$T\overline{T}$-deformation of q-Yang-Mills theory

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Abstract

We derive the $T\overline{T}$-perturbed version of two-dimensional q-deformed Yang-Mills theory on an arbitrary Riemann surface by coupling the unperturbed theory in the first order formalism to Jackiw-Teitelboim gravity. We show that the $T\overline{T}$-deformation results in a breakdown of the connection with a Chern-Simons theory on a Seifert manifold, and of the large N factorization into chiral and anti-chiral sectors. For the $U(N)$ gauge theory on the sphere, we show that the large N phase transition persists, and that it is of third order and induced by instantons. The effect of the $T\overline{T}$-deformation is to decrease the critical value of the 't Hooft coupling, and also to extend the class of line bundles for which the phase transition occurs. The same results are shown to hold for (q,t)-deformed Yang-Mills theory. We also explicitly evaluate the entanglement entropy in the large N limit of Yang-Mills theory, showing that the $T\overline{T}$-deformation decreases the contribution of the Boltzmann entropy.

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1 Introduction

Two-dimensional quantum field theories provide a playground for the study of exactly solvable models, and for testing the relationships and dualities with other areas such as integrable systems, statistical mechanics and string theory. Recent broad interest has been attracted to the solvable irrelevant deformation by the $T\overline{T}$ operator, which is present in any local relativistic two-dimensional quantum field theory, that was identified by [1, 2] (see [3] for a review). The novelty of this deformation is marked by two key properties. First, it does not alter the integrability of a system. Second, when the field theory is compactified on a circle, the evolution of the energy levels with the parameter τ, encoding the strength of the deformation, is described by a first order inhomogeneous differential equation of Burgers type. The deformation induced by the operator $T\overline{T}$, henceforth called the '$T\overline{T}$-deformation', was interpreted in [4] as a random fluctuation of the background geometry. A further step in this direction was made in [5, 6], where a path integral formulation of the $T\overline{T}$-deformed theory was put forward: it was proven in [5, 6] that the deformation of a given quantum field theory by the $T\overline{T}$ operator is equivalent to coupling the undeformed theory to flat space Jackiw-Teitelboim (JT) gravity. These ideas are similar in spirit to the interpretation of the $T\overline{T}$-deformation as a field-dependent spacetime coordinate transformation [7].

After the original formalism for $T\overline{T}$-deformed field theories considered by [1, 2], a wide range of other aspects of the $T\overline{T}$-deformation have been investigated. The original flat space deformation was extended to two-dimensional quantum field theories on AdS_2 in [8]. Their role in $\mathrm{AdS}_3/\mathrm{CFT}_2$ holography was investigated in [9, 10, 11, 12, 13, 14, 15, 16, 17]. The $T\overline{T}$-deformation of Wess-Zumino-Witten (WZW) models was studied from the string theory perspective in the target space theory [18] and also in the gauged worldsheet sigma-model [19]. Supersymmetric extensions were considered by [20, 21, 22, 23, 24, 25]. Generalized $T\overline{T}$-deformations were discussed in [26, 27, 28], while the extensions to higher-dimensional field theories is considered by [29] using holography, and more recently in [30] from direct analysis of the renormalization group flow equation. Other facets of the perturbation by the $T\overline{T}$ operator considered recently include the study of the modular properties of the partition functions of deformed theories [31, 32], the extension of the $T\overline{T}$-deformation to non-relativistic systems [33], and correlation functions in conformal field theories on curved manifolds [34, 35]. A bridge between the Polyakov loop and the $T\overline{T}$-deformation of a bosonic field theory has been established in [36].

In this paper we are concerned with the $T\overline{T}$-deformations of two-dimensional gauge theories. A simple proposal for the $T\overline{T}$-deformation of Yang-Mills theory on a Riemann surface was advocated by [37]: due to the simple form of the evolution of the two-dimensional Yang-Mills Hamiltonian with the deformation parameter τ, the $T\overline{T}$-deformed version of the theory simply amounts to replacing the quadratic Casimir that appears in the usual heat kernel expansion of the partition function according to

$$C_2 \longmapsto \frac{C_2}{1-\tau\,\frac{g_{\mathrm{YM}}^2}{2}\,C_2} \ , \tag{1.1}$$

where g_{YM} is the Yang-Mills coupling constant. Following this proposal, the phase structure of the $U(N)$ theory on the sphere was studied at large N in [38], using standard field theory techniques. The prescription (1.1) was derived by [39] by directly coupling the heat kernel expansion, which depends on the area of the Riemann surface, to JT gravity and performing the gravitational path integral.

The aim of the present paper is to study the analogous features of the q-deformation of two-dimensional Yang-Mills theory. To justify the effect of the $T\overline{T}$-deformation given by (1.1), we follow a different route than [39]. We consider the first order formalism for two-dimensional Yang-Mills theory, which rewrites it as a deformation of BF theory, and hence as an example of an 'almost' topological gauge theory, in the sense which we make precise in Section 2. In this latter general class of theories we can study the effect of the $T\overline{T}$-deformation precisely through its coupling to JT gravity, and we reproduce the prescription of [37] as a corollary of a more general result, using standard abelianisation techniques to evaluate the path integral. At the same time, thanks to the almost topological nature of these two-dimensional theories, we can employ cutting and gluing techniques of topological quantum field theory to rigorously justify the extension of the $T\overline{T}$-deformation, which is only well-defined on flat space, to curved Riemann surfaces such as the sphere, which was not addressed by [37, 39]. In a certain sense, the two deformations are compatible: the q-deformation results from a modification of the path integral measure, leaving the quadratic Casimir unchanged, while the $T\overline{T}$-deformation modifies only the Hamiltonian, i.e. the Casimir, but nothing else.

With our techniques, we are able to explore how various facets of two-dimensional Yang-Mills theory are affected by the $T\overline{T}$-deformation. For example, we obtain closed expressions for Wilson loop observables as well as the partition functions on Riemann surfaces with marked points.

However, a number of noteworthy features are lost under the deformation. For example, the $T\overline{T}$-deformation of q-deformed Yang-Mills theory is no longer related to Chern-Simons theory (or a deformation thereof) on a circle bundle over the Riemann surface. Moreover, the large N factorization property, which splits the $U(N)$ gauge theory into chiral and anti-chiral sectors, no longer holds after deformation. This splitting is a crucial ingredient in the derivation of the large N string theory dual of two-dimensional Yang-Mills theory, thus casting doubt on the existence of such a string theory description of the $T\overline{T}$-deformed theory. This can be physically understood by mapping the $T\overline{T}$-deformed gauge theory onto a system of N non-relativistic fermions on a circle, which are now subjected to non-local interactions leading to long-range correlations between the fermions.

A central point of this paper is a generalization of the analysis of [38] to the large N limit of the $T\overline{T}$-deformation of q-deformed $U(N)$ Yang-Mills theory on the sphere. We show that the main features of the undeformed theory are preserved, namely there is a third order phase transition induced by instantons. Furthermore, the $T\overline{T}$-deformation has the same general features as in the case of ordinary Yang-Mills theory, and in particular the critical line is lowered as the strength τ of the deformation is increased. On the other hand, it extends the class of line bundles for which the phase transition occurs. We also show that these results continue to hold in the refinement of the theory, known as (q,t)-deformed Yang-Mills theory, whereby the region of the small coupling phase is reduced by the refinement.

The exact solvability of two-dimensional Yang-Mills theory also makes it an interesting testing ground for exploring concepts of quantum information, such as quantum entanglement, in quantum field theory. Generally, the entanglement entropy of codimension one spatial subregions provides a powerful tool for investigating new perspectives in quantum field theory, but it is notoriously difficult to compute. Most examples involve only free fields or explicit conformal symmetry, whereas the highly interesting examples involve the renormalization group (RG) flow of the entanglement entropy. For the RG flow triggered by the $T\overline{T}$-deformation, the entanglement entropy was computed in [39] for finite N, where it was found that the effect of the deformation is relatively mild, which is anticipated from the ultraviolet finiteness of two-dimensional Yang-Mills theory. In this paper we study the entanglement entropy of the $T\overline{T}$-deformed gauge theory at large N, where we find that the contribution from the Shannon entropy vanishes, while the contribution from the Boltzmann entropy, per point of the entangling surface, is explicitly evaluated and shown to decrease as the strength τ of the deformation is increased.

Organization of the paper. In Section 2 we present our formalism for generic almost topological gauge theories. In Section 3 we then focus on Yang-Mills theory in two dimensions together with its q-deformation and subsequent refinement which depend, among other continuous moduli, on a discrete parameter $p \in \mathbb{Z}$; we present their $T\overline{T}$-deformation and study how their well-known properties are changed by the deformation. Section 4 is dedicated to the study of the phase structure at large N, where we find that the expected phase transition extends to $p < 2$ as a consequence of the $T\overline{T}$-deformation. In Section 5 we present some results for the entanglement entropy of these theories. We conclude with possible avenues for future research in Section 6. Two appendices at the end of the paper contain some technical details that supplement the analyses of the main text.

Conventions. To avoid excessive repetition of the word 'deformation', we will only explicitly state it when using the terminology '$T\overline{T}$-deformation'. The q-deformed Yang-Mills theory and its refinement, (q,t)-deformed Yang-Mills theory, will be henceforth simply called 'q-Yang-Mills theory' and '(q,t)-Yang-Mills theory', respectively.

2 $T\overline{T}$-deformation of almost topological gauge theories

Consider the partition function of a gauge theory $\mathscr{T}$ with compact connected gauge group G on a Riemann surface Σ which is described by the insertion of a non-local operator $\mathcal{O}(\Phi)$ in the path integral of a two-dimensional topological quantum field theory:

$$\mathcal{Z}_{\mathscr{T}}[\Sigma] = \int \mathscr{D}\Phi \ \mathrm{e}^{-S_{\mathrm{TQFT}}(\Phi)} \, \mathcal{O}(\Phi) =: \big\langle \mathcal{O}(\Phi) \big\rangle_{\mathrm{TQFT}} \ .$$

Here Φ collectively denotes the fields of the theory and $\mathscr{D}\Phi$ is a gauge-invariant measure on the space of fields, while the action $S_{\mathrm{TQFT}}(\Phi)$ defines a topological field theory. Notable examples of such theories are two-dimensional Yang-Mills theory and its relatives, which arise from a BF-type topological gauge theory through a deformation that is precisely of this type, as we will review in Section 3. These theories will be the focus of subsequent sections. Nevertheless, one may also consider deformations of the topologically twisted sigma-models of [40, 41] or of other classes of two-dimensional topological field theories [42] by some non-local operator, and our considerations in this section also pertain to these more general gauge theories.

The spacetime, on which our field theory is defined, is a Riemann surface Σ, possibly with s marked points decorated with representations $R_1, \dots, R_s$ of the gauge group G, in which case the partition function is denoted by

$$\mathcal{Z}_{\mathscr{T}}[\Sigma; R_1, \dots, R_s] \ .$$

The surface Σ is allowed to have boundaries, and the partition function will be understood as a function of suitable boundary conditions, which in particular include the holonomies of the gauge connection along the one-dimensional boundaries. Field theories without gauge symmetries can be considered as well in this framework as a special case with trivial gauge group.

Theories of this class are amenable to the $T\overline{T}$-deformation, albeit defined on a curved spacetime Σ, thanks to their "almost" topological nature. According to [5, 6] (see also [43]), $T\overline{T}$-deformation is equivalent to coupling the field theory to two-dimensional topological gravity. If the theory we start with is topological, the gravitational sector of the path integral can be integrated out with no effect. However, if non-local operators have been inserted, they couple to the gravitational sector and the $T\overline{T}$-deformation is represented symbolically as

$$\mathcal{Z}_{\mathscr{T}}[\Sigma] \xrightarrow{\ T\overline{T}\text{-deformation}\ } \mathcal{Z}^{T\overline{T}}_{\mathscr{T}}[\Sigma]$$

with

$$\mathcal{Z}^{T\overline{T}}_{\mathscr{T}}[\Sigma] = \left\langle \frac{1}{\mathcal{Z}_{\mathrm{JT}}} \int \mathscr{D}\boldsymbol{e} \ \mathcal{O}(\Phi;\boldsymbol{e}) \ \exp\left(\tfrac{1}{2\tau} \textstyle\int_\Sigma (\boldsymbol{e}-\boldsymbol{f}) \wedge (\boldsymbol{e}-\boldsymbol{f})\right) \right\rangle_{\mathrm{TQFT}} \ .$$

This is the path integral of JT gravity, normalized by the pure gravity partition function $\mathcal{Z}_{\mathrm{JT}}$. The integration is over the coframe field $\boldsymbol{e}$ of the target space, with $\boldsymbol{f}$ the coframe field of the worldsheet Σ, and the path integral measure is induced by the metric

$$\delta s^2 = \int_\Sigma \delta\boldsymbol{e} \wedge \delta\boldsymbol{e} \ .$$

The notation $\mathcal{O}(\Phi;\boldsymbol{e})$ means that the non-local operator has the same form as before, but now lives in the manifold with coframe field $\boldsymbol{e}$. For derivative-free operators $\mathcal{O}(\Phi)$ this simply means that, in every integral, we have to replace the original volume form ω on Σ, written in terms of the coframe field of Σ as $\boldsymbol{f} \wedge \boldsymbol{f}$, by the target space volume form $\boldsymbol{e} \wedge \boldsymbol{e}$. This presentation is equivalent to the change of variables described in [7], but for the purposes of the present work we find it convenient to use the explicit path integral presentation.

The proof of equivalence with the $T\overline{T}$-deformation presented in [6] relies on showing that the gravitational path integral is one-loop exact, and reproduces the $T\overline{T}$-deformed partition function. This ties in nicely with the arguments of [5] that the gravitational dressing provided by the $T\overline{T}$-deformation is a semi-classical effect. We shall see this explicitly for the class of non-local operators that we ultimately consider below.

We always consider the surface Σ to be equipped with a Riemannian metric, and therefore use the Euclidean gravity path integral, following the conventions of [43] (which agree with those of [37, 7]). The deformation parameter τ in the present paper then differs by a sign from the conventions of [6, 44] which work in Lorentzian signature.

It was noted in [39] (see also [44, Appendix A] for relevant discussion) that there is a subtlety in the normalization of the path integral measure $\mathscr{D}\boldsymbol{e}$: as will be manifest below, assuming the naive normalization of the measure and performing the gravity path integral, one does not recover the undeformed theory in the limit $\tau \to 0$. Imposing the latter condition instead leads to a choice of normalization for the path integral measure $\mathscr{D}\boldsymbol{e}$ which depends on $\mathcal{O}(\Phi)$. In particular, the order of the path integrations do not commute, and the topological gravity degrees of freedom should always be introduced inside the correlator $\langle\,\cdot\,\rangle_{\text{TQFT}}$. Below we will provide a more extensive comparison between the present analysis and that of [39], and this technical aspect will play an important role.

With the application to two-dimensional Yang-Mills theory along with its generalizations and deformations in mind, we now specialize our analysis to the case in which the functional dependence of $\mathcal{O}(\Phi)$ is through operators of the form

$$\mathcal{O}(\Phi) = \exp\left(-\frac{\lambda}{2N}\int_\Sigma V(\phi,\psi)\,\omega\right) ,$$

where ω is the normalized volume form on Σ and the potential $V(\phi,\psi)$ is a scalar functional of scalar fields ϕ and possibly spinor fields ψ. The coupling is $\frac{\lambda}{N}$, where λ is a 't Hooft parameter and N is the rank of the gauge group G. When coupled to topological gravity, this operator becomes

$$\mathcal{O}(\Phi;\boldsymbol{e}) = \exp\left(-\frac{\lambda}{2N}\int_\Sigma V(\phi,\psi)\;\boldsymbol{e}\wedge\boldsymbol{e}\right) ,$$

and the $T\overline{T}$-deformed partition function reads

$$\mathcal{Z}_{\mathscr{T}}^{T\overline{T}}[\Sigma] = \int \mathscr{D}\Phi\,\frac{1}{\mathcal{Z}_{\text{JT}}}\int \mathscr{D}\boldsymbol{e}\,\exp\Big(-S_{\text{TQFT}}(\Phi) - \lambda\textstyle\int_\Sigma\big(\frac{1}{2N}\,V(\phi,\psi)\;\boldsymbol{e}\wedge\boldsymbol{e} - \frac{1}{2\tau}\,(\boldsymbol{e}-\boldsymbol{f})\wedge(\boldsymbol{e}-\boldsymbol{f})\big)\Big) .$$

We have chosen a non-standard definition of the parameter τ, including an overall factor λ, which will be convenient in the forthcoming discussion. After simple manipulation and integrating over $\boldsymbol{e}' = \boldsymbol{e}-\boldsymbol{f}$, one obtains

$$\begin{aligned}\mathcal{Z}_{\mathscr{T}}^{T\overline{T}}[\Sigma] &= \int \mathscr{D}\Phi\;\exp\Big(-S_{\text{TQFT}}(\Phi) - \tfrac{\lambda}{2}\textstyle\int_\Sigma N^{-1}\,V(\phi,\psi)\;\boldsymbol{f}\wedge\boldsymbol{f} \\ &\qquad\qquad + \tfrac{\lambda}{2}\textstyle\int_\Sigma \frac{\tau\,N^{-1}}{1-\tau\,N^{-1}\,V(\phi,\psi)}\big(N^{-1}\,V(\phi,\psi)\,\boldsymbol{f}\big)\wedge\big(N^{-1}\,V(\phi,\psi)\,\boldsymbol{f}\big)\Big) \\ &= \int \mathscr{D}\Phi\;\exp\Big(-S_{\text{TQFT}}(\Phi) - \tfrac{\lambda}{2N}\textstyle\int_\Sigma \frac{V(\phi,\psi)}{1-\frac{\tau}{N}\,V(\phi,\psi)}\,\omega\Big) ,\end{aligned} \tag{2.1}$$

which correctly reproduces the prescription of [37] for the $T\overline{T}$-deformation given by

$$\frac{\lambda}{2N}\,V(\phi,\psi)\longmapsto\frac{\frac{\lambda}{2N}\,V(\phi,\psi)}{1-\frac{\tau}{N}\,V(\phi,\psi)} . \tag{2.2}$$

In (2.1) we use a $V(\phi,\psi)$-dependent normalization of the gravity path integral that cancels a factor from the Gaussian integration. Had we not done so, we would not recover the undeformed partition function in the limit $\tau \to 0$. This is the incarnation of the subtleties in the normalization of the path integral measure discussed in [39, 44].

$T\overline{T}$-deformation in curved spacetime

The $T\overline{T}$-deformation is only well-defined in flat space. However, in the present setting, we argue that, since the underlying theory is topological, for suitable insertions $\mathcal{O}(\Phi)$ we can define the $T\overline{T}$-deformation on flat space, and then put the theory on a curved manifold Σ. We now explain this point more rigorously.

Thanks to the cutting and gluing property of topological quantum field theories [45, 46], one can decompose Σ into disks, cylinders and pairs of pants, obtaining the same theory on each piece (see Figure 1). Such components have boundaries, and one should impose suitable boundary conditions on the fields. The disk, the cylinder and the pair of pants are homeomorphic respectively to the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}^{\times}$ and the doubly-punctured plane $\mathbb{C}^{\times\times}$. Therefore, we reduce the topological quantum field theory on flat components, which are many copies of the complex plane $\mathbb{C}$, with either zero, one or two holes. At this point, we insert $\mathcal{O}(\Phi)$ on each component, and turn on gravity. Since each component is flat, the $T\overline{T}$-deformation prescription is well-defined on each component.

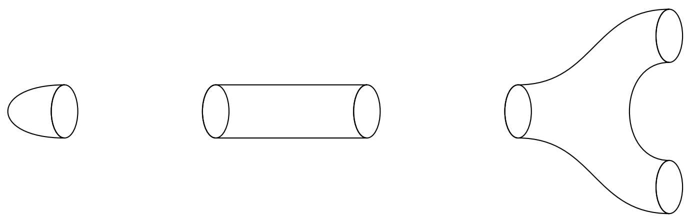


Figure 1: The disk (left), the cylinder (center) and the pair of pants (right), homeomorphic to the complex plane with respectively zero, one or two holes.

After performing the JT gravity path integral, we can glue back together the pieces and reassemble Σ (see Figure 2). Topological gravity couples to bulk geometry and does not change the boundary data, at least for operator insertions $\mathcal{O}(\Phi)$ of the form in (2.1) (we will briefly comment on the most general case shortly). Hence the gluing goes exactly as without $T\overline{T}$-deformation, and we obtain a $T\overline{T}$-deformed theory on Σ.

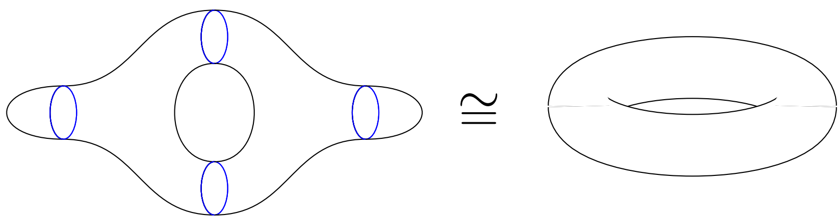


Figure 2: Obtaining a torus from elementary pieces. On the left, the gluing is an integration over boundary conditions (in blue).

At this point it is worthwhile mentioning the proposal [44] that $T\overline{T}$-deformation in curved spacetime corresponds to massive gravity. In the present setting, we could equivalently take the (Euclidean version of the) proposal of [44] as the definition of $T\overline{T}$-deformation on curved two-dimensional manifolds. A step towards a rigorous definition of generic $T\overline{T}$-deformed theories on curved manifolds has also been taken in [47].

Boundary quantum mechanics

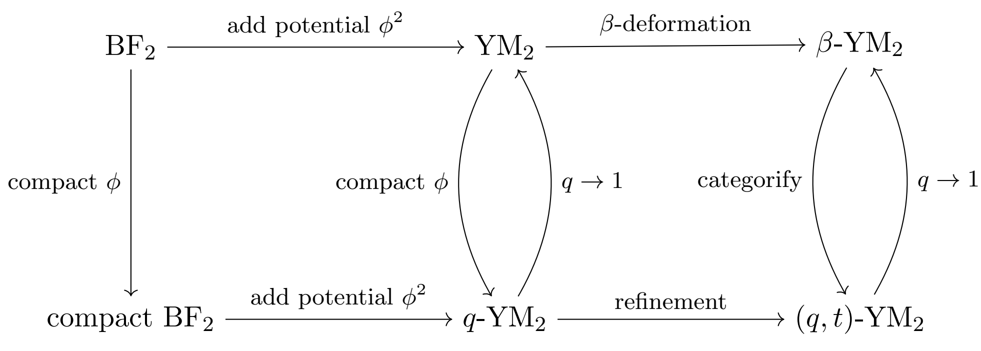
In the procedure of putting the theory on Σ, we exploited the underlying topological quantum field theory and a suitable form of the insertion $\mathcal{O}(\Phi)$. However, a generic operator $\mathcal{O}(\Phi)$ may in principle couple dynamically to the boundary conditions, spoiling the gluing technique. We find it appropriate to say a few more words about this more general scenario, although it will play no role in the rest of our discussion.

First, we notice that there will never be any one-dimensional interfaces, since we insert the same operator on each two-dimensional component. In the general setting, one may need to deal with a dynamical one-dimensional boundary theory. Thinking of the $T\overline{T}$-deformation as a field-dependent change of variables [7], we expect the $T\overline{T}$-deformed boundary theory to be a rewriting of the original theory in the new variables, at least at the classical level. We notice that no interface will arise in any case, since we are putting the same operator $\mathcal{O}(\Phi)$ in each component. Therefore we should be able, at least in principle, to glue the pieces together, at the price of solving the gluing theory [48], which will be a quantum mechanics deformed by the effect of the change of variables on the boundary modes.

3 Two-dimensional Yang-Mills and q-Yang-Mills theories

In this paper we study two-dimensional Yang-Mills theory, its q-deformation and its subsequent refinement to two-dimensional (q,t)-deformed Yang-Mills theory. These are examples of almost topological gauge theories of the type discussed in Section 2, where the underlying topological field theory is two-dimensional BF theory,[1] whose fields are a scalar field ϕ on Σ in the adjoint representation of the gauge algebra $\mathfrak{g}$ and the curvature F^A of a gauge connection A on (a trivial principal G-bundle over) Σ. Ordinary Yang-Mills theory on Σ corresponds to a deformation of this BF theory by a non-local operator $\mathcal{O}(\phi)$ which adds a potential $V(\phi) = \operatorname{Tr}\phi^2$ to the BF action. This theory can be β-deformed by modifying the discrete matrix model which arises for $\beta = 2$ to a general β-ensemble. One can further deform the underlying BF theory by making the field ϕ compact, that is, taking it to be valued in the adjoint representation of the gauge group G. Adding the potential $V(\phi)$ deforms this theory to q-Yang-Mills theory which can be subsequently refined to (q,t)-Yang-Mills theory, that is a categorification of the β-ensemble. The initial theories, with non-compact ϕ, can then be regarded as classical limits $q \to 1$ of the theories with compact scalar ϕ.

We depict these relationships between the various incarnations of Yang-Mills theory on Σ through the diagram



In this section we use the formalism developed in Section 2 to study the $T\overline{T}$-deformation of the Yang-Mills theories appearing in this diagram.

[1]This theory is sometimes also refered to as two-dimensional topological Yang-Mills theory.

3.1 $T\overline{T}$-deformation of two-dimensional Yang-Mills theory

In [37] the $T\overline{T}$-deformation of two-dimensional Yang-Mills theory on Σ was obtained, through explicit solution of the flow equation

$$\frac{\partial\mathcal{L}(\tau)}{\partial\tau} = \det_{\mu,\nu=1,2}\left[T_{\mu\nu}(\tau)\right] ,$$

to all orders in $\tau\in[0,\infty)$. This equation is to be solved with the initial condition on the deformed Lagrangian $\mathcal{L}(\tau)$ that requires $\mathcal{L}(0)=\mathcal{L}_{\rm YM}$ to be the Yang-Mills Lagrangian, and $T_{\mu\nu}(\tau)$ is the Hilbert energy-momentum tensor of the two-dimensional field theory. The same deformation has recently been obtained in [39] through the coupling with JT gravity. We rederive the result by exploiting the equivalent first order formulation as a deformation of BF theory. The argument is as follows: Yang-Mills theory is a pure gauge theory, but it is equivalent to a BF theory with additional Gaussian term for the scalar $\phi\in\Omega^0(\Sigma,\mathfrak{g})$ given by (see [49])

$$S_{\rm YM} = \frac{N}{2\lambda}\int_\Sigma \operatorname{Tr} F^A * F^A = \int_\Sigma \operatorname{Tr}\Big(\mathrm{i}\,\phi\,F^A + \frac{\lambda}{2N}\,\phi^2\,\omega\Big) ,$$

where the equality is understood to hold on-shell. Here $F^A\in\Omega^2(\Sigma,\mathfrak{g})$ is the curvature of a gauge connection A on Σ, Tr is an invariant quadratic form on the Lie algebra $\mathfrak{g}$, ω is the symplectic structure on Σ and $*$ is the Hodge operator constructed from the Riemannian metric compatible with ω. The Yang-Mills coupling is

$$g_{\rm YM}^2 = \lambda\,N^{-1} .$$

The first term is the action of two-dimensional BF theory which is topological, thus the $T\overline{T}$-deformation of the first order formulation only changes the potential $V(\phi)=\operatorname{Tr}\phi^2$ as in (2.2). From this point, the derivation of the heat kernel expansion using abelianization of the path integrals goes exactly as in [49]: one conjugates the scalar field ϕ into a Cartan subalgebra of $\mathfrak{g}$ using gauge invariance and the Weyl integral formula, and then integrates over the root components A_α of the gauge connections with respect to the root space decomposition of the Lie algebra $\mathfrak{g}$. Then two-dimensional Yang-Mills theory can be $T\overline{T}$-deformed by replacing the quadratic Casimirs of representations R of G according to[2]

$$C_2(R)\longmapsto C_2^{T\overline{T}}(R,\tau) := \frac{C_2(R)}{1-\frac{\tau}{N^3}\,C_2(R)} . \tag{3.1}$$

This derivation immediately extends to the generalized Yang-Mills theory of [50], where higher order Casimir operators are included by adding higher degree terms to the potential $V(\phi)$. These can include multi-trace terms, since the derivation does not rely on the explicit form of $V(\phi)$. The $T\overline{T}$-deformation of the generalized two-dimensional Yang-Mills theory is then directly obtained from (2.2). The final answer for the partition function of generalized Yang-Mills theory is then

$$\mathcal{Z}^{T\overline{T}}_{\text{gen-YM}}[\Sigma] = \sum_R \dim(R)^{\chi(\Sigma)}\,\exp\Big(-\frac{\lambda}{2N}\,\frac{C_{\rm gen}(R)}{1-\frac{\tau}{N^3}\,C_{\rm gen}(R)}\Big) , \tag{3.2}$$

where $C_{\rm gen}(R)$ includes the quadratic and higher Casimir operators. The sum runs over isomorphism classes of irreducible representations R of G [51] with dimension $\dim(R)$, the coupling λ is identified with the area A of the surface Σ, and $\chi(\Sigma)$ is the Euler characteristic of Σ.

[2]We are slightly changing the normalization of the topological gravity action, $\frac{1}{\tau}\mapsto\frac{N^2}{\tau}$, to make the right-hand side well-defined at all τ for every N.

Comparison with the literature

Since the partition function of $T\overline{T}$-deformed two-dimensional Yang-Mills theory has been derived in different ways in the literature [37, 39, 52], it is appropriate to now pause and discuss our result.

Formula (3.2), or more precisely its original version with $C_{\mathrm{gen}}(R) = C_2(R)$, was first proposed in [37], although it was not rigorously justified for curved surfaces Σ. The proposal of [37] was also the starting point of previous work [38] studying the phase structure of the $T\overline{T}$-deformed theory. Here we have provided the derivation, following an argument similar to that of [39] but with a few important differences.

In [39] topological gravity is introduced after integrating out the gauge fields. In particular, as carefully explained there, the JT gravity path integral is representation-dependent and is inserted inside the sum over irreducible representations. Schematically

$$\sum_R Z_R(\omega) \xrightarrow{T\overline{T}\text{-deformation of [39]}} \sum_R \int \mathscr{D}_R \boldsymbol{e}\ Z_R(\boldsymbol{e} \wedge \boldsymbol{e}) \ ,$$

where $Z_R(\omega)$ is the summand in (3.2), and we have stressed its dependence on the volume form ω of Σ. The normalization of the measure $\mathscr{D}_R \boldsymbol{e}$ is taken to be R-dependent.

Therefore, the procedure of [39] does not deform the original path integral, but deforms each summand in the expression obtained after abelianization [49]. The technique we adopted, instead, describes a deformation of the full path integral, and proves that the abelianization takes place also in the $T\overline{T}$-deformed theory. The two results coincide, as expected. Indeed the gauge fields do not enter in the definition of the operator $\mathcal{O}(\phi)$, which couples to gravity. For this reason, the integration over the coframe field is expected to commute with the integration over the gauge fields.

3.2 $T\overline{T}$-deformed q-Yang-Mills theory

We can extend the argument above to q-deformed Yang-Mills theory: this deformation modifies the domain of integration, making the scalar field ϕ compact, i.e. taking $\phi \in \Omega^0(\Sigma, G)$ to be valued in the Lie group G instead of its Lie algebra $\mathfrak{g}$, without altering the action [53, 54]. In this case abelianization proceeds by conjugating ϕ into the maximal torus of G. The $T\overline{T}$-deformation thus changes the potential for the (now compact) scalar, exactly as in the case of ordinary two-dimensional Yang-Mills theory. The final answer for the $T\overline{T}$-deformed partition function of q-Yang-Mills theory on a surface Σ of genus g_Σ with s boundaries is

$$\mathcal{Z}^{T\overline{T}}_{q\text{-YM}}[\Sigma; g_1, \ldots, g_s] = \sum_R \dim_q(R)^{\chi(\Sigma)}\, q^{\frac{p}{2}\, C_2^{T\overline{T}}(R,\tau)}\ \chi_R(g_1) \cdots \chi_R(g_s) \ , \tag{3.3}$$

with the identification of the q-parameter

$$q = \mathrm{e}^{-\lambda/N} \ .$$

Here $p \in \mathbb{Z}$ is a discrete parameter, the $T\overline{T}$-deformed Casimir is defined in (3.1), and

$$\chi(\Sigma) = 2 - 2g_\Sigma - s$$

is the Euler characteristic of Σ. The boundary conditions $g_1, \ldots, g_s \in G$ are the holonomies of the gauge connection around the boundaries, with characters χ_R in the representation R, and $\dim_q(R)$ is the quantum dimension of R. For closed surfaces Σ, the formula (3.3) is simply

$$\mathcal{Z}^{T\overline{T}}_{q\text{-YM}}[\Sigma] = \sum_R \dim_q(R)^{2-2g_\Sigma}\, q^{\frac{p}{2}\, C_2^{T\overline{T}}(R,\tau)} \ .$$

Again, the argument straightforwardly extends to generalized q-deformed Yang-Mills theory, with additional higher degree terms added to the potential $V(\phi)$.

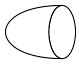
The partition function of the ordinary $T\overline{T}$-deformed Yang-Mills theory from Section 3.1 above is recovered by taking the limit

$$p \to \infty \qquad \text{and} \qquad \lambda \to 0 \qquad \text{with} \quad \lambda\, p = A \text{ fixed}\ , \tag{3.4}$$

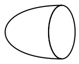
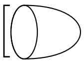
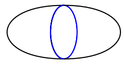
where A is the area of Σ.

q-Yang-Mills theory on the disk and on the cylinder

As we have shown, the procedure of $T\overline{T}$-deformation works for every Riemann surface Σ, possibly with boundary.[3] Special roles are played by the disk and cylinder partition functions. On the disk we have

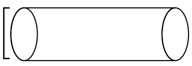
$$\mathcal{Z}^{T\overline{T}}_{q\text{-YM}} \Big[$$

$$; g \Big] = \sum_R \dim_q(R)\ q^{\frac{p}{2}\, C_2^{T\overline{T}}(R,\tau)}\ \chi_R(g)\ .$$

Gluing two disks whose boundaries have opposite orientations and using the orthogonality of the characters we get the $T\overline{T}$-deformed partition function on the sphere $\mathbb{S}^2$:

$$\int_G \mathrm{d}g\ \mathcal{Z}^{T\overline{T}}_{q\text{-YM}} \Big[$$

$$; g \Big]\ \mathcal{Z}^{T\overline{T}}_{q\text{-YM}} \Big[$$

$$; g^{-1} \Big] = \mathcal{Z}^{T\overline{T}}_{q\text{-YM}} \Big[$$

$$\Big] = \mathcal{Z}^{T\overline{T}}_{q\text{-YM}} \big[\mathbb{S}^2\big]\ ,$$

where $\mathrm{d}g$ is the invariant Haar measure on G.

The cylinder partition function is

$$\mathcal{Z}^{T\overline{T}}_{q\text{-YM}} \Big[$$

$$; g_{\text{in}}, g_{\text{out}} \Big] = \sum_R q^{\frac{p}{2}\, C_2^{T\overline{T}}(R,\tau)}\ \chi_R(g_{\text{in}}^{-1})\, \chi_R(g_{\text{out}})\ ,$$

where we have already taken into account the orientation in the definition of the boundary condition g_{in}. In the topological limit $\lambda = 0$, it serves as a propagator: attaching it to any surface Σ replaces the holonomy g_{in} by g_{out}. At non-zero area though, attaching a cylinder has a notable effect which effectively increases the coupling. With our choice of normalization for τ, the effect of gluing a cylinder to Σ is precisely the same as in the theory without $T\overline{T}$-deformation.

Supersymmetry

An additional consistency check for our formulas comes from the minimal supersymmetric extension of Yang-Mills theory. Two-dimensional Yang-Mills theory and its q-deformation are equivalent to their supersymmetric counterparts. The BRST multiplet is (A, ϕ, ψ), with ψ a Grassmann-odd one-form on Σ with values in the Lie algebra $\mathfrak{g}$, and the action is schematically modified as

$$S_{\text{YM}} \longmapsto S_{\text{YM}} + \int_\Sigma \mathrm{Tr}(\psi \wedge \psi)\ .$$

The equivalence is straightforwardly checked by integrating out ψ. On the other hand, the new term is topological and hence is insensitive to the $T\overline{T}$-deformation. We can thus first $T\overline{T}$-deform and integrate out ψ afterwards, obtaining again the result (3.3). So regardless of the route followed, the $T\overline{T}$-deformation of two-dimensional Yang-Mills theory and its generalizations always provides the same answer with or without supersymmetry.

[3] Two-dimensional Yang-Mills theory can also be defined on surfaces with corners [55, 56], and it is likely that our technique extends to that case.

Refinement

Let us now consider the refinement of q-deformed Yang-Mills theory [57], also known as (q,t)-deformed Yang-Mills theory. The refinement leaves the action unchanged but modifies the path integral measure [57]. Therefore we can $T\overline{T}$-deform the theory and the abelianization technique continues to work, hence the $T\overline{T}$-deformation modifies the partition function of (q,t)-Yang-Mills theory only in the Gaussian potential, according to (2.2). We will give more details later on in Section 4.6.

θ-angle

In q-Yang-Mills theory, the θ-angle term is introduced in the path integral as a linear term in the potential $V(\phi)$, and it descends from a chemical potential for D2-branes in the construction of [58, 53]. Therefore it will also couple to the gravitational path integral after $T\overline{T}$-deformation, and will enter in the denominator of the deformed potential through

$$-\frac{\lambda}{2N}\,C_2(R)+\mathrm{i}\,\theta\,C_1(R)\longmapsto\frac{-\frac{\lambda}{2N}\,C_2(R)+\mathrm{i}\,\theta\,C_1(R)}{1-\frac{\tau}{N^3}\left[C_2(R)-\frac{2\,\mathrm{i}\,\theta\,N}{\lambda}\,C_1(R)\right]}\ ,$$

where C_1 is the first Casimir of G, which is non-zero only for non-simply connected gauge groups.

3.3 Wilson loops, marked points and q a root of unity

One may also include Wilson loop operators in an irreducible representation R of G along a closed curve $\mathcal{C}$ on Σ:

$$W_R(\mathcal{C})=\mathrm{Tr}_R\,\mathcal{P}\exp\oint_{\mathcal{C}}A\ .$$

We assume for simplicity that $\mathcal{C}$ does not wind around any handle of Σ.

The expectation value of a collection of s Wilson loops in $T\overline{T}$-deformed two-dimensional Yang-Mills theory, both the ordinary and q-deformed versions, is computed as follows. Cut Σ along the s cycles $\mathcal{C}_1,\dots,\mathcal{C}_s$, obtaining $s+1$ components: s of them have disk topology and the last is the remainder of Euler characteristic $\chi(\Sigma)-s$. The next step is to compute the $T\overline{T}$-deformed partition function on each component, which is a wavefunction of the holonomies along the boundaries $\mathcal{C}_1,\dots,\mathcal{C}_s$. Then glue the components pairwise back together by integrating over G. The example of $\mathbb{S}^2$ with three loops is depicted in Figure 3.

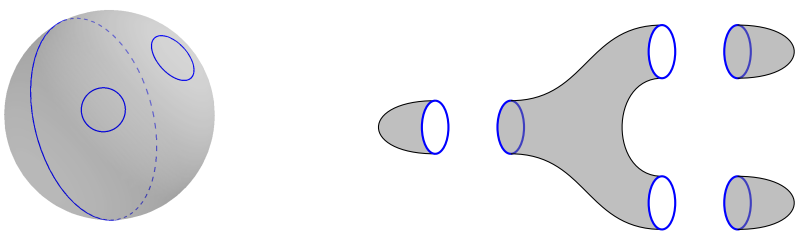


Figure 3: A sphere with three Wilson loops is cut into three disks plus a remaining pair of pants.

In this way we find the normalized expectation value

$$\left\langle W_{R_1}(\mathcal{C}_1)\cdots W_{R_s}(\mathcal{C}_s)\right\rangle = \frac{1}{\mathcal{Z}^{T\overline{T}}_{q\text{-YM}}[\Sigma]} \sum_R \dim_q(R)^{\chi(\Sigma)-s}\, q^{\frac{p-A_W}{2}\, C_2^{T\overline{T}}(R,\tau)} \times \prod_{i=1}^{s} \sum_{\widetilde{R}_i} \dim_q(\widetilde{R}_i)\, q^{\frac{a_i}{2}\, C_2^{T\overline{T}}(\widetilde{R}_i,\tau)}\, N^{\widetilde{R}_i}{}_{R_i R} \ ,$$

where a_i is the area enclosed by the loop $\mathcal{C}_i$, R_i is the representation label of the i-th loop, and $\widetilde{R}_i$ is a summation variable denoting an irreducible representation associated to the quantization on the i-th component. Geometrically, R is associated to the remainder, R_i to the i-th cut and $\widetilde{R}_i$ to the i-th disk. We have also denoted

$$A_W = \sum_{i=1}^{s} a_i \ ,$$

and we assume $A_W < p$ to ensure convergence of the first series. The quantities $N^{R_1}{}_{R_2R_3}$ are fusion coefficients obtained from the integration over the holonomies; for unitary gauge group they are the Littlewood-Richardson coefficients. This formula differs from the original theory simply in the replacement of the Casimir as in (3.1).

In the limit in which the loops shrink to a point, $a_i \to 0$, we obtain (after dropping the normalization $\mathcal{Z}^{T\overline{T}}_{q\text{-YM}}[\Sigma]^{-1}$) the partition function of $T\overline{T}$-deformed q-Yang-Mills theory on a surface Σ with s marked points decorated with irreducible representations $R_1, \dots, R_s$:

$$\mathcal{Z}^{T\overline{T}}_{q\text{-YM}}[\Sigma; R_1, \dots, R_s] = \sum_R \dim_q(R)^{\chi(\Sigma)}\, q^{\frac{p}{2}\, C_2^{T\overline{T}}(R,\tau)} \prod_{i=1}^{s} \sum_{\widetilde{R}_i} \frac{\dim_q(\widetilde{R}_i)}{\dim_q(R)}\, N^{\widetilde{R}_i}{}_{R_i R} \ .$$

Analogous formulas hold for the $T\overline{T}$-deformation of ordinary Yang-Mills theory.

Lost connection with Chern-Simons theory

For $\tau = 0$ and $0 < |q| < 1$ (with possibly $q \in \mathbb{C}$), when $|q| \to 1$ at roots of unity, the sum over representations terminates for gauge group $G = U(N)$ [59]. After $T\overline{T}$-deformation, the quadratic Casimir part is modified and the cancellations that truncate the series no longer take place.

The usual connection between q-Yang-Mills theory and Chern-Simons theory thus no longer holds. For the same reason, even when q is a root of unity, one cannot understand Wilson loops in the $T\overline{T}$-deformed version in terms of observables in Chern-Simons theory, or some deformation thereof, living in the total space of a degree p circle bundle over Σ.

Furthermore, when looking for the modular matrices S and T of $PSL(2,\mathbb{Z})$ in expressions such as (3.3), we recall that matrix elements like

$$\mathsf{S}_{R\widetilde{R}} \qquad \text{and} \qquad \mathsf{T}_{R\widetilde{R}}$$

are defined for integrable representations R and $\widetilde{R}$, while in our case the sum runs over all the irreducible representations. In this sense the $T\overline{T}$-deformation spoils the modularity properties of the theory at q a root of unity, as could have been foreseen from the explicit form of (3.1).[4]

[4]Gauging the $T\overline{T}$-deformed WZW model of [19] does not yield a connection with Chern-Simons theory, or some deformation thereof. The relation with Chern-Simons theory is indeed a special property of the conformal fixed point [60].

3.4 Breakdown of factorization

We have seen that the usual connection with Chern-Simons theory is lost as soon as the $T\overline{T}$-deformation is turned on. In the following we discuss another well-known central feature of two-dimensional Yang-Mills theory that does not hold after $T\overline{T}$-deformation: the factorization of the partition function $\mathcal{Z}_{q\text{-YM}}$ into chiral and anti-chiral sectors [58, 53, 61, 62]. This strongly suggests that the usual large N string theory picture of two-dimensional Yang-Mills theory breaks down after $T\overline{T}$-deformation.

Let $\mathscr{H}$ be the Hilbert space of states of the theory, and endow it with the basis $\{|R\rangle\}$ in one-to-one correspondence with isomorphism classes of irreducible unitary representations of G [51]. Adopting a common shorthand, we call it the representation basis. The normalization is

$$\langle R|\widetilde{R}\,\rangle = \dim(R)^{\chi(\Sigma)}\ \delta_{R\widetilde{R}}\ .$$

The factorization property relies on being able to make a replacement

$$q^{\frac{p}{2}\,C_2(R)} \longrightarrow q^{\frac{p}{2}\,C_2(R_+)}\,q^{\frac{p}{2}\,C_2(R_-)}\ ,$$

where R_+ and R_- are known as "chiral" and "anti-chiral" representations, which correspond to states

$$|R_\pm\rangle \in \mathscr{H}_\pm\ ,$$

in the factorized Hilbert space $\mathscr{H}_+ \otimes \mathscr{H}_-$. It is clear from (3.1)–(3.3) that the factorization breaks down at $\tau \neq 0$.

Quantization of the $T\overline{T}$-deformed theory

Consider the unitary gauge group $G = U(N)$ and the surface Σ as a fibration over $\mathbb{S}^1$, with the circle interpreted as the Euclidean time direction. By definition, the partition function of q-Yang-Mills theory on Σ is given by

$$\mathcal{Z}_{q\text{-YM}}(\lambda) = \mathrm{Tr}_{\mathscr{H}}\ \mathrm{e}^{-\frac{\lambda\,p}{2N}\,\widehat{H}_{q\text{-YM}}} = \sum_R \langle R|\,\mathrm{e}^{-\frac{\lambda\,p}{2N}\,\widehat{H}_{q\text{-YM}}}|R\rangle\ ,$$

where $\widehat{H}_{q\text{-YM}}$ is the Hamiltonian, and we have taken the trace over the Hilbert space $\mathscr{H}$ in the representation basis, which diagonalizes $\widehat{H}_{q\text{-YM}}$ with eigenvalues $C_2(R)$. A generic deformation controlled by a parameter τ which triggers an RG flow would produce

$$\mathcal{Z}^{\mathrm{def}}_{q\text{-YM}}(\lambda,\tau) = \mathrm{Tr}_{\mathscr{H}(\tau)}\ \mathrm{e}^{-\frac{\lambda\,p}{2N}\,\widehat{H}(\tau)} = \sum_{R(\tau)} \langle R(\tau)|\,\mathrm{e}^{-\frac{\lambda\,p}{2N}\,\widehat{H}(\tau)}|R(\tau)\rangle\ ,$$

deforming both the Hamiltonian to $\widehat{H}(\tau)$ and the Hilbert space to $\mathscr{H}(\tau)$. The basis $\{|R(\tau)\rangle\}$ would reduce to the representation basis when sending $\tau \to 0$. Note that, in this general framework, since the Hilbert space changes, one may need to include additional states. However, when the deformation is by the composite operator $T\,\overline{T}$, the explicit form of the deformed eigenvalues of $\widehat{H}(\tau)$ is known, and in particular no new eigenvalues arise. For q-Yang-Mills theory we obtain explicitly

$$\mathcal{Z}^{T\overline{T}}_{q\text{-YM}}(\lambda,\tau) = \sum_{R(0)} \langle R(0)|\,\mathrm{e}^{-\frac{\lambda\,p}{2N}\,\widehat{H}(\tau)}|R(0)\rangle\ ,$$

with the deformed Hamiltonian

$$\widehat{H}(\tau) = \frac{\widehat{H}_{q\text{-YM}}}{1-\frac{\tau}{N^3}\,\widehat{H}_{q\text{-YM}}}$$

diagonalized by the representation basis $\{|R\rangle\} = \{|R(0)\rangle\}$ for all $\tau \geq 0$. Only the eigenvalues $C_2^{T\overline{T}}(R,\tau)$ are different. Therefore, although in general from the knowledge of the eigenvalues one cannot exclude that additional degenerate states arise in the $T\overline{T}$-deformed theory, we see that this is not the case for two-dimensional Yang-Mills theory and its relatives. Indeed, having found explicitly the $T\overline{T}$-deformed partition function, the presence of additional states at $\tau > 0$ should have a null net contribution, but this is not possible from the explicit, strictly positive form of the eigenvalues.

In conclusion, the partition function of $T\overline{T}$-deformed q-Yang-Mills theory, and hence also ordinary Yang-Mills theory through the limit (3.4), is the trace of the exponential of the $T\overline{T}$-deformed Hamiltonian found in [37] taken in an undeformed Hilbert space of states.

Free fermion formulation

Let us now focus on ordinary (without q-deformation) two-dimensional Yang-Mills theory for definiteness. While the factorization structure of q-Yang-Mills theory is richer, in the sense that already at finite N one sees a factorization into chiral and anti-chiral building blocks, the breakdown of these properties happens at a fundamental level, which is more clearly seen by looking directly at $T\overline{T}$-deformed ordinary Yang-Mills theory.

It is well-known that the Hilbert space $\mathscr{H}$ factorizes at large N as [63]

$$\mathscr{H} \xrightarrow{N\to\infty} \mathscr{H}_+ \otimes \mathscr{H}_- \ , \tag{3.5}$$

with the representation basis factorizing accordingly as

$$|R\rangle \xrightarrow{N\to\infty} |R_+\rangle \otimes |R_-\rangle \ ,$$

with $|R_\pm\rangle \in \mathscr{H}_\pm$. The Hilbert spaces $\mathscr{H}_+$ and $\mathscr{H}_-$ are known as the "chiral" and "anti-chiral" sectors, respectively. From the factorization (3.5) one has [63]

$$\begin{aligned} \lim_{N\to\infty} \mathcal{Z}_{\mathrm{YM}}(A) &= \left(\sum_{R_+} \langle R_+ |\, \mathrm{e}^{-\frac{A}{2N}\, \widehat{H}_{\mathrm{YM}}} | R_+ \rangle \right) \left(\sum_{R_-} \langle R_- |\, \mathrm{e}^{-\frac{A}{2N}\, \widehat{H}_{\mathrm{YM}}} | R_- \rangle \right) \\ &= \left(\mathrm{Tr}_{\mathscr{H}_+}\ \mathrm{e}^{-\frac{A}{2N}\, \widehat{H}_{\mathrm{YM}}} \right) \left(\mathrm{Tr}_{\mathscr{H}_-}\ \mathrm{e}^{-\frac{A}{2N}\, \widehat{H}_{\mathrm{YM}}} \right) \end{aligned}$$

where we dropped overall constants. In the $T\overline{T}$-deformed theory, the large N factorization of the Hilbert space (3.5) continues to hold according to the discussion above, but the trace can no longer be factorized into a product of traces.

We will now further elucidate this point through the equivalence with a system of N non-relativistic fermions [64, 65] (non-perturbative corrections were studied in [66]). In the mapping of two-dimensional $U(N)$ Yang-Mills theory to a system of N free fermions on $\mathbb{S}^1$ [64, 65], the ground state corresponds to the state where the fermions occupy the N lowest energy levels, as in the left panel of Figure 4. In the representation basis, the ground state is described by the trivial representation, while higher-dimensional representations are mapped to excited states, in which fermions have jumped to higher energy levels.

At finite N, excitations above the positive Fermi surface, or below the negative Fermi surface, may arise from any fermion, as depicted in the central and right panels of Figure 4. In the large N limit the two Fermi surfaces decouple, and excitations above the positive Fermi level (respectively below the negative Fermi level) correspond to fermions close to that surface, thus with positive (respectively negative) energy, jumping to higher (respectively lower) unoccupied levels.

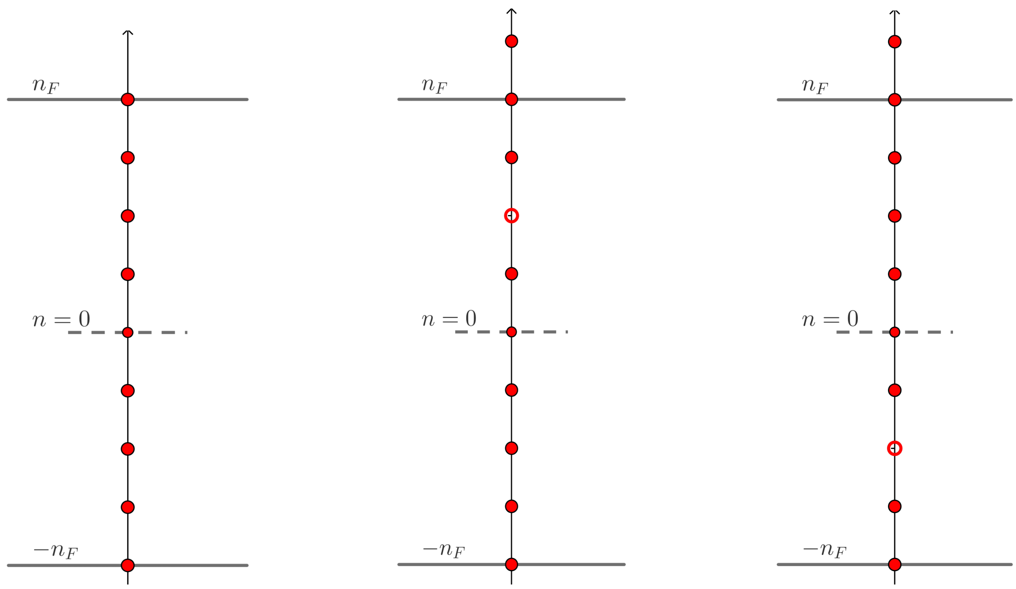


Figure 4: Non-relativistic fermions: ground state (left) and two excited states (center, right). The two excited states are excitations over the positive Fermi surface. In the center, a fermion occupying a positive energy level jumps above the positive Fermi surface: this will correspond to a chiral state at large N. On the right, a fermion occupying a negative energy level jumps above the positive Fermi surface: this will be exponentially suppressed at large N.

Jumps of order N sites are exponentially suppressed with N, and can only be seen from a non-perturbative analysis [66]. Therefore the factorization is interpreted as a disentanglement of the Fermi surfaces.

In the $T\overline{T}$-deformed theory the two Fermi surfaces remain entangled even in the large N limit. The crucial difference between the picture of [64] and its $T\overline{T}$-deformed version lays in the interpretation of the Casimir in terms of free fermions. While at $\tau = 0$ it is a confining quadratic potential, this interpretation is lost at $\tau > 0$. Indeed, by expanding the $T\overline{T}$-deformed potential (2.2) in a geometric series, we do not obtain a confining potential for fermions, but instead infinitely many non-local interaction terms which introduce long-distance correlations.

A consequence of these additional interactions is that, even at $N \to \infty$, the energy required for a fermion to jump to another level does not only depend on the energy separation between the initial and final state, but it is also a function of the levels occupied by all of the other fermions. From a conformal field theory perspective, this casts doubt on the existence of a string theory dual to $T\overline{T}$-deformed two-dimensional Yang-Mills theory, unless it is a highly exotic one.

4 Phase transitions in $T\overline{T}$-deformed q-Yang-Mills theory

Two-dimensional $U(N)$ Yang-Mills theory on $\mathbb{S}^2$ undergoes a third order phase transition [67], henceforth called the Douglas-Kazakov (DK) transition, which is induced by instanton instabilities [68]. The same third order phase transition is experienced by the q-deformed theory [69, 70, 61] when $p > 2$. The critical value of the coupling λ_{cr} decreases monotonically with increasing p and one eventually recovers the DK transition in the limit $p \to \infty$ [69, 70, 61], see Figure 5. This means that the q-deformation extends the region in parameter space corresponding to a weak coupling phase.

In [38] it was shown that $T\overline{T}$-deformed (but not q-deformed) Yang-Mills theory also undergoes

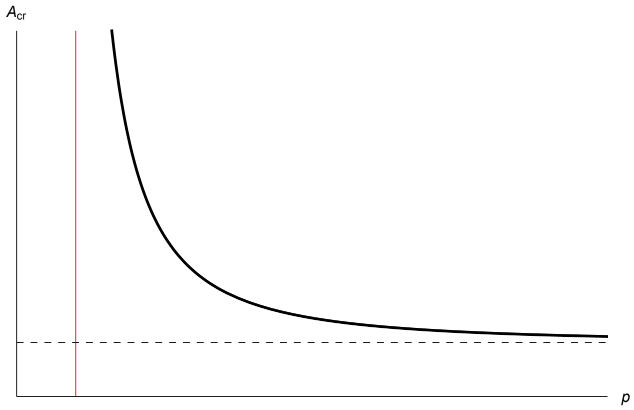


Figure 5: The critical curve of q-Yang-Mills theory, in terms of the parameter $A = \lambda/p$ as a function of p. The horizontal asymptote (dashed) is the DK critical point $A_{\rm cr} = \pi^2$. The vertical asymptote (red) is the point $p = 2$. This plot is inspired by [69].

a DK-type transition for $0 \leq \tau < \tau_{\rm max}$, with

$$\frac{1}{\tau_{\rm max}} = \frac{1}{\pi^2} - \frac{1}{12} \,. \tag{4.1}$$

The critical value of the area parameter decreases with increasing τ, and eventually no weak coupling phase exists when τ approaches $\tau_{\rm max}$ [38]. Therefore the $T\overline{T}$-deformation reduces the region of parameter space corresponding to a weak coupling phase.

The goal of this section is to analyze the large N phase structure when both the q-deformation and the $T\overline{T}$-deformation are turned on. In the following we summarize our large N results. Then we proceed in Section 4.1 to briefly recall the strategy of [38] to extend the analysis of [67, 68] to the $T\overline{T}$-deformed setting. In Section 4.2 we present the large N formalism when both deformations are turned on and study the weak coupling regime, while Section 4.3 is dedicated to a study of the critical surface. Section 4.4 discusses the role of instanton contributions, while Section 4.5 is dedicated to a study of the phase transition and the strong coupling regime. Finally, in Section 4.6 we comment on the large N limit of the refined theory.

Large N results

Before diving into the detailed analysis of the large N phase structure, we summarize here our main findings:

- $T\overline{T}$-deformed q-Yang-Mills theory undergoes a third order phase transition when $p > p_0$. Remarkably, we find that $p_0 < 2$.
- The slice of parameter space giving a weak coupling phase is extended, relative to the pure $T\overline{T}$-deformation, and is reduced only relative to the q-deformation. This interpolates perfectly between the effects discovered respectively in [69, 70, 61] and [38], recovering the single-deformation scenarios as limiting cases.
- The phase transition is induced by instantons. The shrinking of the weak coupling region is explained by the fact that in the $T\overline{T}$-deformed theory the suppression factor of the instantons is smaller, hence their effect becomes relevant at lower values of the coupling λ.

4.1 Large N limit of $T\overline{T}$-deformed Yang-Mills theory

Here and in the rest of this section we take $G = U(N)$ and $\Sigma = \mathbb{S}^2$. We also continue the Euler characteristic χ to real values close to $\chi(\mathbb{S}^2) = 2$,

$$0 < 2 - \varepsilon_- < \chi < 2 + \varepsilon_+ \ , \tag{4.2}$$

with $\varepsilon_\pm > 0$, which will also be useful later on in Section 5.

Irreducible $SU(N)$ representations up to isomorphism are in one-to-one correspondence with Young diagrams, labelled by partitions $\mathfrak{R} = (\mathfrak{R}_1, \dots, \mathfrak{R}_{N-1})$ with

$$\mathfrak{R}_1 \geq \mathfrak{R}_2 \geq \cdots \geq \mathfrak{R}_{N-1} \geq \mathfrak{R}_N := 0 \ .$$

Using the short exact sequence of groups

$$1 \longrightarrow SU(N) \longrightarrow U(N) \longrightarrow U(1) \longrightarrow 1$$

the irreducible $U(N)$ representations are obtained from those of $SU(N)$ summing over the $U(1)$ sector. Therefore a $U(N)$ representation R corresponds to a partition

$$+\infty > R_1 \geq R_2 \geq \cdots \geq R_N > -\infty \ , \tag{4.3}$$

with $R_i = \mathfrak{R}_i + \mathfrak{r}$ for all $i = 1, \dots, N$ and $\mathfrak{r} \in \mathbb{Z}$. It is useful to change variables

$$h_i = -R_i + i - \frac{N+1}{2} \qquad \text{for} \quad i = 1, \dots, N \ . \tag{4.4}$$

In these variables the partition function of $T\overline{T}$-deformed $U(N)$ Yang-Mills theory reads

$$\mathcal{Z}_{\mathrm{YM}}^{T\overline{T}}(A,\tau) = \frac{1}{N!\, G(N+1)^\chi} \sum_{\vec{h}\in\mathbb{Z}^N} \Delta(\vec{h})^\chi \ \exp\left(- \frac{\frac{A}{2N}\left(\sum\limits_{j=1}^{N} h_j^2 - \frac{N\,(N^2+1)}{12}\right)}{1 - \frac{\tau}{N^3}\left(\sum\limits_{j=1}^{N} h_j^2 - \frac{N\,(N^2+1)}{12}\right)} \right) , \tag{4.5}$$

where we used the symmetry of the sum to lift the restriction (4.3) to the principal Weyl chamber, letting the sum run over unordered $\vec{h} = (h_1, \dots, h_N) \in \mathbb{Z}^N$. The shift proportional to $-\frac{1}{12}$ in the Casimir would simply give an overall factor at $\tau = 0$, but it becomes relevant at $\tau > 0$. Here G is the Barnes G-function, which for integer argument can be written as

$$G(N+1) = \prod_{j=1}^{N-1} j! \tag{4.6}$$

and $\Delta(\vec{h})$ is the Vandermonde determinant

$$\Delta(\vec{h}) = \prod_{1\leq i<j\leq N} (h_i - h_j) \ . \tag{4.7}$$

When sending $N \to \infty$ the leading contribution to the partition function comes from the saddle point configuration. This is the one that minimizes the action

$$S_{\mathrm{YM}}[h] = \frac{A}{2} \sum_{k=0}^{\infty} \tau^k \left(\int_0^1 h(x)^2 \, \mathrm{d}x - \frac{1}{12} \right)^{k+1} - \frac{\chi}{2} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, \log|h(x) - h(y)| \ ,$$

where $x = i/N$ and $h(x) = h_i/N$ have become continuous variables at large N, and we have expanded the $T\overline{T}$-deformed Casimir in a geometric series, which is allowed as long as the $T\overline{T}$-deformed theory is well-defined. We are also taking a 't Hooft limit, since we already reabsorbed the gauge coupling in the definition of A/N.

Introduce the eigenvalue density $\rho(h)$, which is defined by

$$\rho(h)\,\mathrm{d}h = \mathrm{d}x$$

and is normalized

$$\int_{\mathrm{supp}(\rho)} \rho(h)\,\mathrm{d}h = 1 \ .$$

Then taking the derivative of the action and setting it equal to zero, we arrive at the saddle point equation

$$⨍_{\mathrm{supp}(\rho)} \mathrm{d}u\, \frac{\rho(u)}{h-u} = \frac{A}{\chi}\, h \sum_{k=0}^{\infty} (k+1)\,\tau^k \left(\mu_2 - \frac{1}{12}\right)^k \ , \tag{4.8}$$

where $⨍_{\mathrm{supp}(\rho)}$ denotes the principal value of the integral over the support and μ_2 is the second moment of the eigenvalue distribution ρ:

$$\mu_2 := \mu_2[\rho] = \int_{\mathrm{supp}(\rho)} \mathrm{d}u\, \rho(u)\, u^2 \ . \tag{4.9}$$

A crucial aspect here is that the original ensemble is discrete, and the condition (4.3) in terms of the new weight variables h_i says that

$$h_{i+1} - h_i \geq 1 \ ,$$

which in the large N limit implies

$$\rho(h) \leq 1 \qquad \text{for} \quad h \in \mathrm{supp}(\rho) \ . \tag{4.10}$$

The solutions we find will have to satisfy this constraint: in a generic discrete ensemble, if $\rho(h)$ ceases to satisfy (4.10) at a codimension one locus in the space of parameters, the system undergoes a phase transition, and the new phase will be governed by a different density $\rho(h)$.

Strategy

Standard methods for solving integral equations [71] do not apply to the saddle point equation (4.8) due to the dependence on ρ on both sides of the equality. However, it was noted in [38] that (4.8) can be solved perturbatively in τ, with the zeroth order being the DK solution [67].

The technique of [38] essentially consists in solving the equation at k-th order using the expression for μ_2 obtained by plugging in the distribution $\rho(h)$ approximated at $(k-1)$-th order, which is the standard method of iteration in perturbation theory. The solution can be found in this way thanks to the especially simple dependence of the right-hand side of (4.8) on $\rho(h)$: it only enters the saddle point equation in a coefficient that renormalizes A. In fact, we can write the saddle point equation at a generic order $O(\tau^k)$ as

$$⨍_{\mathrm{supp}(\rho)} \mathrm{d}u\, \frac{\rho(u)}{h-u} = \frac{A}{\chi}\, c_k\, h \ ,$$

where the coefficient c_k is given by

$$c_k := c_k(A,\tau) = \sum_{j=0}^{k} (j+1)\,\tau^j \left(\mu_2^{(k-1)} - \frac{1}{12}\right)^j \ ,$$

with the second moment $\mu_2^{(k-1)}$ computed using the distribution $\rho(h)$ approximated at order $k-1$. Therefore one simply has to solve the same equation order by order, by finding at k-th order exactly the same solutions as in [67] but with renormalized area

$$A \longmapsto \frac{2A\,c_k}{\chi} \ .$$

Since the system presents two phases [67, 68], we have to repeat the procedure in each phase. As shown in [38], we can go to arbitrarily high order and eventually find

$$c_\infty = \begin{cases} b_\infty \ , & 0 < A < A_{\rm cr}(\tau) \ , \\ d_\infty \ , & A > A_{\rm cr}(\tau) \ , \end{cases}$$

where the parameters b_∞ and d_∞ are defined through the equation [38]

$$c_\infty = \left(1 - \tau\left(\mu_2 - \frac{1}{12}\right)\right)^{-2}$$

with μ_2 a function of $\frac{2A\,c_\infty}{\chi}$ which is evaluated at weak coupling for b_∞ and at strong coupling for d_∞. In particular, $c_\infty \to 1$ in the limit $\tau \to 0$, reproducing the undeformed case.

The critical value is found by imposing the condition (4.10), which is translated into the condition $\frac{2A\,b_\infty}{\chi} < \pi^2$ on the parameters. This gives the critical line [38]

$$A_{\rm cr}(\tau) = \frac{\chi}{2}\,\pi^2 \left(1 - \tau\left(\frac{1}{\pi^2} - \frac{1}{12}\right)\right)^2 \ . \tag{4.11}$$

In particular, the domain of A corresponding to a small coupling phase shrinks as τ increases, and eventually disappears at the value $\tau = \tau_{\rm max}$ defined in (4.1).

Recall that in all formulas we kept track of χ for later convenience, but its physical value is $\chi = 2$.

4.2 Large N limit of $T\overline{T}$-deformed q-Yang-Mills theory

At this stage we are ready to apply the formalism reviewed in Section 4.1 above to the harder problem of q-Yang-Mills theory.

In terms of the shifted weights $\vec{h} \in \mathbb{Z}^N$ introduced in (4.4), the partition function of $T\overline{T}$-deformed $U(N)$ q-Yang-Mills theory is

$$\mathcal{Z}^{T\overline{T}}_{q\text{-YM}}(\lambda,\tau) = \frac{1}{N!} \sum_{\vec{h}\in\mathbb{Z}^N} \left(\frac{\Delta_q(\vec{h})}{\Delta_q(\emptyset)}\right)^\chi \exp\left(- \frac{\frac{\lambda p}{2N}\left(\sum_{j=1}^N h_j^2 - \frac{N(N^2+1)}{12}\right)}{1 - \frac{\tau}{N^3}\left(\sum_{j=1}^N h_j^2 - \frac{N(N^2+1)}{12}\right)} \right) \ , \tag{4.12}$$

where $\Delta_q(\vec{h})$ is a q-deformation of the Vandermonde determinant

$$\Delta_q(\vec{h}) = \prod_{1\le i<j\le N} 2\sinh\frac{\lambda\,(h_i - h_j)}{2N} \ ,$$

and we used the shorthand notation $\Delta_q(\emptyset) := \Delta_q(h_i = i)$. Here $\Delta_q(\emptyset)$ plays the role of a q-deformation of the Barnes G-function defined in (4.6). We have also continued χ beyond its physical value $\chi = \chi(\mathbb{S}^2) = 2$ as in (4.2).

We now take the large N limit of (4.12), which we stress is a 't Hooft limit with 't Hooft coupling λ, while the Yang-Mills coupling is λ/N. In this limit, the contribution of $\Delta_q(\emptyset)^{-\chi}$ is given by [69]

$$\lim_{N\to\infty} \chi \log \Delta_q(\emptyset) = -\frac{\chi}{\lambda^2}\, F_0^{\rm CS}(\lambda) \ ,$$

where $F_0^{\rm CS}(\lambda)$ is the planar free energy of $U(N)$ Chern-Simons theory on the three-sphere $\mathbb{S}^3$:

$$F_0^{\rm CS}(\lambda) = \frac{\lambda^3}{12} - \frac{\pi^2}{6}\,\lambda - \mathrm{Li}_3(\,\mathrm{e}^{\,-\lambda}) + \zeta(3) \ .$$

The analogue of the saddle point equation (4.8) in this q-deformed setting is

$$⨍_{\mathrm{supp}(\rho)} \mathrm{d}u\, \rho(u) \coth\frac{\lambda\,(h-u)}{2} = \frac{2p}{\chi}\, h \sum_{k=0}^{\infty} (k+1)\, \tau^k \left(⨍_{\mathrm{supp}(\rho)} \mathrm{d}u\, \rho(u)\, u^2 - \frac{1}{12}\right)^k \ . \tag{4.13}$$

The solution will be a function

$$\rho(h) := \rho\big(h; \lambda, \tfrac{2p}{\chi}, \tau\big)$$

depending parametrically on the couplings, which is normalized and satisfies the constraint from (4.10): $\rho(h) \le 1$. We notice also that χ only enters the large N limit in the combination $\frac{2p}{\chi}$, and hence is simply a rescaling of p.

We solve the saddle point equation (4.13) perturbatively, as in [38] and reviewed in Section 4.1 above. Assuming a one-cut solution, the zeroth order solution is [69, 70, 61]

$$\rho^{(0)}(h) = \frac{2p}{\pi\,\chi}\, \tan^{-1} \sqrt{\frac{\mathrm{e}^{\,\chi\,\lambda/2p}}{\cosh^2 \frac{\lambda\,\chi}{4p}\, h} - 1} \ ,$$

with support

$$\mathrm{supp}\big(\rho^{(0)}\big) = \big[-\alpha^{(0)}, \alpha^{(0)}\big] \qquad \text{where} \quad \alpha^{(0)} = \frac{2}{\lambda}\cosh^{-1}\, \mathrm{e}^{\,\chi\,\lambda/4p} \ .$$

We have put the superscript $^{(0)}$ everywhere to remind us that this is the zeroth order solution in a perturbative expansion in τ. The second moment of this distribution is

$$\mu_2^{(0)} = \int_{-\alpha^{(0)}}^{\alpha^{(0)}} \mathrm{d}u\, \rho^{(0)}(u)\, u^2 = \frac{\chi^2}{12p^2} + \frac{1}{3\lambda^2}\Big(\pi^2 + 6\,\mathrm{Li}_2\big(\,\mathrm{e}^{\,-\chi\,\lambda/2p}\big)\Big) + \frac{8p}{\chi\,\lambda^3}\Big(\mathrm{Li}_3\big(\,\mathrm{e}^{\,-\chi\,\lambda/2p}\big) - \zeta(3)\Big) \ .$$

The next order approximation of (4.13) is

$$⨍_{\mathrm{supp}(\rho)} \mathrm{d}u\, \rho(u) \coth\frac{\lambda\,(h-u)}{2} = \frac{2p}{\chi}\, b_{q,1}\, h \ ,$$

where

$$b_{q,1} := b_{q,1}\big(\lambda, \tfrac{2p}{\chi}, \tau\big) = 1 + \tau\left(\mu_2^{(0)} - \frac{1}{12}\right) \ .$$

The solution at this order then will be again as in [69, 70, 61], but with a renormalized value of p given by

$$p \longmapsto \frac{2p}{\chi}\, b_{q,1} \ .$$

Iterating this argument, at a generic order $O(\tau^k)$ the saddle point equation is the same but the renormalization of p at this order is

$$\frac{2p}{\chi} b_{q,k} \ .$$

The parameter $b_{q,k}$ is obtained using the approximation $\mu_2^{(k-1)}$, which is itself a function of $b_{q,k-1}$. We find

$$b_{q,k} = \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1 - x^{k+1}}{1-x} - 1 \right) \Bigg|_{x=\tau \left(\mu_2^{(k-1)} - \frac{1}{12} \right)} \ ,$$

and the convergence at $k \to \infty$ is guaranteed by the convergence of the geometric series defining the $T\overline{T}$-deformation. Although the study of the limiting value $b_{q,\infty}$ is based on exactly the same arguments as for b_∞ in [38], it is difficult to find explicit formulas due to the q-deformation. We provide more details on $b_{q,\infty}$ and an approximate study in the large p regime in Appendix A.

Even without an explicit expression, we can extract information about $b_{q,\infty}$ from its defining equation

$$b_{q,\infty} = \left(1 - \tau \left(\mu_2 - \frac{1}{12} \right) \right)^{-2} \ , \tag{4.14}$$

with μ_2 depending itself on $b_{q,\infty}$. From this equation we already see that $b_{q,\infty} \geq 1$, with equality only at $\tau = 0$.

We also have to check the consistency of the $T\overline{T}$-deformation. Looking back at (3.1), we have to find for what values of τ the inequality

$$\tau \left(\mu_2 - \frac{1}{12} \right) < 1$$

is satisfied, so that the deformed Casimir is a well-defined (positive) deformation of the quadratic Casimir of $U(N)$. From (4.14), the left-hand side of this inequality is $1 - 1/\sqrt{b_{q,\infty}}$, with $b_{q,\infty} \geq 1$, hence the $T\overline{T}$-deformation is well-posed for all non-negative values of τ. Thus the theory is well-defined at large N all along the RG flow triggered by the $T\overline{T}$-deformation. This was not obvious from (3.1) and we regard it as a strong consistency check.

4.3 Critical curves

We now set χ equal to its physical value $\chi = \chi(\mathbb{S}^2) = 2$.

The solution we found in Section 4.2 above holds as long as ρ satisfies the requirement (4.10). From the formulas above, and the property $|\tan^{-1}(x)| \leq \frac{\pi}{2}$, we have

$$\rho(h) < \frac{p \, b_{q,\infty}}{2}$$

and the system undergoes a phase transition only for those values of $p > p_0$, where $p_0 := p_0(\tau)$ is implicitly defined by

$$p_0 \, b_{q,\infty}(p_0, \tau) = 2 \ .$$

At $\tau = 0$, this was used in [69, 70, 61] to show that for $p \leq \chi(\mathbb{S}^2) = 2$ there is only one phase, while two phases separated by a critical line in the (λ, p)-plane appear for $p > 2$.

We have seen in Section 4.2 above that $b_{q,\infty} \geq 1$, which implies that for all $\tau > 0$ a phase transition takes place whenever $p > p_0$ with $p_0 < 2$. As soon as the $T\overline{T}$-deformation is turned on, the theory with $p = 2$ develops a strong coupling phase, with critical line descending from ∞ to a finite value of λ.

For $p > p_0(\tau)$, the theory presents two phases, separated by a codimension one critical surface in the octant $(\lambda > 0, p > 0, \tau \geq 0)$, parametrized by $\lambda = \lambda_{\rm cr}(p,\tau)$. This surface should be seen as a one-parameter family of critical curves, parameterized by $\tau \geq 0$, describing the evolution of the critical curve of Figure 5 along the RG flow induced by the $T\overline{T}$-deformation, with τ playing the role of the "time". See Figure 6 for a schematic picture. (Note that Figure 6 represents just a rough illustration of the actual critical surface.)

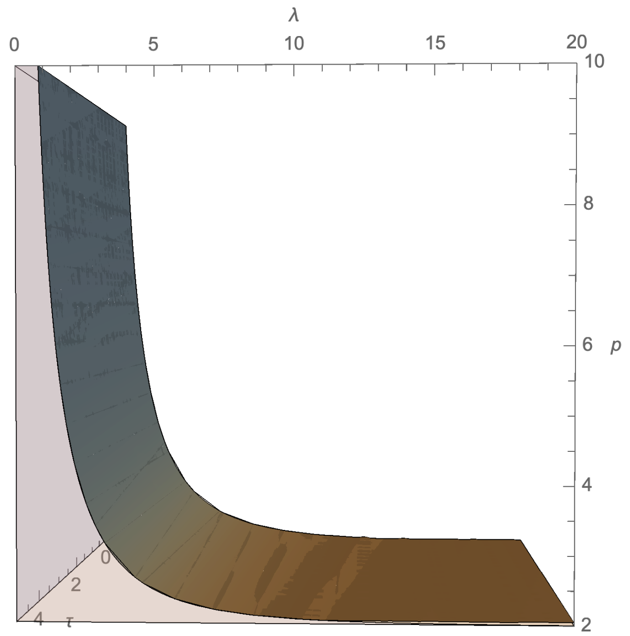


Figure 6: Schematic plot of the critical surface. The gray region represents the weak coupling phase.

This critical surface is defined implicitly by the equation

$$\lambda_{\rm cr} = p\, b_{q,\infty} \log\left(1 + \tan^2 \frac{\pi}{p\, b_{q,\infty}}\right) . \tag{4.15}$$

It is important to bear in mind that $b_{q,\infty}$ depends on λ, and it must be evaluated at the critical value $\lambda_{\rm cr}$ in the right-hand side of (4.15).

Without an explicit expression for $b_{q,\infty}$ we cannot provide a formula for $\lambda_{\rm cr} = \lambda_{\rm cr}(p,\tau)$ describing the critical surface. Nevertheless, the lessons learned from the study of Appendix A are that $b_{q,\infty}$ is a monotonically decreasing function of p, eventually approaching b_∞ from above as $p \to \infty$. In conclusion, from Appendix A we find that $b_{q,\infty}$ decreases with p, and from (4.14) we see that $b_{q,\infty}$ increases with τ. This matches precisely with the known effects of the two deformations taken separately. This, together with (4.15), implies

$$p^{-1} A_{\rm cr}(\tau) \leq \lambda_{\rm cr}(p,\tau) \leq \lambda_{\rm cr}(p,0) ,$$

which means that the volume in the octant $(\lambda > 0, p > 0, \tau \geq 0)$ of the parameter space describing a weak coupling phase is reduced, relative to only the $T\overline{T}$-deformation, and is enhanced relative to only the q-deformation.

$p = 1$ case

Since $b_{q,\infty}$ is an increasing function of τ, one may expect that, for τ sufficiently large, the phase transition also takes place at $p = 1$. In other words, one may wonder whether eventually $p_0(\tau) < 1$

for sufficiently large τ. Unfortunately, our analysis of $b_{q,\infty}$ is not reliable in the limit $p \to 1$, and we cannot draw any conclusions in this direction.

The parameter p has a geometric meaning as the degree of the holomorphic line bundle

$$\mathcal{O}(-p) \longrightarrow \Sigma \ .$$

For every $p > 1$, the total space is a resolution of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_p$, singling out $p = 1$ as a special case. See also [72] for a discussion on the physical relevance of $p > 1$.

On the other hand, there is nothing special about $p = 1$ compared to $p = 2$ in the original construction of [58, 53]. It would be interesting to understand better the fate of the theory for $p = 1$ with $T\overline{T}$-deformation.

4.4 Instanton analysis

A classic result of two-dimensional Yang-Mills theory on $\mathbb{S}^2$ is that the phase transition is triggered by instantons [68]; by 'instanton' here we mean a solution to the classical Yang-Mills equation of motion which is gauge-inequivalent to the trivial connection. In the weak coupling phase, the Boltzmann weight of the saddle point configuration dominates the partition function at large N. However, beyond a critical value of the coupling, non-perturbative contributions cease to be suppressed and compete with the Boltzmann weight, inducing a phase transition.

The same mechanism is at work in the $T\overline{T}$-deformed theory [38]. Here the deformation reduces the suppression factor of the instantons, and therefore the phase transition takes place at lower values of the coupling.

In q-Yang-Mills theory, again the unstable instantons are the cause of the phase transition [69, 61]. Here we show that the same arguments apply to the $T\overline{T}$-deformed theory. We find that, as in the situation without q-deformation, instantons are less suppressed in the $T\overline{T}$-deformed theory.

We start by rewriting the sphere partition function as

$$\mathcal{Z}^{T\overline{T}}_{q\text{-YM}}(\lambda, \tau) = \frac{1}{N!} \sum_{\vec{\ell} \in \mathbb{Z}^N} Z_{\vec{\ell}}(\lambda, \tau) \ , \tag{4.16}$$

where $Z_{\vec{\ell}}$ encodes the instanton contributions, and can be obtained from a modular transformation of the partition function written in the representation basis. See Appendix B for further details. The complete analysis for q-Yang-Mills theory was carried out in [69, 61], and we show how it is adapted to the $T\overline{T}$-deformed theory in Appendix B. Although we cannot get a closed expression, we show that it can be evaluated order by order in τ.

As pointed out already in [68], focusing on the first instanton sector gives clearer insights into understanding how the non-perturbative effects kick in. We therefore consider the one-instanton sector, for which

$$\vec{\ell} = (\ell_1, 0, \dots, 0) \qquad \text{with} \quad \ell_1 = \pm 1 \ .$$

Its contribution to the partition function $\mathcal{Z}^{T\overline{T}}_{q\text{-YM}}$ is

$$\begin{aligned} Z_{(\ell_1,0,\dots,0)}(\lambda, \tau) = \frac{1}{\Delta_q(\emptyset)^2} \int_{\mathbb{R}^N} \mathrm{d}\vec{h} \ \mathrm{e}^{-2\pi\,\mathrm{i}\,\ell_1\, h_1} \prod_{1 \le i < j \le N} 4 \sinh^2 \frac{\lambda\,(h_i - h_j)}{2N} & \\ \times \ \exp\left(-\frac{\lambda\, p}{2N} \, \frac{\sum_{i=1}^N h_i^2 - \frac{N\,(N^2-1)}{12}}{1 - \tau\left(\sum_{i=1}^N h_i^2 - \frac{N\,(N^2-1)}{12}\right)} \right) & \ . \end{aligned}$$

In the large N limit, the contribution of ℓ_1 to the eigenvalue density is of order $O(N)$, hence sub-leading against the $O(N^2)$ contributions from the rest of the action. Therefore in the large N 't Hooft limit we can integrate over the eigenvalues $h_2, \dots, h_N$ using the eigenvalue density $\rho(h)$ found in Section 4.2 above. Rescaling the integration variable h_1 to $h = h_1/N$, we obtain

$$Z_{(\ell_1,0,\dots,0)}(\lambda,\tau) = \frac{\mathcal{Z}_{N-1}(\lambda,\tau)}{\Delta_q(\emptyset)^2} \int_{\mathbb{R}} \mathrm{d}h\ \mathrm{e}^{\,-N\, S_{\text{eff}}[h]}\ ,$$

where $\mathcal{Z}_{N-1}$ comes from integrating out the remaining $N-1$ eigenvalues, and is equal to the partition function of the $U(N-1)$ theory in the zero-instanton sector.[5] The effective action functional is given by

$$S_{\text{eff}}[h] = -2 \int_{\text{supp}(\rho)} \mathrm{d}u\, \rho(u) \log\left|\sinh \frac{\lambda\,(h-u)}{2}\right| + \frac{\lambda\, p\, b_{q,\infty}}{2}\, h^2 - 2\pi\,\mathrm{i}\,\ell_1\, h\ ,$$

where we used the definition (4.14) of $b_{q,\infty}$ to simplify the expression. We obtain the saddle point equation for the first eigenvalue given by

$$p\, b_{q,\infty}\, h - \frac{2\pi\,\mathrm{i}\,\ell_1}{\lambda} = \fint_{\text{supp}(\rho)} \mathrm{d}u\, \rho(u) \coth \frac{\lambda\,(h-u)}{2}\ . \tag{4.17}$$

This is a saddle point equation for h, with $\rho(u)$ known. At this point we notice that, as expected, (4.17) is the same equation found in [69, 70], except for the renormalization $p \mapsto p\, b_{q,\infty}$. We can therefore read off the solution from [69, 70] to get

$$h = \begin{cases} \frac{2\,\mathrm{i}\,\ell_1}{\lambda} \tan^{-1} \sqrt{\frac{\mathrm{e}^{\,-\lambda/p\, b_{q,\infty}}}{\cos^2 \frac{\pi}{p\, b_{q,\infty}}} - 1}\ , & p\, b_{q,\infty} > 2\ , \\ \frac{2\pi\,\mathrm{i}\,\ell_1}{\lambda}\ , & p\, b_{q,\infty} \leq 2\ , \end{cases} \tag{4.18}$$

where we used $|\ell_1| = 1$ to simplify

$$\cosh\left(\frac{\lambda}{2p\, b_{q,\infty}}\, \frac{2\pi\,\mathrm{i}\,|\ell_1|}{\lambda}\right) = \cos \frac{\pi}{p\, b_{q,\infty}}\ .$$

From (4.18) it follows that there is no phase transition for p below a critical value $p_0(\tau)$, defined such that

$$p\, b_{q,\infty}(\lambda, p, \tau) \leq 2 \qquad \text{for} \quad \lambda > 0 \qquad \text{when} \quad p \leq p_0(\tau)\ ,$$

because then the instanton contributions are suppressed for all values of λ. On the other hand, when $p > p_0(\tau)$ and $p\, b_{q,\infty} > 2$ we have (dropping an irrelevant overall constant)

$$\frac{Z_{(\ell_1,0,\dots,0)}(\lambda,\tau)}{Z_{(0,0,\dots,0)}(\lambda,\tau)} = \exp\left(-\frac{N}{\lambda\, p}\, \gamma(\lambda, p\, b_{q,\infty})\right)$$

where γ is the function defined in [69], which in turn is a one-parameter deformation of the suppression function found in [68]. When

$$\frac{\mathrm{e}^{\,-\lambda/p\, b_{q,\infty}}}{\cos^2 \frac{\pi}{p\, b_{q,\infty}}} - 1 > 0\ ,$$

corresponding to $\lambda < \lambda_{\text{cr}}$, the function γ is a positive decreasing function of λ, for any fixed p. However, it becomes purely imaginary when $\lambda > \lambda_{\text{cr}}$, implying that the one-instanton sector is no longer suppressed and its contribution becomes relevant.

[5]The symmetry breaking $U(N) \to U(1) \times U(N-1)$ in the one-instanton sector is explained in Appendix B.

4.5 Strong coupling phase

For values of λ such that, for given p and τ, the eigenvalue density $\rho(h)$ found in Section 4.2 above does not satisfy the constraint (4.10), we have to drop the assumption of a one-cut solution and find another distribution $\rho(h)$ satisfying the bound.

The strategy is the same as that followed in Section 4.2 above for the weak coupling phase: expanding the saddle point equation (4.13) as a power series in τ, we can solve it iteratively. We do not spell out the technical details here, as they are exactly as in [69, 70, 61], up to the renormalization $p \mapsto \frac{2p}{\chi}\, d_{q,k}$. The coefficient

$$d_{q,k} = \sum_{j=0}^{k} (j+1)\, \tau^j \left(\mu_2^{(k-1)} - \frac{1}{12} \right)^j \tag{4.19}$$

is formally the same as $b_{q,k}$, but they differ in that $b_{q,k}$ is computed using the second moment μ_2 of the distribution $\rho(h)$ at weak coupling in (4.14), whereas for $d_{q,k}$ the moment μ_2 corresponds to $\rho(h)$ at strong coupling in (4.19). We distinguish the weak and strong coupling solutions by ρ_{weak} and ρ_{strong}. The complete solution would require finding $d_{q,\infty}$.

Third order phase transition

Even without closed expressions available for $b_{q,\infty}$ and $d_{q,\infty}$, we can extract information from the form of the eigenvalue density and from what is known at $\tau = 0$. In fact, at $\tau = 0$ the phase transition is of third order [69, 70, 61], which means that $\log \mathcal{Z}_{q\text{-YM}}$ is twice continuously differentiable along the critical curve $\lambda = \lambda_{\text{cr}}(p)$.

The crucial feature is that one can extract the derivative $\frac{\partial}{\partial\lambda} \log \mathcal{Z}^{T\overline{T}}_{q\text{-YM}}$ from the second moment $\mu_2[\rho]$ in each phase, and a third order phase transition implies that $\mu_2[\rho_{\text{weak}}]$ and $\mu_2[\rho_{\text{strong}}]$ agree up to their first derivatives at $\tau = 0$. We now exploit this fact to describe the behaviours of $b_{q,\infty}$ and $d_{q,\infty}$ close to the critical curve. Using the defining expressions (4.14) and (4.19) for $b_{q,\infty}$ and $d_{q,\infty}$, we can expand in λ around $\lambda = \lambda_{\text{cr}}$ to get

$$\begin{aligned} b_{q,\infty} &= f_0^{\text{weak}}(b_{q,\infty}|_{\lambda=\lambda_{\text{cr}}}) + (\lambda - \lambda_{\text{cr}})^2\ f_2^{\text{weak}}(b_{q,\infty}|_{\lambda=\lambda_{\text{cr}}}) + O\big((\lambda - \lambda_{\text{cr}})^3\big) \ , \\ d_{q,\infty} &= f_0^{\text{strong}}(d_{q,\infty}|_{\lambda=\lambda_{\text{cr}}}) + (\lambda - \lambda_{\text{cr}})^2\ f_2^{\text{strong}}(d_{q,\infty}|_{\lambda=\lambda_{\text{cr}}}) + O\big((\lambda - \lambda_{\text{cr}})^3\big) \ . \end{aligned}$$

The agreement of μ_2 at weak and strong coupling at $\tau = 0$ can be used to show that $f_0^{\text{weak}} = f_0^{\text{strong}}$ at the critical point for all $\tau \geq 0$, which in turn guarantees that $b_{q,\infty}$ and $d_{q,\infty}$ agree up to the first derivative. Essentially, by direct inspection one finds that the defining equation for $d_{q,\infty}$ is exactly the same as for $b_{q,\infty}$ at order $O(\lambda - \lambda_{\text{cr}})$.

On the other hand, one can check that, after inclusion of the renormalization of p at weak or strong coupling, the first derivative $\frac{\partial}{\partial\lambda} \log \mathcal{Z}^{T\overline{T}}_{q\text{-YM}}$ depends only on $b_{q,\infty}$ (respectively $d_{q,\infty}$) at weak (respectively strong) coupling, and not on their derivatives. This is done again by expanding the integral formula for μ_2 close to λ_{cr}, and is a consequence of the very simple way in which the parameters $b_{q,\infty}$ and $d_{q,\infty}$ enter.

As an immediate consequence, the second derivative $\frac{\partial^2}{\partial\lambda^2} \log \mathcal{Z}^{T\overline{T}}_{q\text{-YM}}\big|_{\lambda=\lambda_{\text{cr}}}$ depends only on the 1-jets of $b_{q,\infty}$ and $d_{q,\infty}$ at λ_{cr}, which we have argued to match. We therefore find that the phase transition is of third order.

We refer to [38] for further discussion, since the details of the argument do not depend on the explicit form of $\rho(h)$ once we zoom in close to the critical curve.

4.6 Refinement

Consider the refinement of $U(N)$ q-Yang-Mills theory [57] with refinement parameter $t = q^\beta$, for $\beta \in \mathbb{Z}_{>0}$; the unrefined limit $t = q$ then corresponds to $\beta = 1$. A refined definition of the shifted weight variables $\vec{h} \in \mathbb{Z}^N$ introduced in (4.4) is given by

$$h_i = -R_i + \beta \left(i - \frac{N+1}{2} \right) \qquad \text{for} \quad i = 1, \dots, N \ .$$

In this basis we find that the $T\overline{T}$-deformed partition function on $\mathbb{S}^2$ is

$$\mathcal{Z}^{T\overline{T}}_{(q,\,t)\text{-YM}}(\lambda, \tau) = \frac{1}{N!} \sum_{\vec{h} \in \mathbb{Z}^N} \frac{\Delta_{(q,t)}(\vec{h})\, \Delta_{(q,t)}(-\vec{h})}{-\Delta_{(q,t)}(\emptyset)^2} \exp \left(- \frac{\frac{\lambda\, p}{2N} \left(\sum_{i=1}^N h_i^2 - \beta^2 \, \frac{N\,(N^2-1)}{12} \right)}{1 - \frac{\tau}{N^3} \left(\sum_{i=1}^N h_i^2 - \beta^2 \, \frac{N\,(N^2-1)}{12} \right)} \right) \ , \tag{4.20}$$

in which the $T\overline{T}$-deformation only changes the refined (or β-deformed) quadratic Casimir, relative to the case $\tau = 0$. The Macdonald measure $\Delta_{(q,t)}(\vec{h})$ in (4.20) is given for $t = q^\beta$ as

$$\Delta_{(q,t)}(\vec{h}) = \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} 2 \sinh \frac{\lambda\,(h_i - h_j + m)}{2N} \ ,$$

and $\Delta_{(q,t)}(\emptyset)$ is obtained by setting $h_i - h_j = \beta\,(i-j)$. This discrete matrix model has been thoroughly studied in [73].

It was proven in [74] that the 't Hooft limit of (q,t)-Yang-Mills theory coincides with that of q-Yang-Mills theory with rescaled coupling $\lambda' = \beta\,\lambda$. The proof of [74] is straightforwardly adapted to the $T\overline{T}$-deformed setting, and we find that q-Yang-Mills theory and its refinement coincide in the 't Hooft limit after rescaling

$$\lambda' = \beta\,\lambda \qquad \text{and} \qquad \tau' = \beta^2\,\tau \ .$$

When $\beta > 1$ the refinement non-trivially modifies the deformation parameter τ, and the weak coupling region in parameter space is drastically reduced relative to the unrefined case $\beta = 1$.

5 Entanglement entropy

The goal of this section is to study the von Neumann entanglement entropy of $T\overline{T}$-deformed Yang-Mills theory [75] in a general state at large N. The entanglement entropy of two-dimensional Yang-Mills theory has been studied in [76, 77] at finite N (see also [78, 79]), and a thorough analysis of the large N limit in both phases and including sub-leading corrections appeared in [80]. The $T\overline{T}$-deformation at finite N was considered in [39]. In this section we give a first step towards the extension of the large N analysis to the $T\overline{T}$-deformed setting.

The formalism of [79] used to study the entanglement entropy (at finite N) is well-suited to the class of almost topological gauge theories considered in the present paper, and can be adapted to q-Yang-Mills theory and its $T\overline{T}$-deformation. In the following we consider only the case of ordinary Yang-Mills theory.

Entanglement entropy at large N

The calculations which follow mimic those of [80]. We start by using the replica trick [81] to write down the formula for the entanglement entropy. The replica trick consists in defining an auxiliary manifold Σ^n, which is an n-sheeted Riemann surface obtained by cyclically gluing n sheets along cuts on a spatial subregion of Σ. This defines a branched cover $\Sigma^n \longrightarrow \Sigma$. One then continues the dependence on n of the partition function of a quantum field theory $\mathscr{T}$ on Σ^n to values $n \in \mathbb{R}_{>0}$. We can then compute the entanglement entropy as

$$S_{\text{entang}}[\Sigma] = -\frac{\partial}{\partial n}\left(\frac{\mathcal{Z}_{\mathscr{T}}[\Sigma^n]}{\mathcal{Z}_{\mathscr{T}}[\Sigma]^n}\right)\Bigg|_{n=1} = \left(1-\frac{\partial}{\partial n}\right)\log \mathcal{Z}_{\mathscr{T}}[\Sigma^n]\Bigg|_{n=1} .$$

This formula may seem problematic, due to the non-uniqueness of analytic continuation from $n \in \mathbb{Z}_{>0}$ to $n \in \mathbb{R}_{>0}$. Usually, one would need to define a continuation to $n \in \mathbb{C}$ and then check, with the aid of Carlson's theorem, that different continuations would differ by an everywhere vanishing function. Nevertheless, as will be manifest below, we can translate this problem into the problem of analytically continuing the Euler characteristic $\chi(\Sigma)$ to a real quantity χ in a neighbourhood of the physical value $\chi = 2$. Throughout Section 4 we have been careful in keeping track of the Euler characteristic χ in all the formulas, and we can continue them to $\chi > 0$.[6]

Following [80], in the case at hand we define

$$P(\vec{h}) = \frac{1}{\mathcal{Z}_{\text{YM}}^{T\overline{T}}(A,\tau)}\left(\frac{\Delta(\vec{h})}{\Delta(\emptyset)}\right)^{\chi} \mathrm{e}^{-\frac{A}{2N}C_2^{T\overline{T}}(\vec{h},\tau)}$$

which is the probability of finding a given configuration of weights $\vec{h} \in \mathbb{Z}^N$, with variables as defined in (4.4), where we recall that $\Delta(\vec{h})$ is the Vandermonde determinant (4.7). Note that $\Delta(\emptyset)^{\chi} = G(N+1)^{\chi}$ does not contribute, since it is cancelled by the normalization. We see that there is no issue in defining $P(h)$ for non-integer $\chi > 0$.

Then the replica trick gives [80]

$$S_{\text{entang}}(A,\tau) = -\sum_{\vec{h}\in\mathbb{Z}^N} P(\vec{h})\log P(\vec{h}) + \sum_{\vec{h}\in\mathbb{Z}^N} P(\vec{h})\log \Delta(\vec{h}) ,$$

where the first series is identified with the Shannon entropy, which measures the entropy due to the fluctuations of the values of the weights $\vec{h}$, while the second series is identified with the Boltzmann entropy, which measures the total entropy due to each sector $\vec{h}$ weighted with their probability (see [77, 78, 80] for further details).

We notice that the Shannon entropy is

$$S_{\text{Shan}}(A,\tau) := -\sum_{\vec{h}\in\mathbb{Z}^N} P(\vec{h})\log P(\vec{h}) = \sum_{\vec{h}\in\mathbb{Z}^N} P(\vec{h})\, S_{\text{YM}}[\vec{h}] .$$

This implies in particular that at large N, when the saddle point configuration dominates and the fluctuations are averaged out, we get

$$\lim_{N\to\infty} S_{\text{Shan}}(A,\tau) = 0 .$$

[6] We stress that we are computing the entanglement entropy of a quantum field theory, in which JT gravity has been integrated out *ab initio*. Therefore the replicas that appear here should not be confused with the (one-dimensional unsewn) replicas used in purely gravitational theories.

This was also found in [80], and it is in fact a general feature of all the deformations of two-dimensional Yang-Mills theory studied in this paper, which follows directly from their matrix model presentations. This of course will not hold after the introduction of sub-leading corrections.

In the remainder of this section we focus on the Boltzmann entropy

$$S_{\mathrm{Boltz}}(A,\tau) := \sum_{\vec{h}\in\mathbb{Z}^N} P(\vec{h}) \log \Delta(\vec{h}) \ .$$

Entanglement entropy in the weak coupling phase

Let us define

$$\mathcal{F}^{T\overline{T}}(A,\tau) := \lim_{N\to\infty} \frac{1}{N^2} \log \mathcal{Z}^{T\overline{T}}_{\mathrm{YM}}(A,\tau) \ .$$

This quantity is relevant for the computation of the entanglement entropy at large N, because the Boltzmann entropy normalized by N^2 and by the number of entangling points is given by

$$\widetilde{S}_{\mathrm{Boltz}}(A,\tau) = \frac{\partial \mathcal{F}^{T\overline{T}}(A,\tau)}{\partial \chi} \ . \tag{5.1}$$

In the large N limit we find

$$\mathcal{F}^{T\overline{T}}(A,\tau) = \frac{3\chi}{4} + \frac{\chi}{2}\int_{\mathrm{supp}(\rho)} \mathrm{d}h\, \rho(h) \log|h| - \frac{A}{4}\left(\frac{C_2 - \frac{1}{12}}{1-\tau\, C_2}\right) \ , \tag{5.2}$$

where

$$C_2 = \mu_2 - \frac{1}{12} = \int_{\mathrm{supp}(\rho)} \mathrm{d}h\, \rho(h)\, h^2 - \frac{1}{12}$$

is the limiting value of the quadratic Casimir at large N. (Note that $-\frac{1}{12}$ appears twice in the last numerator of (5.2).) This expression is now to be evaluated in each phase using the corresponding eigenvalue density $\rho(h)$ obtained in [38].

In the weak coupling (small area) phase we obtain

$$\mathcal{F}^{T\overline{T}}(A<A_{\mathrm{cr}},\tau) = \frac{\chi}{2} - \frac{\chi}{4}\log\left(\frac{2A\, b_\infty}{\chi}\right) - \frac{A}{4}\, \frac{\frac{\chi}{2A\, b_\infty} - \frac{1}{6}}{1 - \frac{\tau\,\chi}{2A\, b_\infty} + \frac{\tau}{12}} \ ,$$

where [38]

$$b_\infty\big(\tfrac{2A}{\chi},\tau\big) = \frac{1 + \frac{\tau\,\chi}{A}\,(1+\frac{\tau}{12}) + \sqrt{1 + \frac{2\tau\,\chi}{A}\,(1+\frac{\tau}{12})}}{2(1+\frac{\tau}{12})^2} \ .$$

Plugging this expression into (5.1) we obtain

$$\begin{aligned}\widetilde{S}_{\mathrm{Boltz}}(A<A_{\mathrm{cr}},\tau) = {}& -\frac{A}{4}\left(1-\frac{\tau}{12}\right)\left(\frac{\big(2A\, b_\infty(\frac{2A}{\chi},\tau)\big)^{-1} + \chi^{-1}\, b_\infty(\frac{2A}{\chi},\tau)^{-2}\, \frac{\partial b_\infty(A,\tau)}{\partial A}}{\left(1-\tau\big(\frac{\chi}{2A\, b_\infty(\frac{2A}{\chi},\tau)} - \frac{1}{12}\big)\right)^2}\right) \\ & + \frac{1}{2} - \frac{1}{4}\log\left(\frac{2A\, b_\infty(\frac{2A}{\chi},\tau)}{\chi}\right) + \frac{3A\, b_\infty(\frac{2A}{\chi},\tau)}{2} + \frac{A^2}{\chi}\, \frac{\partial b_\infty(A,\tau)}{\partial A} \ .\end{aligned}$$

We have not inserted the explicit expressions for b_∞ and $\frac{\partial b_\infty}{\partial A}$ in order to avoid clutter, but the formula can be straightforwardly evaluated. The normalized Boltzmann entropy is plotted in Figure 7 for $\chi = 2$ and for various values of τ.

This describes the evolution of the result of [80] along the RG flow triggered by the $T\overline{T}$-deformation of two-dimensional Yang-Mills theory, which shows that the entropy decreases with increasing deformation parameter τ.

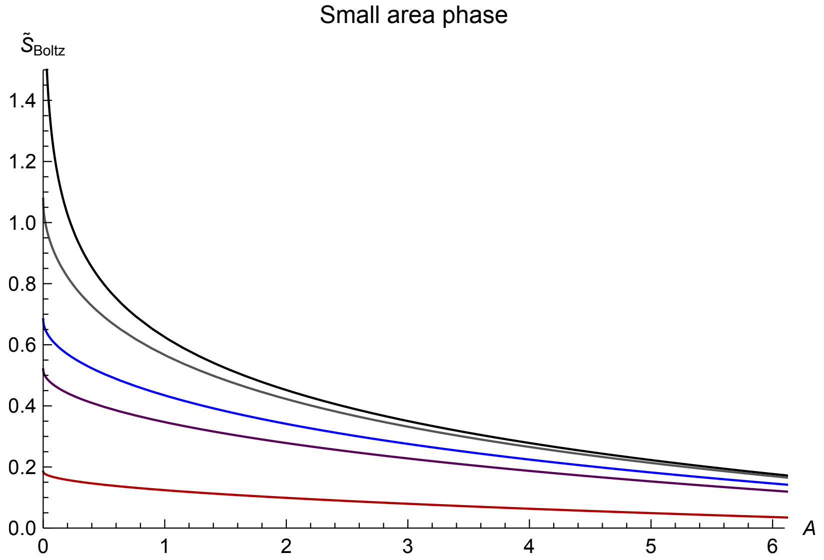


Figure 7: Entanglement entropy of $T\overline{T}$-deformed Yang-Mills theory at large N, normalized by N^2 and per number of entangling points, as a function of the area A in the weak coupling phase. The different curves correspond to the values $\tau = 0$ (black), $\tau = 0.1$ (gray), $\tau = 0.5$ (blue), $\tau = 1$ (purple), and $\tau = 5$ (red).

6 Outlook

There are many intriguing open problems that could be tackled, but a treatment of them is out of the scope of the present paper. In this final section we point out four directions for which the present study has laid the groundwork for further investigation.

Pullback of $T\overline{T}$-deformation to $\mathcal{N} = 4$ Yang-Mills theory

From the construction of [58, 53] (see also [61, 62, 82]), q-deformed Yang-Mills theory on Σ descends from $\mathcal{N} = 4$ supersymmetric Yang-Mills theory living on a non-compact four-manifold X_4 which is the total space of the degree p holomorphic line bundle

$$\mathcal{O}(-p) \longrightarrow \Sigma \ .$$

To implement the boundary conditions on each fibre $\mathbb{C}_z$ over points $z \in \Sigma$, one introduces a group element $g(z, \bar{z}) \in G$ encoding the holonomy of the gauge field at infinity in $\mathbb{C}_z$:

$$g(z, \bar{z}) = \mathrm{e}^{\,\mathrm{i}\,\phi(z,\bar{z})} := \exp\left(\oint_{S^1_\infty(z)} A^{(4)} \right) ,$$

where $A^{(4)}$ is the four-dimensional gauge connection and $S^1_\infty(z)$ is the circle at infinity in the fibre over $z \in \Sigma$.

For $p > 0$ the line bundle $\mathcal{O}(-p)$ is non-trivial and presents p topological obstructions to defining the section $\phi(z, \bar{z})$ globally. This produces exactly p copies of the action

$$\int_\Sigma \mathrm{Tr}(\phi^2)\, \omega \ ,$$

which, together with the compactness of $\phi(z, \bar{z})$, deforms the topological Yang-Mills theory on Σ, descending in the bulk from the four-dimensional gauge theory on X_4, to q-deformed Yang-Mills theory [58, 53].

From our point of view it is then natural to ask what would be the effect and the geometrical meaning of the $T\overline{T}$-deformation, once the theory is embedded into the four-dimensional setting. In particular, it would be interesting to determine whether a four-dimensional picture emerges when the gauge theory on Σ is dynamically coupled to JT gravity.

Two-dimensional Yang-Mills-Higgs theory

In [83, 84] a two-dimensional Yang-Mills-Higgs theory was considered, and it was shown that the wavefunctions of the $U(N)$ theory coincide with the wavefunctions of the N-particle Hamiltonian of nonlinear Schrödinger theory. We shall now briefly present the setup of [83].

Consider the Yang-Mills action and take its minimal supersymmetric extension, with BRST multiplet (A, ϕ, ψ), which adds a term $\int_\Sigma \mathrm{Tr}\,(\psi \wedge \psi)$ to the action. Denote by Q the supercharge under which the extended Yang-Mills action is invariant. The Yang-Mills-Higgs theory is built by including the Higgs field Φ, a bosonic $\mathfrak{g}$-valued one-form in the adjoint representation, and its fermionic superpartner Ψ, as well as two pairs of scalar auxiliary fields (φ_+, χ_+) and (φ_-, χ_-), with $\varphi_\pm$ Grassmann-even and $\chi_\pm$ Grassmann-odd $\mathfrak{g}$-valued scalars in the adjoint representation. The action is then further modified as

$$S_{\mathrm{YM}} + \{\mathsf{Q}, S_{\mathrm{H}}\} \ ,$$

where [83, 84]

$$S_{\mathrm{H}} = \int_\Sigma \mathrm{Tr}\left(\frac{1}{2}\,\Phi \wedge \Psi + \alpha\,(\chi_+\,\varphi_- + \chi_-\,\varphi_+)\;\omega\right) \ ,$$

for arbitrary $\alpha \in \mathbb{R}$. Acting with Q one finds [83] that the action is quadratic in all of the fields $(\psi, \Phi, \Psi, \varphi_\pm, \chi_\pm)$, so they can be integrated out and produce one-loop determinants. The abelianization technique works in that case, and the one-loop determinants coming from the integration of the Higgs sector modify the partition function of the pure Yang-Mills theory on Σ expressed in the representation basis.

It would be very interesting to study the $T\overline{T}$-deformation of such theories: besides the deformation of the quadratic potential in the scalar ϕ, also the quadratic term involving the fields $\varphi_\pm$ and $\chi_\pm$ is affected by the $T\overline{T}$-deformation as in (2.2), and the fields can no longer be integrated out. Nonetheless, the gauge connection A does not appear in any of the deformation terms, and the abelianization should work for the path integral over ϕ. Yet, we are left with the path integral over $\varphi_\pm$ and $\chi_\pm$.

On the other hand, it is well-known that $T\overline{T}$-deformation preserves integrability. Therefore, since the wavefunctions of the Yang-Mills-Higgs theory are those of an integrable quantum mechanics, it would be appropriate to ask whether the wavefunctions of the $T\overline{T}$-deformation of Yang-Mills-Higgs theory are still related to an integrable system.

Defects

Another aspect worth pursuing is the inclusion of defects. Here we show how to $T\overline{T}$-deform two-dimensional Yang-Mills theory with defects [85].

Let $P \longrightarrow \Sigma$ be a principal G-bundle and $D \longrightarrow \Sigma$ an $\mathsf{Out}(G)$-bundle on Σ, where $\mathsf{Out}(G)$ is the group of outer automorphisms of the Lie group G. Following [85] we define the D-twisted G-bundle

$$P' \longrightarrow \Sigma \ ,$$

which is a G'-bundle with structure group

$$G' \cong G \rtimes \mathsf{Out}(G) \ .$$

The partition function of two-dimensional Yang-Mills theory with symmetry defects is then [85]

$$\mathcal{Z}_{\text{YM-def}}[\Sigma] = \int_{\mathscr{A}_G^D(\Sigma)} \mathscr{D}(P', A) \int \mathscr{D}\phi \ \mathrm{e}^{-S_{\text{YM}}(P',A,\phi)} \ ,$$

where $\mathscr{A}_G^D(\Sigma)$ is the space of pairs (P', A) of D-twisted G-bundles $P' \longrightarrow \Sigma$ with connection A. The scalar field ϕ can only be introduced after fixing P', while the action is the standard Yang-Mills action, where now we have stressed the dependence on the choice of D-twisted bundle.

As extensively discussed in Section 2, to turn on the $T\overline{T}$-deformation, we must insert JT gravity as the innermost path integral here. It is immediately seen that we can again integrate out the dynamical coframe field, reproducing the deformation of the potential (2.2). We stress that one should be very careful with the order of path integration: since ϕ is only introduced after fixing a choice of bundle P', one must first compute the gravitational path integral, followed by the path integral over ϕ as a function of the choice $(P', A) \in \mathscr{A}_G^D(\Sigma)$, and only then eventually integrate over $\mathscr{A}_G^D(\Sigma)$.

The outcome of this analysis is that $T\overline{T}$-deforming two-dimensional Yang-Mills theory with defects modifies the action exactly as without defects. On the other hand, it would be interesting to investigate further the interplay between $T\overline{T}$-deformation and symmetry defects, which requires the introduction of defects in the already $T\overline{T}$-deformed field theory. This would entail a reformulation of the $T\overline{T}$-deformed quantum field theory in the formalism of [86].

$T\overline{T}$-deformation in higher dimensions

We conclude on a rather speculative note. Recalling that a univocal definition of what the analogues of the $T\overline{T}$-deformation would be in higher dimensions is still lacking, a relevant question is the following: If we take a two-dimensional theory $\mathscr{T}_A^{(2)}$ with a known relationship to a higher-dimensional field theory $\mathscr{T}_B^{(d>2)}$ and deform it, what would be the effect on $\mathscr{T}_B^{(d>2)}$? This question can be represented symbolically by the following diagram:

$$\begin{array}{ccc} \mathscr{T}_A^{(2)} & \longleftrightarrow & \mathscr{T}_B^{(d>2)} \\ {\scriptstyle T\overline{T}\text{-deformation}}\ \Big\downarrow & & \vdots\ {\scriptstyle \text{deformation?}} \\ \mathscr{T}_A^{T\overline{T}(2)} & \leftarrow\text{-----}\rightarrow & \mathscr{T}_B^{\bullet(d>2)} \end{array} \tag{6.1}$$

As a first step, we can ask the following questions: does the relation between the theories survive along the RG flow triggered by the $T\overline{T}$-deformation of $\mathscr{T}_A^{(2)}$? If so, what would be the theory on the other side, denoted $\mathscr{T}_B^{\bullet(d>2)}$ in the diagram (6.1)? Does it admit a description as a deformation of the theory $\mathscr{T}_B^{(d>2)}$?

The answers to these questions within a general framework seem out of reach at the moment. Nevertheless, we hope that the study of the $T\overline{T}$-deformed q-Yang-Mills theory will lead to exciting discoveries in this direction. Indeed, q-Yang-Mills theory is intimately connected to four-dimensional supersymmetric Yang-Mills theory in many ways. As described above, one very direct connection is that it descends from $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on a non-compact four-manifold X_4. There are, however, other tight and direct links between q-deformed two-dimensional Yang-Mills theory and four-dimensional supersymmetric Yang-Mills theory, for example in the equivalence with the superconformal index [87] in the context of Gaiotto's $4d/2d$ dualities [88]. Two-dimensional Yang-Mills theory and its deformations appear as well from direct localization computations on the four-sphere $\mathbb{S}^4$ [89, 90, 91] and on the four-dimensional hemisphere [92], possibly with the inclusion of defects [93].

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A Approximate solution for $b_{q,\infty}$

In Section 4.2 we have studied the weak coupling phase in the large N limit of $T\overline{T}$-deformed q-Yang-Mills theory. Turning on the $T\overline{T}$-deformation amounts to replacing $p \mapsto p\, b_{q,\infty}$, where $b_{q,\infty}$ depends on p, λ and τ, and is implicitly determined by (4.14). In this appendix we analyze (4.14) in the small q-deformation regime, that is, $p \to \infty$ with $\lambda\, p = A$ fixed. We set $\chi = 2$; the χ-dependence is eventually reinstated by replacing p with $\frac{2p}{\chi}$.

Expanding (4.14) at large p we get

$$b_{q,\infty} = \left(1 - \tau \left(\frac{1}{A\, b_{q,\infty}} + \frac{1}{6p^2\, b_{q,\infty}^2} + \frac{A}{72p^4\, b_{q,\infty}^3} + O\big(p^{-6}\big) - \frac{1}{12}\right)\right)^{-2} . \tag{A.1}$$

At $O(p^{-2})$ this equation admits only one solution satisfying

$$\lim_{\tau \to 0} b_{q,\infty} = 1 ,$$

as required in order to recover the correct behaviour when the $T\overline{T}$-deformation is turned off. This solution is a decreasing function of p, for fixed A and τ, and converges from above to the corresponding quantity b_∞ of $T\overline{T}$-deformed Yang-Mills theory as p becomes large. The explicit expression for this solution is rather lengthy and cumbersome, hence we do not write it explicitly. Instead, we plot the solution as a function of p, for different values of A and τ, in Figures 8, 9 and 10.

From these plots we see the qualitative behaviour of $b_{q,\infty}$, and for all fixed choices of A and τ it converges asymptotically to the value b_∞. This guarantees that $b_{q,\infty}$ indeed arises as a q-deformation of b_∞, and b_∞ is correctly recovered in the limit (3.4). In particular, when $\tau > 0$ we have

$$1 < b_\infty < b_{q,\infty} ,$$

that is, $b_{q,\infty}$ is a decreasing function of p which converges to b_∞ from above.

We now look at how these features are altered when the $O(p^{-4})$ contribution is taken into account. Going to the next non-trivial order introduces a dependence on $b_{q,\infty}^{-3}$ in the right-hand side of (A.1). The values of $b_{q,\infty}$ are thus found by solving a degree seven polynomial equation, but six solutions will be spurious. We do not dive into an analytic approach, and instead numerically illustrate the behaviour of the solution as a function of p, for different values of τ. From what we have learnt at $O(p^{-2})$, it is sufficient to limit ourselves to $A = 1$ and a few values of τ. The solutions are plotted in Figures 11 and 12 for $\tau = 0.1$ and $\tau = 1$, respectively. We see that the solution is

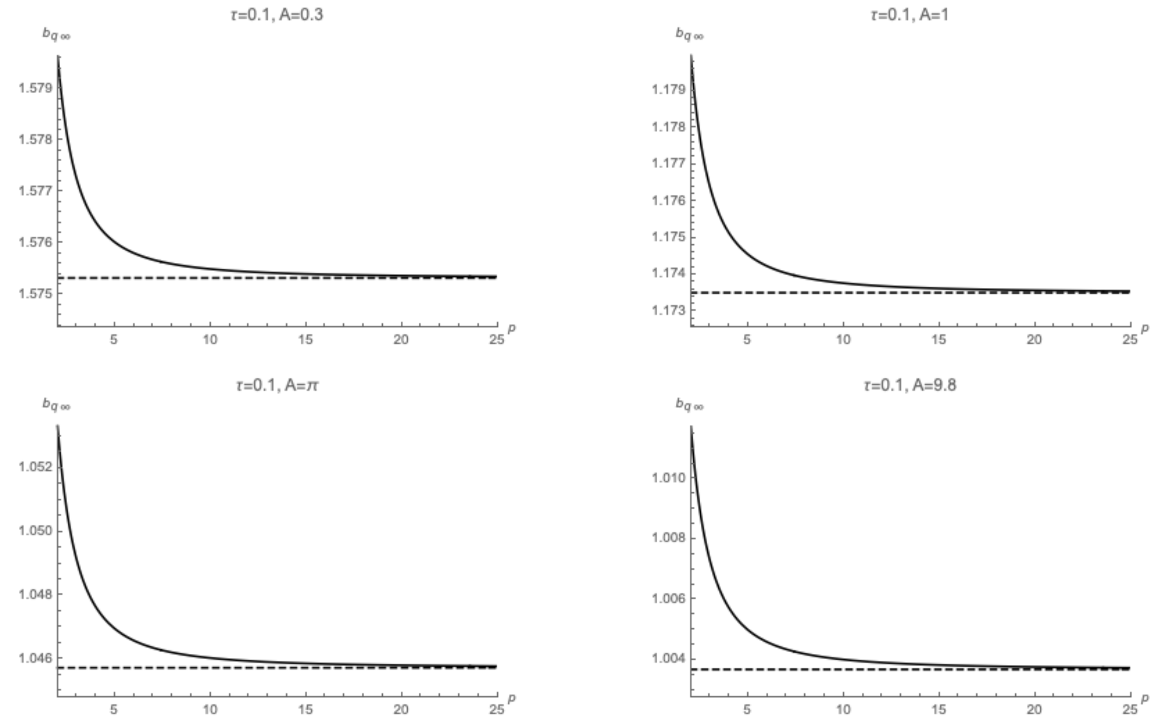

Figure 8: Plot of $b_{q,\infty}$ as a function of p at $\tau = 0.1$. The four plots correspond to $A = 0.3, 1, \pi, 9.8$. The dashed horizontal line is the corresponding value of b_∞ in $T\overline{T}$-deformed Yang-Mills theory without q-deformation.

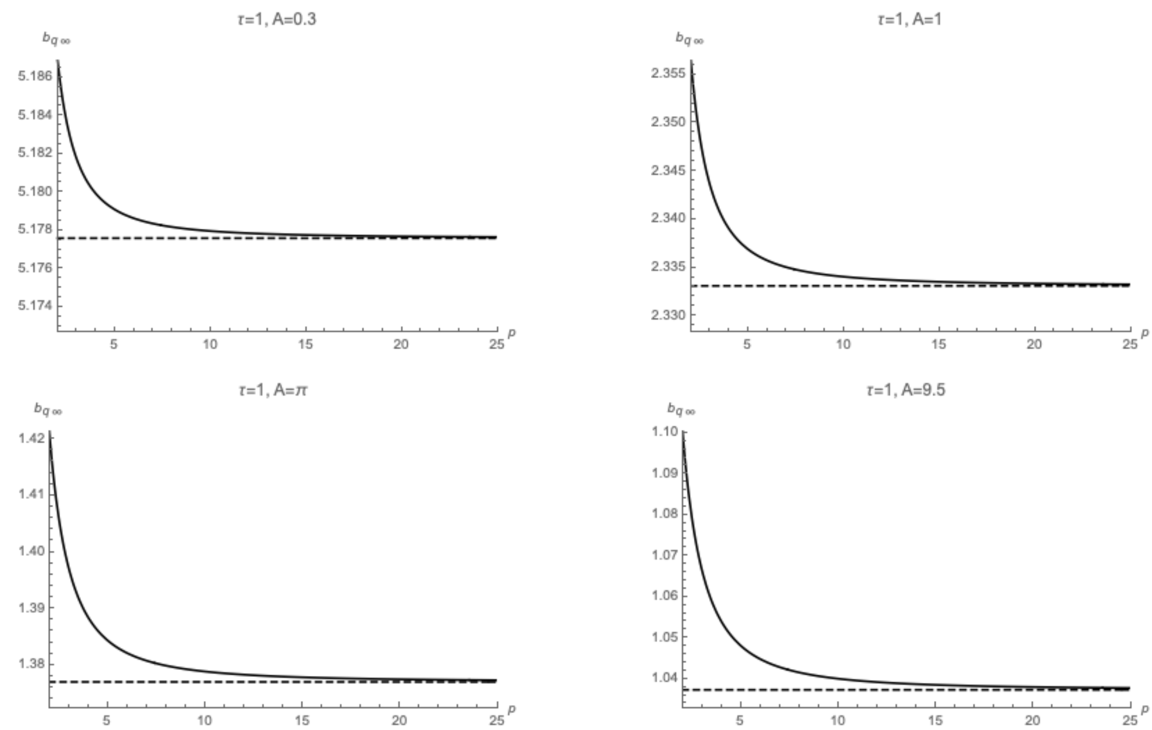

Figure 9: Plot of $b_{q,\infty}$ as a function of p at $\tau = 1$. The four plots correspond to $A = 0.3, 1, \pi, 9.5$. The dashed horizontal line is the corresponding value of b_∞ in $T\overline{T}$-deformed Yang-Mills theory without q-deformation.

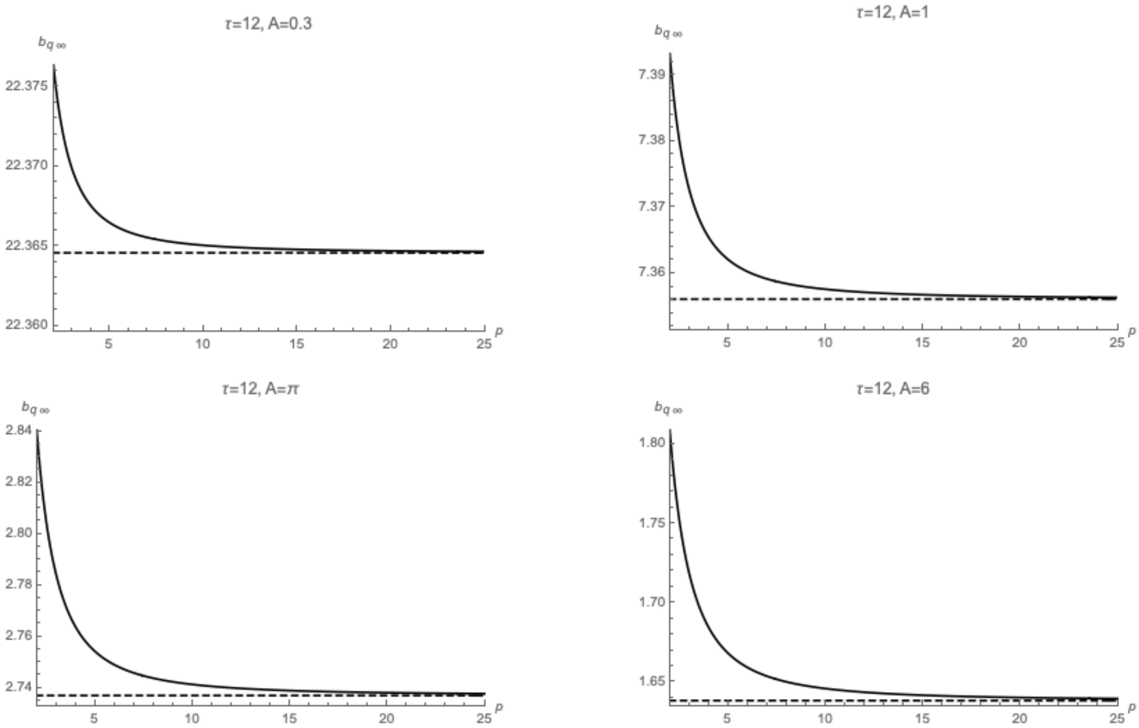

Figure 10: Plot of $b_{q,\infty}$ as a function of p at $\tau = 12$. The four plots correspond to $A = 0.3, 1, \pi, 6$. The dashed horizontal line is the corresponding value of b_∞ in $T\overline{T}$-deformed Yang-Mills theory without q-deformation.

again a decreasing function of p, which changes rapidly for small p and is almost constant at large p. We also check that the solution approaches 1 as $\tau \to 0$, in agreement with our analytic study. The conclusions therefore remain unchanged after the inclusion of $O(p^{-4})$ corrections.

B Instantons in $T\overline{T}$-deformed q-Yang-Mills theory

In the analyses of [69, 61], where the Poisson resummation of the heat kernel expansion of the partition function is done explicitly for the q-deformed theory, it was found that the instantons are responsible for the phase transition also in the q-deformed case. An analogous result was presented in [70], where only the first instanton sector was taken into account. From the results of [38], we expect that this property is not affected by the $T\overline{T}$-deformation.

Two-dimensional Yang-Mills instantons

Let us start with some generalities concerning instantons in two-dimensional Yang-Mills theory. We start by rewriting the partition function of $T\overline{T}$-deformed q-Yang-Mills theory on $\mathbb{S}^2$ as in (4.16):

$$\mathcal{Z}^{T\overline{T}}_{q\text{-YM}}(\lambda, \tau) = \frac{1}{N!} \sum_{\vec{\ell} \in \mathbb{Z}^N} Z_{\vec{\ell}}(\lambda, \tau) \ , \tag{B.1}$$

where $Z_{\vec{\ell}}$ encodes the contribution of an instanton labelled by $\vec{\ell} = (\ell_1, \ell_2, \ldots, \ell_N) \in \mathbb{Z}^N$. Two-dimensional $U(N)$ Yang-Mills instantons are given by diagonal $\mathfrak{u}(N)$-valued gauge fields

$$A = \mathrm{diag}\left(A_{\ell_1}, A_{\ell_2}, \ldots, A_{\ell_N}\right)$$

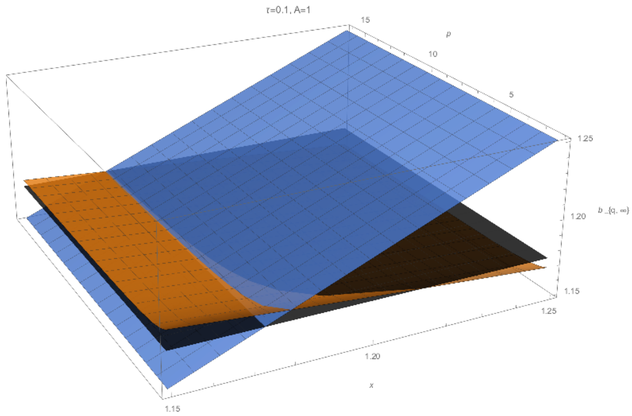

Figure 11: Plot of $b_{q,\infty}$ as a function of p, at $A = 1$ and $\tau = 0.1$. The blue surface is the function x on the left-hand side of (A.1). The orange surface is the right-hand side of (A.1) as a function of p and x. $b_{q,\infty}$ is determined by the intersection of these two surfaces. The black horizontal surface is the asymptotic value b_∞.

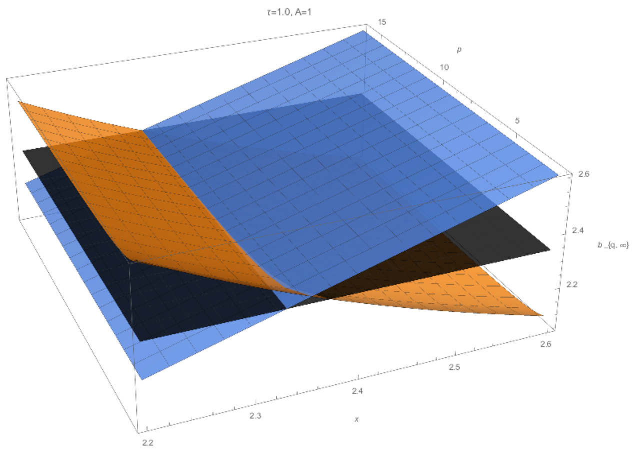

Figure 12: Plot of $b_{q,\infty}$ as a function of p, at $A = 1$ and $\tau = 1.0$. The blue surface is the function x on the left-hand side of (A.1). The orange surface is the right-hand side of (A.1) as a function of p and x. $b_{q,\infty}$ is determined by the intersection of these two surfaces. The black horizontal surface is the asymptotic value b_∞.

where A_{ℓ_i} is a Dirac monopole potential of charge $\ell_i \in \mathbb{Z}$. Each entry is a gauge connection in the monopole bundle over $\mathbb{S}^2$ of magnetic charge ℓ_i,

$$\mathcal{L}^{\otimes \ell_i} \longrightarrow \mathbb{S}^2 ,$$

where $\mathcal{L}$ is the canonical line bundle of $\mathbb{P}^1$ (we identify $\mathbb{P}^1 \cong \mathbb{S}^2$). Each instanton configuration determines a splitting of the $U(N)$ gauge bundle on $\mathbb{S}^2$, which in turn describes a symmetry breaking

$$U(N) \longrightarrow \prod_{l\in\mathbb{Z}} U(N_l) ,$$

where N_l encodes the degeneracy of the magnetic charges: N_l counts how many times the integer $l \in \mathbb{Z}$ appears in the string $\vec{\ell} = (\ell_1, \ell_2, \dots, \ell_N) \in \mathbb{Z}^N$ (we omit factors in the product with $N_l = 0$). A generic configuration breaks the gauge group $U(N)$ to its maximal torus $U(1)^N$, while the one-instanton sector describes a soft breaking $U(N) \to U(1) \times U(N-1)$. The trivial connection $A = 0$ (the zero-instanton sector) is the only gauge field preserving the full $U(N)$ symmetry.

The restriction to gauge-inequivalent configurations reduces the coweights $\vec{\ell} \in \mathbb{Z}^N$ to the Weyl chamber

$$\ell_1 \geq \ell_2 \geq \cdots \geq \ell_N ,$$

but from the symmetry of the partition function we can drop this restriction at the cost of an overall factor $(N!)^{-1}$.

Complete instanton partition function

We will now focus on the instanton expansion (B.1) and give the first steps towards understanding the complete instanton partition function, including all instanton contributions. However, due to the sophistication of the $T\overline{T}$-deformation, we cannot provide a complete answer and so we only present a partial analysis here.

Each summand $Z_{\vec{\ell}}$ in (B.1) can be computed as the Fourier transform of the contribution Z_R of an irreducible $U(N)$ representation R:

$$Z_{\vec{\ell}}(\lambda,\tau) = \frac{1}{\Delta_q(\emptyset)^2} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N \mathrm{d}h_i \, \mathrm{e}^{-2\pi\,\mathrm{i}\,\ell_i\,h_i} \right) \prod_{1\leq i<j\leq N} 4\sinh^2 \frac{\lambda\,(h_i - h_j)}{2N} \times \exp\left(- \frac{\frac{\lambda p}{2N} \left(\sum_{j=1}^N h_j^2 - \frac{N\,(N^2+1)}{12} \right)}{1 - \frac{\tau}{N^3} \left(\sum_{j=1}^N h_j^2 - \frac{N\,(N^2+1)}{12} \right)} \right) .$$

We can expand the effect of the $T\overline{T}$-deformation in a double power series, finding

$$Z_{\vec{\ell}}(\lambda,\tau) = K_{N,\lambda,p} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N \mathrm{d}h_i \, \mathrm{e}^{-\frac{\lambda p}{2N}\,(h_i^2 + \frac{4\pi\,\mathrm{i}\,N}{\lambda p}\,\ell_i\,h_i)} \right) \prod_{1\leq i<j\leq N} 4\sinh^2 \frac{\lambda\,(h_i - h_j)}{2N} \times \sum_{n=0}^\infty \frac{1}{n!} \left(-\frac{\lambda p}{2N} \right)^n \sum_{k=0}^\infty c_k(n) \left(\frac{\tau}{N^3} \right)^k \left(\sum_{i=1}^N h_i^2 - \frac{N\,(N^2-1)}{12} \right)^k ,$$

where $\{c_k(n)\}$ are the coefficients of the Taylor expansion of the function $\left(\frac{x}{1-x}\right)^n$ around $x = 0$, and

$$K_{N,\lambda,p} = \mathrm{e}^{\frac{\lambda\,p\,(N^2-1)}{24N}} \, \Delta_q(\emptyset)^{-2}$$

is an overall factor. A direct computation of $Z_{\vec{\ell}}$ is difficult already for $\tau = 0$. However, by exploiting the Weyl denominator formula, we can compute a different function $\tilde{Z}_{\vec{\ell}}$, which we define in the same way as $Z_{\vec{\ell}}$ but with only a single power of the q-deformed Vandermonde determinant $\Delta_q(\vec{h})$, instead of the square $\Delta_q(\vec{h})^2$ which enters the expression for $Z_{\vec{\ell}}$. Explicitly,

$$\begin{aligned}\tilde{Z}_{\vec{\ell}}(\lambda,\tau) = K_{N,\lambda,p} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} \mathrm{d}h_i \ \mathrm{e}^{-\frac{\lambda p}{2N}\left(h_i^2 + \frac{4\pi\,\mathrm{i}\,N}{\lambda p}\,\ell_i\,h_i\right)} \right) \prod_{1\leq i<j\leq N} 2\sinh\frac{\lambda\,(h_i-h_j)}{2N} \\ \times \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\lambda p}{2N}\right)^n \sum_{k=0}^{\infty} c_k(n)\left(\frac{\tau}{N^3}\right)^k \left(\sum_{i=1}^{N} h_i^2 - \frac{N\,(N^2-1)}{12}\right)^k .\end{aligned} \tag{B.2}$$

At $\tau = 0$, the Weyl denominator formula gives [69, 61]

$$\tilde{Z}_{\vec{\ell}}(\lambda,0) = K'_{N,\lambda,p}\ \mathrm{e}^{-\frac{2\pi^2 N}{\lambda p}\sum_{i=1}^{N}\ell_i^2} \sum_{1\leq i<j\leq N} \sigma_{ij} \sin\left(\frac{\ell_i-\ell_j}{2p}\right) ,$$

where $\sigma_{ij} = +1$ if the permutation of the first N integers which sends i and j to the first and second positions respectively is even, and $\sigma_{ij} = -1$ if the permutation is odd. Here $K'_{N,\lambda,p}$ is another overall constant that we will not keep track of.

Turning on the $T\overline{T}$-deformation corresponds to introducing powers of the original quadratic Casimir, which in the expansion in (B.2) corresponds to introducing the terms with $k > 0$. The Fourier transform of each summand in (B.2) then gives

$$\begin{aligned}\tilde{Z}_{\vec{\ell}}(\lambda,\tau) = \tilde{K}_{N,\lambda,p}\ \mathrm{e}^{-\frac{2\pi^2 N}{\lambda p}\sum_{i=1}^{N}\ell_i^2} \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\lambda p}{2N}\right)^n \sum_{k=0}^{\infty} c_k(n)\left(\frac{\tau}{N^3}\right)^k \\ \times \sum_{1\leq i<j\leq N} \sigma_{ij}\left(P_{2k}^{\mathrm{s}}(\vec{\ell})\sin\left(\frac{\ell_i-\ell_j}{2p}\right) - (\ell_i-\ell_k)\,P_{2k}^{\mathrm{c}}(\vec{\ell})\cos\left(\frac{\ell_i-\ell_j}{2p}\right)\right) ,\end{aligned} \tag{B.3}$$

where P_{2k}^{s} and P_{2k}^{c} are totally symmetric polynomials of degree $2k$ in N variables, which can be explicitly computed order by order in τ.

At this point, the expression for $Z_{\vec{\ell}}$ could be obtained by the Fourier convolution of two functions of the form (B.3). The explicit calculation is rather cumbersome and should be performed order by order in τ; we do not attempt it here. However, we stress that, for each instanton sector $\vec{\ell}$, every order in the perturbative expansion can in principle be evaluated with the generalization of the strategy of [69, 61] that we have just sketched.

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