.20}$$

(ii) the solutions $\eta(\rho, \phi)$ are real-valued for real ρ and ϕ, and finite everywhere on the cone $\mathcal{C}_{2\pi/3M}$, except for the apex $\rho = 0$;

(iii) large-ρ asymptotic:

$$\eta(\rho, \phi) = -2M \ln \rho + o(1) \quad \text{as } \rho \to \infty; \tag{2.21}$$

(iv) small-ρ asymptotic:

$$\eta(\rho, \phi) = -2g \ln(\rho) + O(1) \quad \text{for } -1 < g < \tfrac{1}{2}. \tag{2.22}$$

The factors $-2M$ and $-2g$ in (2.21) and (2.22) have been chosen, for future convenience, to be consistent with earlier studies [13,28].

As in Lukyanov & Zamolodchikov [28], one can start from equation (2.22) and develop an expansion for $z, \bar{z} \sim 0$ of the form

$$\eta = -g\ln(z\bar{z}) + \eta_0 + \sum_{k=1}^{\infty} \gamma_k(z^{3Mk} + \bar{z}^{3Mk}) - \frac{e^{-\eta_0}}{(g+1)^2}(z\bar{z})^{g+1} - \frac{s^{6M}e^{2\eta_0}}{(-2g+1)^2}(z\bar{z})^{-2g+1} + \cdots, \quad (2.23)$$

where η_0 γ_k are integration constants, all remaining terms omitted in the expansion being uniquely determined once these constants are given. These constants are not new parameters, but should be fixed by demanding the consistency of this expansion with the remaining conditions (i)–(iii).

The expansion (2.23) remains valid with z and $\bar{z}$ regarded as independent complex variables and, for later analysis, it will be useful to record the form of (2.22) in the so-called *light-cone* limit $\bar{z} \to 0$ (with fixed z),

$$\eta \sim -g\ln(z\bar{z}) + \eta_0 + \gamma(z), \quad (2.24)$$

where $\gamma(z) = \sum_{k=1}^{\infty} \gamma_k z^{3Mk}$.

At this stage, it is possible to define the linear problem associated with the mBD equation

$$\mathbf{D}\boldsymbol{\Psi} = 0, \qquad \bar{\mathbf{D}}\boldsymbol{\Psi} = 0, \quad (2.25)$$

where $\mathbf{D}$ and $\bar{\mathbf{D}}$ are components of an $sl(3)$ connection and can be found by rearranging, in light-cone coordinates, the matrix operators (2.10) with an appropriate redefinition of the spectral parameter λ and after the introduction of the potential $p(z)$,

$$\mathbf{D} = \partial_z + \begin{pmatrix} \frac{1}{2}\partial_z\eta & 0 & \lambda\,e^{\eta}p(z) \\ -\lambda\,e^{-(1/2)\eta} & 0 & 0 \\ 0 & -\lambda\,e^{-(1/2)\eta} & -\frac{1}{2}\partial_z\eta \end{pmatrix} \quad (2.26)$$

and

$$\bar{\mathbf{D}} = \partial_{\bar{z}} + \begin{pmatrix} -\frac{1}{2}\partial_{\bar{z}}\eta & \lambda^{-1}e^{-(1/2)\eta} & 0 \\ 0 & 0 & \lambda^{-1}e^{-(1/2)\eta} \\ \lambda^{-1}p(\bar{z})\,e^{\eta} & 0 & \frac{1}{2}\partial_{\bar{z}}\eta \end{pmatrix}. \quad (2.27)$$

It is possible to deal directly with this system of six equations on $\boldsymbol{\Psi}$ to get information about the solutions and their asymptotics, but it turns out to be simpler to deal with a reduction of (2.26) and (2.27) to two third-order differential equations. Moreover, this step will be necessary for the continuation of the analysis and to show the connection between the original nonlinear problem and a previously studied spectral problem. This reduction can be implemented defining a vector solution to equations (2.26) and (2.27):[2]

$$\boldsymbol{\Psi} = \begin{pmatrix} \lambda^{-1/2}e^{\eta/2}\partial_z(e^{\eta}\partial_z(e^{-\eta}\psi)) \\ -\lambda^{1/2}e^{\eta}\partial_z(e^{-\eta}\psi) \\ \lambda^{3/2}e^{-\eta/2}\psi \end{pmatrix} = \begin{pmatrix} \lambda^{-3/2}e^{-\eta/2}\bar{\psi} \\ -\lambda^{-1/2}e^{\eta}\partial_{\bar{z}}(e^{-\eta}\bar{\psi}) \\ \lambda^{1/2}e^{\eta/2}\partial_{\bar{z}}(e^{\eta}\partial_{\bar{z}}(e^{-\eta}\bar{\psi})) \end{pmatrix}. \quad (2.28)$$

The equality between each row of the two parentheses in (2.28) must always hold, and will be useful to find large-ρ asymptotics of (2.25). Applying $\mathbf{D}$ and $\bar{\mathbf{D}}$ to the first and the second vectors in (2.28), respectively, it is simple to find the system of two third-order linear differential equations that constitutes a compatibility condition on ψ and $\bar{\psi}$ so that $\boldsymbol{\Psi}$ really is a solution of (2.25). The two equations are

$$\partial_z^3\psi - ((\partial_z\eta)^2 + 2\partial_z^2\eta)\partial_z\psi + (\lambda^3 p(z) - \partial_z\eta\partial_z^2\eta - \partial_z^3\eta)\psi = 0 \quad (2.29)$$

and

$$\partial_{\bar{z}}^3\bar{\psi} - ((\partial_{\bar{z}}\eta)^2 + 2\partial_{\bar{z}}^2\eta)\partial_{\bar{z}}\bar{\psi} + (\lambda^{-3}p(\bar{z}) - \partial_{\bar{z}}\eta\partial_{\bar{z}}^2\eta - \partial_{\bar{z}}^3\eta)\bar{\psi} = 0. \quad (2.30)$$

The next step is to determine the $\rho \to 0$ asymptotics of solutions to (2.25). To do this, it is convenient to focus attention on equation (2.29), though the result would not change if

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[2]This is simply recovered solving each of the systems (2.25) with respect to two out of three components of the vector $\boldsymbol{\Psi}$.

equation (2.30) was taken as the starting point instead. This equation is entirely in z, except for the appearance of $\bar{z}$ in $\eta(z, \bar{z})$ that, for now, can be considered as a simple parametric dependence. Finding an asymptotic solution $\psi(z)$ enables a solution $\boldsymbol{\Psi}$ of (2.25) for $\rho \to 0$ to be determined through (2.28). Substituting the asymptotic form (2.22) of η into (2.29) and considering the $\rho \to 0$ limit (actually $z \to 0$) brings equation (2.29) into the form

$$\partial_z^3 \psi - \frac{1}{z^2} g(g+2) \partial_z \psi + \frac{1}{z^3} g(g+2) \psi = 0,$$

and seeking solutions of the form $\psi = z^\mu + \cdots$ the problem is reduced to the solution of the following indicial equation:

$$(\mu - 1)[\mu(\mu - 2) - g(g+2)] = 0.$$

This equation gives three different leading behaviours in the $z \to 0$ limit,

$$\psi_+ \sim z^{-g}, \quad \psi_0 \sim z \quad \text{and} \quad \psi_- \sim z^{g+2}. \tag{2.31}$$

These expressions give the leading behaviours and can be adjusted with appropriate multiplicative constants; once inserted into (2.28) in the usual $\rho \to 0$ limit, they provide the three asymptotic solutions $\boldsymbol{\Psi}$ to (2.25). They are

$$\boldsymbol{\Psi}_+ \sim \begin{pmatrix} 0 \\ 0 \\ e^{(-i\phi - \theta)g} \end{pmatrix}, \quad \boldsymbol{\Psi}_0 \sim \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Psi}_- \sim \begin{pmatrix} e^{(i\phi + \theta)g} \\ 0 \\ 0 \end{pmatrix} \quad (\rho \to 0) \tag{2.32}$$

with $\lambda = e^\theta$. In order to fix the values of the constants introduced earlier, a particular symmetry condition has been considered. In fact, although $\eta(\rho, \phi)$ is a single-valued function on the cone (2.20), the connection components (2.26) and (2.27) are not. Instead, the linear problem (2.25) is invariant with respect to the transformation

$$\Omega : \phi \to \phi + \frac{2\pi}{3M}, \quad \theta \to \theta - \frac{2\pi i}{3M}, \tag{2.33}$$

involving the shift of the spectral parameter θ (from now on θ and λ will be referred to as spectral parameters interchangeably, keeping in mind that $\lambda = e^\theta$). Imposing this symmetry, it is straightforward to find (2.32).

On the other hand, it is also possible to give the large-ρ asymptotics, and for this it is best to use a semiclassical approximation. As will be explained more formally in the following, assuming that θ is real, a generic solution of (2.29) grows exponentially at $\rho \to \infty$, but [2,13] there are special ('subdominant') solutions that decay in the sector

$$-\frac{4\pi}{3M + 3} < \phi < \frac{4\pi}{3M + 3}. \tag{2.34}$$

In order to build such a solution to (2.25), the first step is the specification of a $\rho \to \infty$ asymptotic for (2.29), just as done before for the solution in the region near $\rho = 0$, again remembering to substitute the large-ρ solution of (2.14). In this particular limit, equation (2.29) reduces to a sum of a third-order derivative and a 'potential' $Q(z)$ that will fix the leading term of the subdominant Wentzel–Kramers–Brillouin (WKB) solution,

$$Q(z) = e^{3\theta} (z^{3M} - s^{3M}). \tag{2.35}$$

It follows from (2.35) that the WKB-like solution has, for $M > \frac{1}{2}$, the form

$$\psi \sim c_1 z^{-M} \exp\left(-\frac{z^{M+1}}{M+1} e^\theta + f(\bar{z}) \right), \tag{2.36}$$

where c_1 is an arbitrary constant, and the term $f(\bar{z})$ in the exponential carries the dependence on $\bar{z}$ of ψ. To retrieve a more exact form of (2.36), it is convenient to seek a solution of (2.30) of the form

$$\bar{\psi} \sim c_2 \bar{z}^{-M} \exp\left(-\frac{\bar{z}^{M+1}}{M+1} e^{-\theta} + g(z)\right) \tag{2.37}$$

and then, by inserting both results in (2.28), the compatibility between the two solutions will fix these integration-generated functions. This yields

$$c_2 = c_1 e^{\theta} = c\, e^{\theta}, \quad f(\bar{z}) = -\frac{\bar{z}^{M+1}}{M+1} e^{-\theta}, \quad g(z) = -\frac{z^{M+1}}{M+1} e^{\theta}, \tag{2.38}$$

and the final semiclassical solution of (2.29) (in polar coordinates) is

$$\psi \overset{\text{WKB}}{\sim} c\, e^{-\theta} z^{-M} \exp\left(-\frac{2\rho^{M+1}}{M+1} \cosh(\theta + i(M+1)\phi)\right). \tag{2.39}$$

By inserting (2.39) into the first of (2.28), it is then straightforward to obtain

$$\Psi \overset{\text{WKB}}{\sim} e^{\theta/2} \begin{pmatrix} e^{i\phi M} \\ 1 \\ e^{-i\phi M} \end{pmatrix} \exp\left(-\frac{2\rho^{M+1}}{M+1} \cosh(\theta + i(M+1)\phi)\right), \quad \rho \to \infty. \tag{2.40}$$

Because the functions $\{\Psi_+, \Psi_0, \Psi_-\}$ form a basis in the space of solutions of linear problem (2.25), a linear relation of the following form must hold:

$$\Psi = Q^+(\theta, s)\Psi_+ + Q^0(\theta, s)\Psi_0 + Q^-(\theta, s)\Psi_-, \tag{2.41}$$

where the coefficients $Q^{\pm,0}$, which are independent of the variables ρ and ϕ, are functions of the spectral parameter θ and of s, as well as of the parameter g (this parameter has been temporarily omitted). These coefficients will coincide (up to an overall constant) with Q-functions of a two-dimensional massive QFT related to the Lie algebra A_2, or $su(3)$.

3. The conformal limit

Before proceeding with the advertised correspondence, it is useful to work a little more on equation (2.29). The problem studied up to now has an important connection with a known spectral problem of a particular third-order differential equation. In order to build an appropriate analysis and to understand where the various steps come from, (2.29) will be reduced to this spectral problem and a brief review of previously obtained results will be given.

Recall that equation (2.29) is of the form

$$\partial_z^3 \psi - ((\partial_z \eta)^2 + 2\partial_z^2 \eta)\partial_z \psi + (\lambda^3 p(z, s^{3M}) - \partial_z \eta \partial_z^2 \eta - \partial_z^3 \eta)\psi = 0,$$

and, as was said already, $\bar{z}$ plays the role of a simple parameter, so it is possible to take the so-called light-cone limit $\bar{z} \to 0$ (or, if (2.30) had been chosen as the starting point, then the light-cone limit would have involved z instead of $\bar{z}$) in which η assumes the form (2.24). After that, the limit $z \sim s \to 0, \theta \to +\infty$ can be taken, with the combinations

$$x = e^{\theta/(M+1)}z, \quad E = s^{3M} e^{3\theta M/(M+1)} \tag{3.1}$$

kept finite, whereas

$$\tilde{x} = e^{-\theta/(M+1)}\bar{z} \to 0, \quad \tilde{E} = s^{3M} e^{-3\theta M/(M+1)} \to 0. \tag{3.2}$$

This limit is a particular scaling limit, because z, s are sent to zero and θ to infinity, but they are rearranged in combinations such that the new 'variables' x and E do not diverge or collapse. This is an important turning point of the theory, because it transforms the correspondence between an ordinary differential equation and a massive QFT into a correspondence with a CFT [13] (which is massless). For this reason, the limit (3.2) can be called the *conformal limit*, with s (or, more precisely, s^{M+1}) taking the role of the mass scale.

Inserting (3.1) in (2.29) and taking the limits just discussed, equation (2.29) reduces to

$$\left[\partial_x^3 - g(g+2) \left(\frac{1}{x^2} \partial_x - \frac{1}{x^3} \right) + p(x,E) \right] y(x,E,g) = 0, \tag{3.3}$$

where y has been used instead of ψ to avoid confusion among the various solutions. Motivated by the results of earlier studies [2,13], a generalization of a theorem due to Sibuya for second-order equations can now be proposed, as follows.

Conjecture 3.1. *Equation (3.3) has a solution $y(x,E,g)$ with the following properties:*

(i) *y is an entire function of (x,E) though, owing to the branch point in the potential at $x=0$, x must in general be considered to live on a suitable cover of the punctured complex plane;*

(ii) *y, $y' = \partial_x y$ and $y'' = \partial_x^2 y$ admit, for $M > \frac{1}{2}$, the asymptotic representations*

$$y \sim x^{-M} e^{-(1/(M+1))x^{M+1}}, \quad y' \sim -e^{-(1/(M+1))x^{M+1}}, \quad y'' \sim x^M e^{-(1/(M+1))x^{M+1}}, \tag{3.4}$$

as $x \to \infty$ in the sector

$$|\arg(x)| < \frac{4\pi}{3M+3}; \tag{3.5}$$

(iii) *y is uniquely fixed by the properties (i) and (ii).*

The behaviour of the solution (3.4) follows from the more general formula

$$y(x,E) \sim Q(x,E)^{-1/3} \exp\left(- \int_{x_0}^{x} Q(x,t)^{1/3}\, dt \right), \tag{3.6}$$

where y solves a generic equation $\partial_x^3 y + Q(x,E)y = 0$. This is the analogue of a WKB approximation for a solution of a Schrödinger equation. It is possible to define rotated solutions, by analogy with Dorey *et al.* [15], in order to construct bases of solutions for (3.1). For general values of k, define

$$y_k(x,E,g) = \omega^k y(\omega^{-k}x, \omega^{-3Mk}E, g), \tag{3.7}$$

with

$$\omega = \exp\left(\frac{2\pi i}{3M+3} \right). \tag{3.8}$$

Substituting in, y_k solves

$$\partial_x^3 y_k - g(g+2) \left(\frac{1}{x^2} \partial_x - \frac{1}{x^3} \right) y_k + e^{-2k\pi i} p(x,E) y_k = 0, \tag{3.9}$$

and so, for $k \in \mathbb{Z}$, (3.7) provides a potentially new solution to (3.1). However, for now, it is convenient to leave k arbitrary because fractional values will also be needed. We also define the *Stokes sectors*

$$\mathcal{S}_k : \left| \arg(x) - \frac{2\pi k}{3M+3} \right| < \frac{\pi}{3M+3}. \tag{3.10}$$

Some discussion of the dominance and subdominance of solutions is also required, because, for a third-order problem, this is a little more complicated than the second-order case. For this, we largely follow Dorey *et al.* [15]. First, the behaviour specified by conjecture 3.1 lies in the sectors $\mathcal{S}_{-3/2} \cup \mathcal{S}_{-1/2} \cup \mathcal{S}_{1/2} \cup \mathcal{S}_{3/2}$. Furthermore, in general, there are three types of asymptotic solution for large $|x|$. Aside from the well-behaved leading term $x^{-M} \exp(-x^{M+1}/(M+1))$, there are also solutions that behave as $x^{-M} \exp(e^{\pm \pi i/3} x^{M+1}/(M+1))$. (This can be traced to the three third roots of -1, which are -1, $e^{\pi i/3}$ and $e^{-\pi i/3}$.) According to the sector being considered, either one or two of these solutions tend to zero for large $|x|$. The *subdominant* solution in any given sector will be defined to be the one that tends to zero fastest in that sector (it might also be called maximally

subdominant). Finally, y_k, up to overall scalar factor, is characterized as the solution of (3.1) which is subdominant in the sector $\mathcal{S}_k$. The asymptotic (3.4) along with the definition (3.7) implies

$$
\left.
\begin{aligned}
y_k &\sim \omega^{k(M+1)} x^{-M} e^{-(x^{M+1}/(M+1))\omega^{-(M+1)k}}, \quad y_k' \sim -e^{-(x^{M+1}/(M+1))\omega^{-(M+1)k}} \\
y_k'' &\sim \omega^{-k(M+1)} x^{M} e^{-(x^{M+1}/(M+1))\omega^{-(M+1)k}},
\end{aligned}
\right\}
\tag{3.11}
$$

and

for $|x| \to \infty$ with

$$
x \in \mathcal{S}_{k-3/2} \cup \mathcal{S}_{k-1/2} \cup \mathcal{S}_{k+1/2} \cup \mathcal{S}_{k+3/2}.
\tag{3.12}
$$

In the region formed by the two sectors $\mathcal{S}_{k+1/2} \cup \mathcal{S}_{k+3/2}$, the three asymptotics y_k, y_{k+1} and y_{k+2} live together and comparing their forms, given by (3.11), they turn out to be independent of each other. A further proof of this fact can be built defining the *Wronskian* of the three solutions,

$$
W_{k_1,k_2,k_3} = W[y_{k_1}, y_{k_2}, y_{k_3}],
\tag{3.13}
$$

where the 3×3 Wronskian is defined as

$$
W[f, g, h] = \det \begin{pmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{pmatrix}.
\tag{3.14}
$$

For f, g and h solving (3.1), $W[f, g, h]$ is independent of x, and f, g and h are linearly independent if and only if $W[f, g, h] \neq 0$. In order to demonstrate the independence of y_k, y_{k+1} and y_{k+2}, it is helpful first to compute $(k_1, k_2, k_3) = (-1, 0, 1)$

$$
W_{-1,0,1} = -3i\sqrt{3}
\tag{3.15}
$$

and then, using the general result

$$
W_{k_1+b, k_2+b, k_3+b}(E) = W_{k_1, k_2, k_3}(\omega^{-3Mb} E),
\tag{3.16}
$$

we see that $W_{k,k+1,k+2}$ is always non-zero, thus confirming the independence of $\{y_k, y_{k+1}, y_{k+2}\}$.

The final step for this section is to show how to build new solutions to (3.1) using the y_ks. Supposing that k_1 and k_2 differ by an integer so that $e^{-2k_1\pi i} = e^{-2k_2\pi i} = e^{-2k\pi i}$, it can be checked that the function

$$
z_{k_1, k_2}(x, E, g) = y_{k_1} y_{k_2}' - y_{k_2} y_{k_1}',
\tag{3.17}
$$

which is actually a 2×2 Wronskian, provides a solution of

$$
\partial_x^3 z_{k_1, k_2} - g(g+2) \left(\frac{1}{x^2} \partial_x - \frac{1}{x^3} \right) z_{k_1, k_2} - e^{-2k\pi i}(x^{3M} - E) z_{k_1, k_2}.
\tag{3.18}
$$

Equation (3.18) is the adjoint of (3.9) and, if k is shifted by a half-integer, then z_{k_1, k_2} becomes a solution of the original equation (3.9)

$$
\partial_x^3 z_{k_1+1/2, k_2+1/2} - g(g+2) \left(\frac{1}{x^2} \partial_x - \frac{1}{x^3} \right) z_{k_1+1/2, k_2+1/2} + e^{-2k\pi i}(x^{3M} - E) z_{k_1+1/2, k_2+1/2}.
\tag{3.19}
$$

For $|k_1 - k_2| < 3$, the regions (3.12) for $k_1 = k_2 = k$ overlap, and it is possible to get an asymptotic for z_{k_1, k_2} from (3.11). In particular, for $k = 1, 2, 3$ we have

$$
z_{-k/2, k/2}(x, E, g) \sim 2i \sin\left(\frac{\pi k}{3} \right) x^{-M} e^{-2\cos(\pi k/3)(1/(M+1))x^{M+1}}, \quad x \to \infty.
\tag{3.20}
$$

Considering (3.18) and (3.20), for $k = 1$ it follows, by uniqueness of solutions, that

$$
z_{-1/2, 1/2}(x, E, g) = i\sqrt{3} y(x, E, g).
\tag{3.21}
$$

This result is limited to the $(k = 1)$-case, as for other values of the parameter it is not possible to retrieve enough information to uniquely pin the function down. On the other hand, (3.21) is useful to determine the *BAEs*, but this will be shown in §4, starting from the original problem

(2.29). Now that the analyticity properties of solutions of (3.1) have been studied, it is possible to return to (2.29) and apply to that, with the appropriate caution, all the results that have been found in this section.

4. Bethe ansatz equations

In this section, BAEs will be built starting from certain functional relations involving the coefficients $Q^\pm$ already introduced at the end of §2. In order to get there, it is necessary to apply the same analysis described in §3 to reproduce similar results for the original equation (2.29), which is repeated below for convenience

$$\partial_x^3 \psi - ((\partial_x \eta)^2 + 2\partial_x^2 \eta)\partial_x \psi + (p(x,E) - \partial_x \eta \partial_x^2 \eta - \partial_x^3 \eta)\psi = 0 \tag{4.1}$$

and

$$\partial_{\tilde{x}}^3 \bar\psi - ((\partial_x \eta)^2 + 2\partial_{\tilde{x}}^2 \eta)\partial_x \bar\psi + (p(\tilde{x},\tilde{E}) - \partial_x \eta \partial_{\tilde{x}}^2 \eta - \partial_{\tilde{x}}^3 \eta)\bar\psi = 0, \tag{4.2}$$

where

$$x = z e^{\theta/(M+1)}, \quad \tilde{x} = \bar{z} e^{-\theta/(M+1)}, \quad E = s^{3M} e^{3\theta M/(M+1)}, \quad \tilde{E} = s^{3M} e^{-3\theta M/(M+1)} \tag{4.3}$$

and

$$\psi \equiv \psi(x,\tilde{x},E,\tilde{E},g), \quad \bar\psi \equiv \bar\psi(x,\tilde{x},E,\tilde{E},g). \tag{4.4}$$

First of all, the rotated solutions, analogous to (3.7), have to be defined. In order to do this, the periodicity of $\eta(\rho,\phi)$ has to be exploited because it is the only information, besides the asymptotic behaviours, which is known. It turns out that the right-hand form is

$$\psi_k(x,\tilde{x},E,\tilde{E},g) = \omega^k \psi(\omega^{-k}x,\omega^k\tilde{x},\omega^{-3Mk}E,\omega^{3Mk}\tilde{E},g), \tag{4.5}$$

where

$$\omega = \exp\left(\frac{2\pi i}{3M+3}\right). \tag{4.6}$$

The function (4.5) solves

$$\partial_x^3 \psi_k - ((\partial_x \eta)^2 + 2\partial_x^2 \eta)\partial_x \psi_k + (e^{-2k\pi i}p(x,E) - \partial_x \eta \partial_x^2 \eta - \partial_x^3 \eta)\psi_k = 0, \tag{4.7}$$

where, analogous to (3.9), a $e^{-2k\pi i}$-term appears. In general, arbitrary integer or half-integer values for k will be considered, and the definition of Stokes sectors remains unchanged (see (3.10)). Following the discussion of §3, it is straightforward to find the $|x| \to \infty$ behaviour of the rotated solutions (4.5) (keeping in mind that $\tilde{x}$ goes to $\omega^k \tilde{x}$) and setting

$$c = e^{\theta/(M+1)} \tag{4.8}$$

and

$$\left. \begin{aligned} \psi_k &\sim \omega^{k(M+1)}x^{-M} \exp\left[-\frac{x^{M+1}}{M+1}\omega^{-k(M+1)} - \frac{\tilde{x}^{M+1}}{M+1}\omega^{k(M+1)}\right], \\ \psi_k' &\sim -\exp\left[-\frac{x^{M+1}}{M+1}\omega^{-k(M+1)} - \frac{\tilde{x}^{M+1}}{M+1}\omega^{k(M+1)}\right] \\ \psi_k'' &\sim \omega^{-k(M+1)}x^M \exp\left[-\frac{x^{M+1}}{M+1}\omega^{-k(M+1)} - \frac{\tilde{x}^{M+1}}{M+1}\omega^{k(M+1)}\right] \end{aligned} \right\} \tag{4.9}$$

(the derivatives are only on x).

Now, the three generic solutions $\{\psi_k, \psi_{k+1}, \psi_{k+2}\}$ form a basis for the solutions of equation (2.29). Again, the proof of this property comes from the evaluation of the Wronskian for $(k_1,k_2,k_3) = (-1,0,1)$

$$W_{-1,0,1} = W[\psi_{-1},\psi_0,\psi_1] = -3i\sqrt{3}, \tag{4.10}$$

which shows that these three solutions are independent.

The next step is to define the analogues of the functions (3.17) that, here, will be denoted as u_{k_1,k_2} so as to avoid confusion with the variable z. They read as

$$u_{k_1,k_2}(x,\tilde{x},E,\tilde{E},g) = [\psi_{k_1}\psi'_{k_2} - \psi_{k_2}\psi'_{k_1}](x,\tilde{x},E,\tilde{E},g), \tag{4.11}$$

where, again, the $\bar{z}$-dependence has been omitted. The solutions (4.11) solve

$$\partial_x^3 u_{k_1,k_2} - ((\partial_x\eta)^2 + 2\partial_x^2\eta)\partial_x u_{k_1,k_2} - (e^{-2\pi ik}p(x,E) + \partial_x\eta\partial_x^2\eta + \partial_x^3\eta)u_{k_1,k_2} = 0, \tag{4.12}$$

and, also in this case, it is possible to evaluate (4.11) for $k_1 = -k_2 = -k/2$

$$u_{-k/2,k/2} \sim 2i\sin\left(\frac{\pi}{3}k\right)x^{-M}\exp\left[-2\cos\left(\frac{\pi}{3}k\right)\left(\frac{x^{M+1}}{M+1} + \frac{\tilde{x}^{M+1}}{M+1}\right)\right], \tag{4.13}$$

which, when compared with (4.9) for $k=1$, gives the identification

$$u_{-1/2,1/2} = i\sqrt{3}\psi. \tag{4.14}$$

To get the BAEs, equation (4.14) is used, exploiting the relation

$$u_{-1/2,1/2} = \psi_{-1/2}\psi'_{1/2} - \psi_{1/2}\psi'_{-1/2} = i\sqrt{3}\psi_0. \tag{4.15}$$

At this point, the solutions ψ_k can be expressed on the basis of the $|z| \to 0$ solutions (2.31) with the appropriate normalization to capture the correct behaviour in the conformal limit

$$\psi_k = Q_k^+(E,\tilde{E})\chi_k^+ + Q_k^0(E,\tilde{E})\chi_k^0 + Q_k^-(E,\tilde{E})\chi_k^+, \tag{4.16}$$

where

$$Q_k^{\pm,0}(E,\tilde{E}) = Q^{\pm,0}(\omega^{-k3M}E, \omega^{k3M}\tilde{E}) \tag{4.17}$$

and

$$\left.\begin{array}{l} \chi_k^+ \sim \omega^{k(g+1)}x^{-g}, \\ \chi_k^0 \sim x \\ \chi_k^- \sim \omega^{-k(g+1)}x^{g+2}. \end{array}\right\} \tag{4.18}$$

The functions $Q^{\pm,0}(E,\tilde{E})$, introduced in (4.16), are the massive off-critical analogue of the CFT-related functions $Q^{\pm,0}(E)$ [10,15]. Inserting the expression (4.16) into (4.15) and considering, at first, the terms proportional to x^{-g}, a functional relation containing Q^+ and Q^0 is obtained

$$i\sqrt{3}Q_0^+ = Q_{-1/2}^+ Q_{1/2}^0 \omega^{-(g+1)/2} - gQ_{1/2}^+ Q_{-1/2}^0 \omega^{(g+1)/2}$$
$$- Q_{1/2}^+ Q_{-1/2}^0 \omega^{(g+1)/2} + gQ_{-1/2}^+ Q_{1/2}^0 \omega^{-(g+1)/2}$$
$$= (g+1)(Q_{-1/2}^+ Q_{1/2}^0 \omega^{-(g+1)/2} - Q_{1/2}^+ Q_{-1/2}^0 \omega^{(g+1)/2}) \tag{4.19}$$

with the appropriate ω rotation factors. Equation (4.19) can be written in terms of the variable $\theta = (M+1)/(3M)\ln E$ as

$$i\sqrt{3}Q^+(\theta) = (g+1)\left(Q^+\left(\theta + i\frac{\pi}{3}\right)Q^0\left(\theta - i\frac{\pi}{3}\right)\omega^{-(g+1)/2}\right.$$
$$\left. - \omega^{(g+1)/2}Q^+\left(\theta - i\frac{\pi}{3}\right)Q^0\left(\theta + i\frac{\pi}{3}\right)\right) \tag{4.20}$$

(with s and g kept constant).

Table 1. The dictionary.

classical modified Bullough–Dodd		quantum field theory on a cylinder
θ	$\Leftrightarrow$	particle rapidity
s^{M+1}	$\Leftrightarrow$	Rm (R, circumference; m, mass gap)
g	$\Leftrightarrow$	twist parameter
M	$\Leftrightarrow$	'q' in the q-state Potts model

Considering (4.20) evaluated at $\theta = \theta_n + i\pi/3$ and $\theta = \theta_n - i\pi/3$ with $Q^+(\theta_n) = 0$ to get

$$i\sqrt{3}Q^+\left(\theta_n + \frac{i\pi}{3}\right) = (g+1)\left(Q^+\left(\theta_n + \frac{i2\pi}{3}\right)Q^0(\theta_n)\omega^{-(g+1)/2}\right) \tag{4.21}$$

and

$$i\sqrt{3}Q^+\left(\theta_n - \frac{i\pi}{3}\right) = -(g+1)\left(\omega^{(g+1)/2}Q^+\left(\theta_n - i\frac{2\pi}{3}\right)Q^0(\theta_n)\right), \tag{4.22}$$

respectively, and taking the ratio, the sought-after BAEs are obtained [13]

$$\frac{Q^+(\theta_n + i2\pi/3)}{Q^+(\theta_n - i2\pi/3)}\frac{Q^+(\theta_n - i\pi/3)}{Q^+(\theta_n + i\pi/3)} = -\omega^{(g+1)}. \tag{4.23}$$

The same method can be used to find BAE for the Q^- functions. In conclusion, by defining $\theta^{\pm}$ the zeroes relative to $Q^{\pm}$, the BAEs for the two spectral functions are

$$\frac{Q^{\pm}(\theta_n^{\pm} + i2\pi/3)}{Q^{\pm}(\theta_n^{\pm} - i2\pi/3)}\frac{Q^{\pm}(\theta_n^{\pm} - i\pi/3)}{Q^{\pm}(\theta_n^{\pm} + i\pi/3)} = -\omega^{\pm(g+1)}. \tag{4.24}$$

The BAEs (4.24) can be transformed into the nonlinear integral equation given in §6 of Dorey & Tateo [13] for the ground-state energy of the Izergin–Korepin model or, equivalently, of the scaling q-state Potts and tricritical Potts models on a cylinder geometry. This family of systems plays a key role in the study of integrable QFTs in 1+1 dimensions and statistical mechanical models in two dimensions, and is related to the $\phi_{1,2}$, $\phi_{2,1}$ and $\phi_{1,5}$ integrable deformations of minimal CFTs. The Ising model in external magnetic field, with its E_8-related mass spectrum [39], is perhaps the most physically interesting system in the family. A schematic of the correspondence between the classical objects introduced here and the 'quantum field theory world' is given in table 1.

5. Conclusions

The main objective of this paper was to discuss a particular generalization of the so-called ODE/IM correspondence to the massive case. The steps taken by Lukyanov & Zamolodchikov [28] clarified how massive generalizations of the ODE/IM correspondence should be built. The results of Lukyanov & Zamolodchikov [28] concern the relation between the classical and the quantum versions of the sinh-Gordon model. Adapting their discussion, it has been possible, here, to relate the classical (Tzitzéica) Bullough–Dodd field equation to the Izergin–Korepin massive QFT. The discussion parallels precisely the conformal case treated in Dorey & Tateo [13] and links a certain linear problem to a Bethe ansatz system associated with the a_2 algebra. The BAEs (4.24) and the demonstration of a link with the CFT limit case of Dorey & Tateo [13], described in §3, are the main results of the paper.

After the first discovery of this type of correspondence, it was striking how certain functional relations, typically emerging in the QFT domain, encoded spectral data. In the work of Lukyanov and Zamolodchikov, this rich structure was also enlarged to contain the relation between certain nonlinear partial differential equations and linear spectral problems, thereby giving further useful insights into the general structure of the theory. Here, further support of the validity of this scheme has been given, enlarging the number of working cases of the correspondence. The BD

equation has been chosen because, with the sinh-Gordon equation, it is the simplest representative of the affine Toda field theories. At this stage, it is fairly clear how a more general correspondence scheme could be developed starting from Toda field theories based on more general Lie algebras. The general scheme is as follows, where the arrows have been annotated with labels of sections in this paper and a reference, to indicate where the corresponding step is further discussed for the particular case of the BD equation:
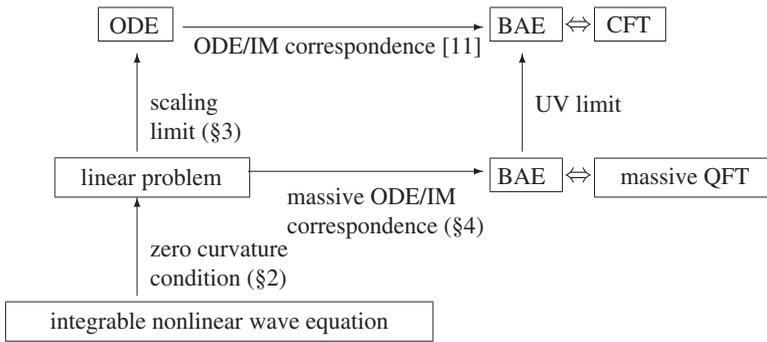
Another direction for future work is the generalization of the correspondence between the Bethe ansatz and classical integrable systems to non-relativistically invariant models such as the KdV equation and its hierarchy, and the generalization from integrable field theories to integrable lattice models.

In conclusion, the connection between these two, originally disconnected, domains of mathematics and theoretical physics gives a hint of a bigger scheme in the wide framework of classical and quantum integrability.

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